A Survey of Parking Functions

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A parking scenario

\[ n \quad \ldots \quad 2 \quad 1 \]

\[ a_1 \quad a_2 \quad \ldots \quad a_n \]
A parking scenario
Parking functions

Car $C_i$ prefers space $a_i$. If $a_i$ is occupied, then $C_i$ takes the next available space. We call $(a_1, \ldots, a_n)$ a parking function (of length $n$) if all cars can park.
Small examples

\[ n = 2 \]: 11 12 21

\[ n = 3 \]: 111 112 121 211 113 131 311 122 212 221 123 132 213 231 312 321
Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of $\alpha$. Then $\alpha$ is a parking function if and only if $b_i \leq i$.

**Corollary.** Every permutation of the entries of a parking function is also a parking function.
Theorem (Pyke, 1959; Konheim and Weiss, 1966). Let $f(n)$ be the number of parking functions of length $n$. Then $f(n) = (n + 1)^{n-1}$.

Proof (Pollak, c. 1974). Add an additional space $n + 1$, and arrange the spaces in a circle. Allow $n + 1$ also as a preferred space.
Pollak’s proof

\[ a_1, a_2, \ldots, a_n \]
Conclusion of Pollak’s proof

Now all cars can park, and there will be one empty space. $\alpha$ is a parking function $\iff$ if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car $C_i$ parking at space $p_i$, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car $C_i$ parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1 + i, a_2 + i, \ldots, a_n + i) \pmod{n+1}$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$
Definition (I. Gessel). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

**Note.** A sequence $b_1 \leq b_2 \leq \cdots \leq b_n$ is an increasing parking function if and only if $1 \leq b_1 \leq \cdots \leq b_n$ is an increasing prime parking function.
Factorization of increasing PF’s

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→ (1, 1), (1, 1, 2, 2), (1), (1, 1, 2, 3)
Factorization of increasing PF’s

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→ (1, 1), (1, 1, 2, 2), (1), (1, 1, 2, 3)

$p(n)$: number of prime parking functions of length $n$

$$\sum_{n \geq 0} (n + 1)^{n-1} \frac{x^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} p(n) \frac{x^n}{n!}}$$
Factorization of increasing PF’s

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 1 & 3 & 3 & 4 & 4 & 7 & 8 \\
\end{array}
\]

→ (1, 1), (1, 1, 2, 2), (1), (1, 1, 2, 3)

**p(n):** number of prime parking functions of length \(n\)

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\sum_{n \geq 0} (n + 1)^{n-1} \frac{x^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} p(n) \frac{x^n}{n!}}
\]

**Corollary.** \(p(n) = (n - 1)^{n-1}\)
Factorization of increasing PF’s

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\begin{array}{cccccccc}
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\]

\[
\rightarrow (1, 1), (1, 1, 2, 2), (1), (1, 1, 2, 3)
\]

\[p(n): \text{number of prime parking functions of length } n\]

\[
\sum_{n \geq 0} (n + 1)^{n-1} \frac{x^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} p(n) \frac{x^n}{n!}}
\]

Corollary. \( p(n) = (n - 1)^{n-1} \)

Exercise. Find a “parking” proof.
Forests

Let $F$ be a rooted forest on the vertex set $\{1, \ldots, n\}$.

Theorem (Sylvester-Borchardt-Cayley). The number of such forests is $(n + 1)^{n-1}$.
The case $n = 3$
A bijection between forests and parking functions

\[ \begin{array}{cccccccccccc}
1 & 2 & 5 & 6 & 4 & 1 & 7 & 9 & 3 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
4 & 1 & 6 & 2 & 1 & 1 & 4 & 6 & 4 & & & & & & & & & \\
\end{array} \]
Inversions

An inversion in $F$ is a pair $(i, j)$ so that $i > j$ and $i$ lies on the path from $j$ to the root.

$$\text{inv}(F) = \#(\text{inversions of } F)$$
Inversions

An inversion in $F$ is a pair $(i, j)$ so that $i > j$ and $i$ lies on the path from $j$ to the root.

$$\text{inv}(F) = \#(\text{inversions of } F)$$

Inversions:

$(5, 4), (5, 2), (12, 4), (12, 8), (3, 1), (10, 1), (10, 6), (10, 9)$

$$\text{inv}(F) = 8$$
The inversion enumerator

Let

\[ I_n(q) = \sum_F q^{\text{inv}(F)}, \]

summed over all forests \( F \) with vertex set \( \{1, \ldots, n\} \). E.g.,

\[
\begin{align*}
I_1(q) &= 1 \\
I_2(q) &= 2 + q \\
I_3(q) &= 6 + 6q + 3q^2 + q^3
\end{align*}
\]
The inversion enumerator

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\[ I_2(q) = 2 + q \]
\[ I_3(q) = 6 + 6q + 3q^2 + q^3 \]

