A Survey of Parking Functions

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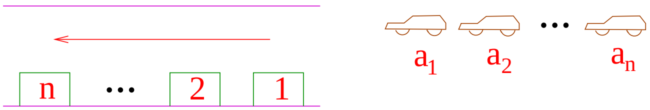
A parking scenario



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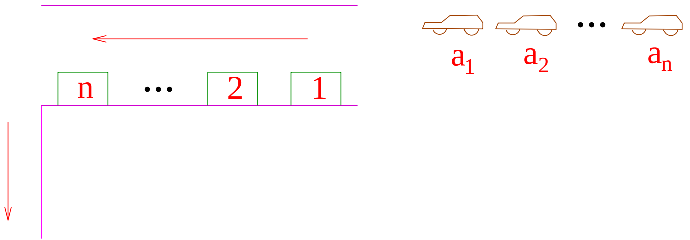


Parking functions

Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

Small examples

$n = 2$: 11 12 21

$n = 3$: 111 112 121 211 113 131 311 122
212 221 123 132 213 231 312 321

Parking function characterization

Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

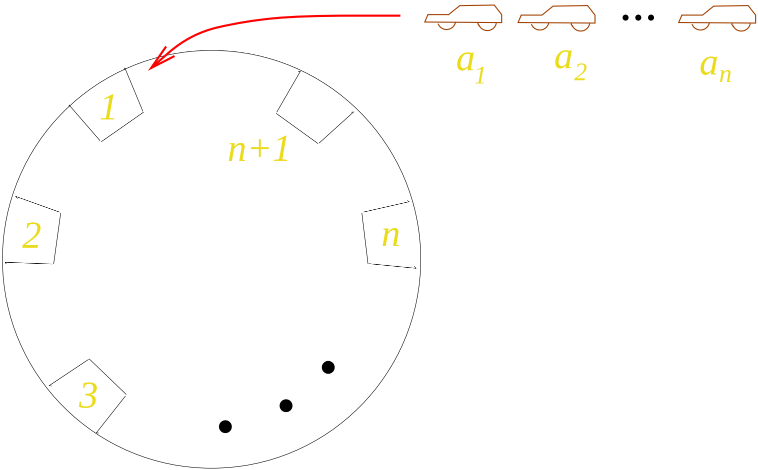
Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Enumeration of parking functions

Theorem (**Pyke**, 1959; **Konheim and Weiss**, 1966). *Let $f(n)$ be the number of parking functions of length n. Then* $\boldsymbol{f(n) = (n+1)^{n-1}}$.

Proof (**Pollak**, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.

Pollak's proof



Conclusion of Pollak's proof

Now all cars can park, and there will be one empty space. α is a parking function $\Leftrightarrow$ if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1 + i, a_2 + i, \ldots, a_n + i) \text{ (modulo } n+1)$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

Prime parking functions

Definition (**I. Gessel**). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

Note. A sequence $b_1 \leq b_2 \leq \cdots \leq b_n$ is an increasing parking function if and only if $1 \leq b_1 \leq \cdots \leq b_n$ is an increasing prime parking function.

Factorization of increasing PF's

1	2	3	4	5	6	7	8	9	10	11
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$$\rightarrow (1,1),\ (1,1,2,2),\ (1),\ (1,1,2,3)$$

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***p*(*n*)**: number of prime parking functions of length n

$$\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!} = \frac{1}{1-\sum_{n\geq 1} p(n)\frac{x^n}{n!}}$$

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Corollary. $p(n) = (n-1)^{n-1}$

Factorization of increasing PF's

$$\begin{array}{|cc|cccc|c|cccc} \color{red}{\mathbf{1}} & 2 & \color{red}{\mathbf{3}} & 4 & 5 & 6 & \color{red}{\mathbf{7}} & \color{red}{\mathbf{8}} & 9 & 10 & 11 \\ \hline \color{red}{\mathbf{1}} & 1 & \color{red}{\mathbf{3}} & 3 & 4 & 4 & \color{red}{\mathbf{7}} & \color{red}{\mathbf{8}} & 8 & 9 & 10 \end{array}$$

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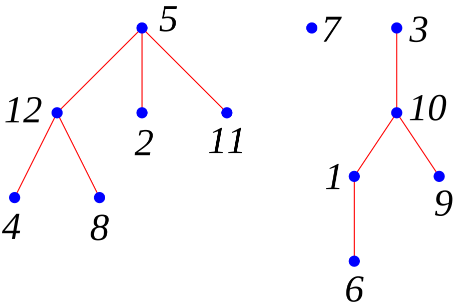
Exercise. Find a "parking" proof.

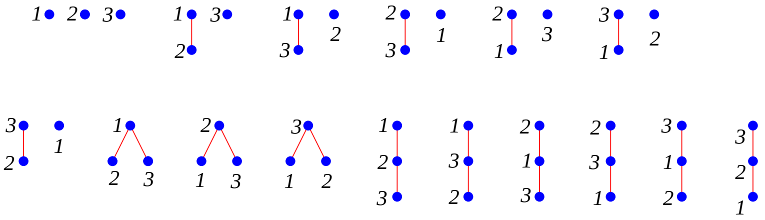
Forests

Let F be a rooted forest on the vertex set $\{1, \ldots, n\}$.

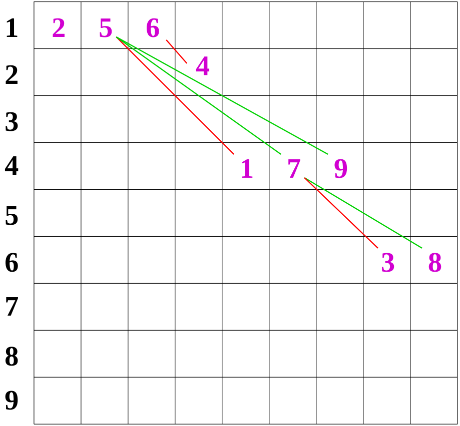


Theorem (**Sylvester-Borchardt-Cayley**). *The number of such forests is* $(n+1)^{n-1}$.

The case $n = 3$



A bijection between forests and parking functions

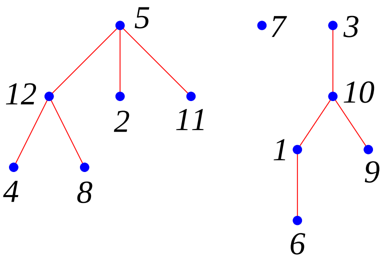


1	2	3	4	5	6	7	8	9
4	1	6	2	1	1	4	6	4

Inversions

An **inversion** in F is a pair (i, j) so that $i > j$ and i lies on the path from j to the root.

$$\mathrm{inv}(F) = \#(\text{inversions of } F)$$



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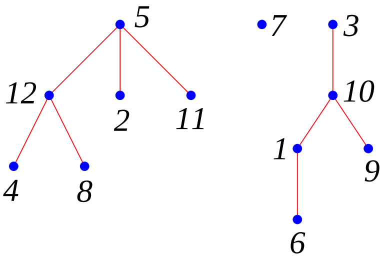

Inversions:

$(5, 4),\ (5, 2),\ (12, 4),\ (12, 8),\ (3, 1),\ (10, 1),\ (10, 6),\ (10, 9)$

$$\mathrm{inv}(F) = 8$$

The inversion enumerator

Let

$$I_n(q) = \sum_F q^{\mathrm{inv}(F)},$$

summed over all forests F with vertex set $\{1,\ldots,n\}$. E.g.,

$$\begin{aligned} I_1(q) &= 1 \\ I_2(q) &= 2+q \\ I_3(q) &= 6+6q+3q^2+q^3 \end{aligned}$$

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Theorem (**Mallows-Riordan** 1968, **Gessel-Wang** 1979) *We have*

$$I_n(1+q) = \sum_G q^{e(G)-n},$$

where G ranges over all connected graphs (without loops or multiple edges) on $n+1$ labelled vertices, and where $e(G)$ denotes the number of edges of G.

Generating function

Corollary.

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}$$

Connection with parking functions

Theorem (**Kreweras**, 1980) *We have*

$$q^{\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ *ranges over all parking functions of length* n*.*

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Note. The earlier bijection between forests and parking functions does not send the number of inversions to the sum of the terms. Such a bijection is more complicated.

The Shi arrangement: background

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$.

$$\begin{aligned} \mathcal{R} &= \text{set of regions of } \mathcal{B}_n \\ \#\mathcal{R} &= \, ?? \end{aligned}$$

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To specify a region, we must specify for each $i < j$ whether $x_i < x_j$ or $x_i > x_j$. Hence the number of regions is the number of ways to linearly order $x_1, \ldots, x_n$.

Labeling the regions

Let R_0 be the **base region**

$$R_0: \; x_1 > x_2 > \cdots > x_n.$$

Labeling the regions

Let $\boldsymbol{R_0}$ be the **base region**

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Label R_0 with

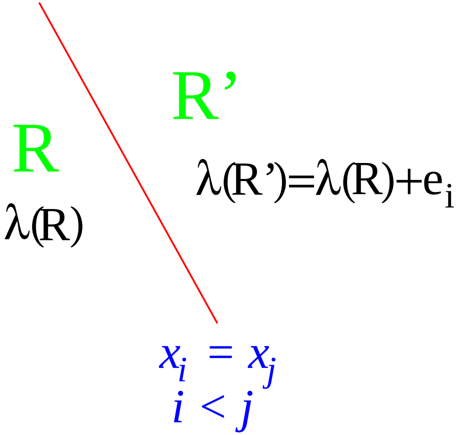
$$\boldsymbol{\lambda(R_0)} = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

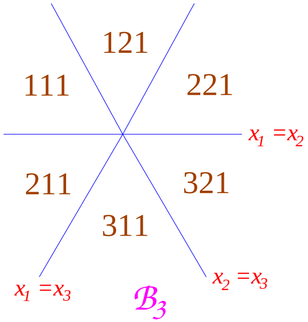
$$\boldsymbol{\lambda(R')} = \lambda(R) + e_i,$$

where $\boldsymbol{e_i}$ = ith unit coordinate vector.

The labeling rule



Description of labels



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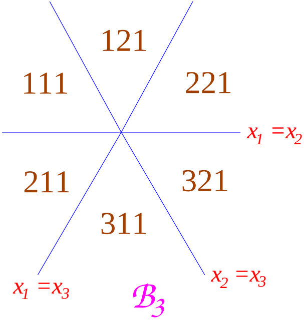

Theorem (easy). *The labels of* $\mathcal{B}_n$ *are the sequences* $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ *such that* $1 \le b_i \le n - i + 1$.

The Shi arrangement

Shi Jianyi

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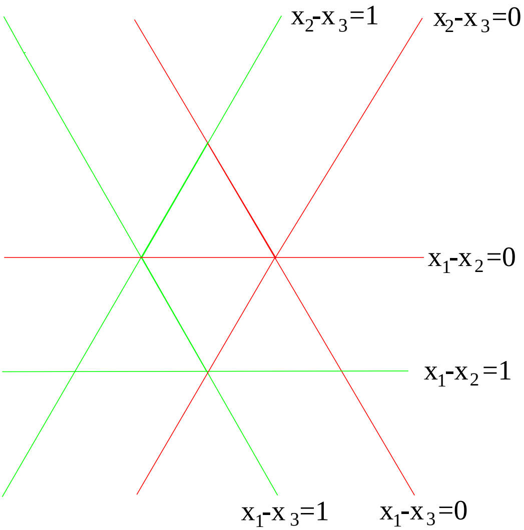
Shi arrangement $\mathcal{S}_n$: the set of hyperplanes

$$x_i - x_j = 0, 1,$$

$1 \leq i < j \leq n$, in $\mathbb{R}^n$.

The case $n = 3$



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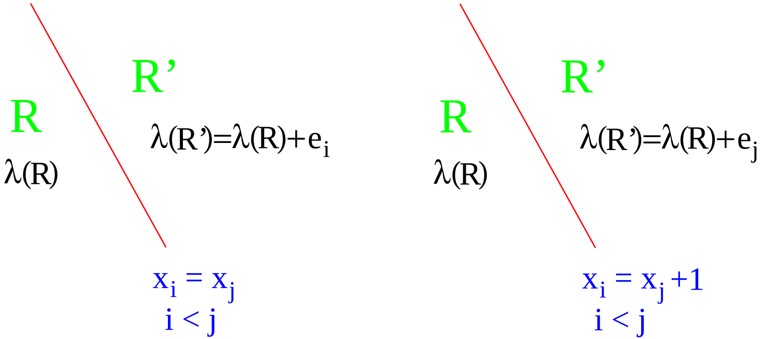
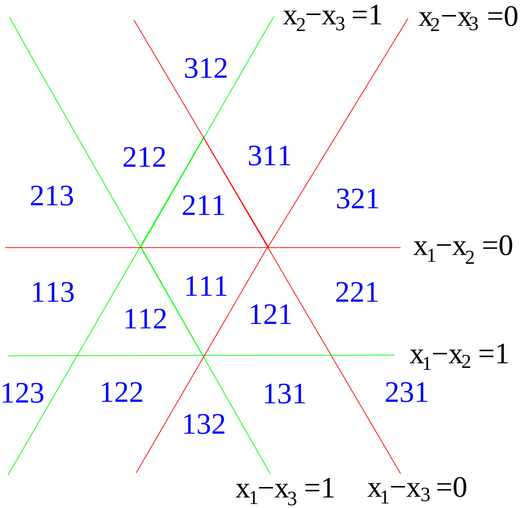
The labeling rule illustrated



The labeling for $n = 3$



Description of the labels

Theorem (**Pak, S.**). *The labels of* $\mathcal{S}_n$ *are the parking functions of length* n *(each occurring once).*

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Corollary (**Shi**, 1986).

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

The parking function polytope

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\boldsymbol{P_n = P(x_1, \ldots, x_n)} \subset \mathbb{R}^n$ by: $(y_1, \ldots, y_n) \in P_n$ if

$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

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for $1 \leq i \leq n$.

(also called **Pitman-Stanley polytope**)

Volume of P

Theorem. *Let* $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. *Then*

$$n!\, V(P_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}.$$

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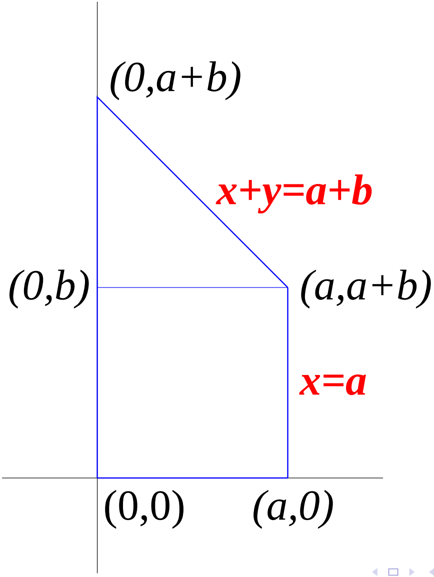
Note. If each $x_i > 0$, then P_n has the combinatorial type of an n-cube.

The case $n = 2$



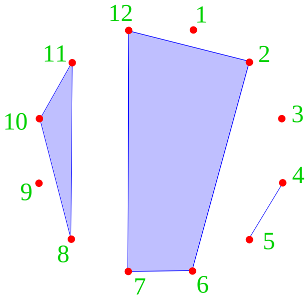
Noncrossing partitions

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$a < b < c < d,\ \ a, c \in B_i,\ \ b, d \in B_j \Rightarrow i = j.$$

$(B_i \neq \emptyset,\ B_i \cap B_j = \emptyset \text{ if } i \neq j, \bigcup B_i = \{1, \ldots, n\})$

Number of noncrossing partitions



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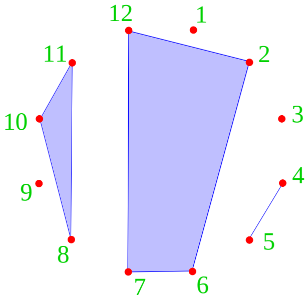
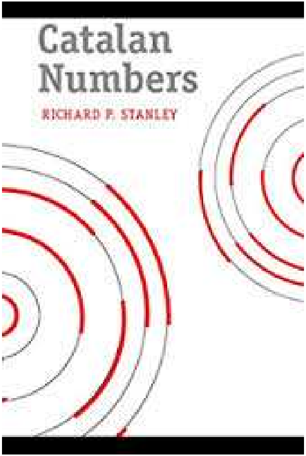


Theorem (**H. W. Becker**, 1948–49). *The number of noncrossing partitions of* $\{1, \ldots, n\}$ *is the* **Catalan number**

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Catalan numbers

214 combinatorial interpretations:



Maximal chains of noncrossing partitions

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \pi_1, \pi_2, \ldots, \pi_n$$

of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one. (Hence π_i has exactly $n+1-i$ blocks.)

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1–2–3–4–5   1–25–3–4   1–25–34

125–34   12345

A maximal chain labeling

Define:

$$\min \mathbf{B} = \text{least element of } B$$

$$\mathbf{j} < \mathbf{B} \;:\; j < k \quad \forall k \in B.$$

Suppose π_i is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$. Define

$$\begin{aligned} \boldsymbol{\Lambda}_{\mathbf{i}}(\mathfrak{m}) &= \max\{j \in B \;:\; j < B'\} \\ \boldsymbol{\Lambda}(\mathfrak{m}) &= (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m})). \end{aligned}$$

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For above example:

1–2–3–4–5   1–25–3–4   1–25–34

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we have

$$\Lambda(\mathfrak{m}) = (2, 3, 1, 2).$$

Labelings and parking functions

Theorem. *Λ is a bijection between the maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ and parking functions of length n.*

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Corollary (**Kreweras**, 1972) *The number of maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ is*

$$(n+1)^{n-1}.$$

The parking function $\mathfrak{S}_n$-module

The symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length n by permuting coordinates.

Sample properties

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

$$n = 3 : \quad \mathbf{111} \quad \mathbf{211} \quad \mathbf{221} \quad \mathbf{311} \quad \mathbf{321}$$

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- Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathfrak{S}_n$ (character value at w):

$$\#\mathrm{Fix}(w) = (n+1)^{(\#\,\text{cycles of } w)-1}$$

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- Multiplicity of the irreducible representation indexed by $\lambda \vdash n$: $\frac{1}{n+1} s_\lambda(1^{n+1})$

Background: invariants of $\mathfrak{S}_n$

The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$R^{\mathfrak{S}_n} = \{f \in R \,:\, w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}.$$

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Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \dots, e_n],$$

where

$$e_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The coinvariant algebra

$\boldsymbol{R_+^{\mathfrak{S}_n}}$: symmetric functions with 0 constant term

(**irrelevant ideal** of $R^{\mathfrak{S}_n}$)

$$\boldsymbol{D} := R/\left(R_+^{\mathfrak{S}_n}\right) = R/(e_1, \ldots, e_n).$$

Then $\dim D = n!$, and $\mathfrak{S}_n$ acts on D according to the **regular representation**.

Diagonal action of $\mathfrak{S}_n$

Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

As before, let

$$\begin{aligned} \boldsymbol{R^{\mathfrak{S}_n}} &= \{f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\} \\ D &= R/\left(R_+^{\mathfrak{S}_n}\right). \end{aligned}$$

Haiman's theorem

Theorem (**Haiman**, 1994, 2001). $\dim D = (n+1)^{n-1}$, *and the action of* $\mathfrak{S}_n$ *on* D *is isomorphic to the action on* $\mathcal{P}_n$, *tensored with the sign representation.*

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Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.

The last slide

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