**Theorem** (Mallows-Riordan 1968, Gessel-Wang 1979) We have

\[ I_n(1 + q) = \sum_G q^{e(G) - n}, \]

where \( G \) ranges over all connected graphs (without loops or multiple edges) on \( n + 1 \) labelled vertices, and where \( e(G) \) denotes the number of edges of \( G \).
Corollary.

\[
\sum_{n \geq 0} I_n(q)(q - 1)^n \frac{x^n}{n!} = \frac{\sum_{n \geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n \geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}
\]
Connection with parking functions

**Theorem (Kreweras, 1980)** We have

\[ q^n \binom{n}{2} I_n(1/q) = \sum_{(a_1, \ldots, a_n)} q^{a_1 + \cdots + a_n}, \]

where \((a_1, \ldots, a_n)\) ranges over all parking functions of length \(n\).
Connection with parking functions

**Theorem** *(Kreweras, 1980)* We have

\[ q^{n \choose 2} I_n(1/q) = \sum_{(a_1, \ldots, a_n)} q^{a_1 + \cdots + a_n}, \]

where \((a_1, \ldots, a_n)\) ranges over all parking functions of length \(n\).

**Note.** The earlier bijection between forests and parking functions does not send the number of inversions to the sum of the terms. Such a bijection is more complicated.
The Shi arrangement: background

**Braid arrangement** $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \ 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$.

$$\mathcal{R} = \text{set of regions of } \mathcal{B}_n$$

$$\# \mathcal{R} = ??$$
**Braid arrangement** $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

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- $\mathcal{R} = \text{set of regions of } \mathcal{B}_n$
- $\#\mathcal{R} = n!$
The Shi arrangement: background

**Braid arrangement** $\mathcal{B}_n$: the set of hyperplanes

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\[ \mathcal{R} = \text{set of regions of } \mathcal{B}_n \]

\[ \#\mathcal{R} = n! \]

To specify a region, we must specify for each $i < j$ whether $x_i < x_j$ or $x_i > x_j$. Hence the number of regions is the number of ways to linearly order $x_1, \ldots, x_n$. 
Labeling the regions

Let $R_0$ be the base region

$$R_0 : x_1 > x_2 > \cdots > x_n.$$
Labeling the regions

Let $R_0$ be the base region

$$R_0 : x_1 > x_2 > \cdots > x_n.$$  

Label $R_0$ with

$$\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$  

If $R$ is labelled, $R'$ is separated from $R$ only by $x_i - x_j = 0$ ($i < j$), and $R'$ is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

where $e_i$ = $i$th unit coordinate vector.
The labeling rule

\[ \lambda(R') = \lambda(R) + e_i \]

\[ x_i = x_j \]

\[ i < j \]
Description of labels

$B_3$

$x_1 = x_3$

$x_2 = x_3$

$x_1 = x_2$
Theorem (easy). The labels of $B_n$ are the sequences $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $1 \leq b_i \leq n - i + 1$. 
The Shi arrangement

Shi Jianyi
The Shi arrangement

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The Shi arrangement

Shi Jianyi (时俭益)

Shi arrangement $S_n$: the set of hyperplanes

$$x_i - x_j = 0, 1,$$

$$1 \leq i < j \leq n, \text{ in } \mathbb{R}^n.$$
The case $n = 3$
Labeling the regions

**base region:**

$$R_0 : \ x_n + 1 > x_1 > \cdots > x_n$$
Labeling the regions

**base region:**

\[ R_0 : \quad x_n + 1 > x_1 > \cdots > x_n \]

\[ \lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n \]
The labeling rule

- If $R$ is labelled, $R'$ is separated from $R$ only by $x_i - x_j = 0$ \((i < j)\), and $R'$ is unlabelled, then set
  \[
  \lambda(R') = \lambda(R) + e_i.
  \]

- If $R$ is labelled, $R'$ is separated from $R$ only by $x_i - x_j = 1$ \((i < j)\), and $R'$ is unlabelled, then set
  \[
  \lambda(R') = \lambda(R) + e_j.
  \]
The labeling rule illustrated

\[ \lambda(R') = \lambda(R) + e_i \]

\[ x_i = x_j \]
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\[ \lambda(R') = \lambda(R) + e_j \]

\[ x_i = x_j + 1 \]
\[ i < j \]
The labeling for $n = 3$
Theorem (Pak, S.). The labels of $S_n$ are the parking functions of length $n$ (each occurring once).
Description of the labels

**Theorem (Pak, S.).** The labels of $S_n$ are the parking functions of length $n$ (each occurring once).

**Corollary (Shi, 1986).**

$$r(S_n) = (n + 1)^{n-1}$$
The parking function polytope

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $P_n = P(x_1, \ldots, x_n) \subset \mathbb{R}^n$ by:

$$(y_1, \ldots, y_n) \in P_n \text{ if } 0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$. 
The parking function polytope

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$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

(also called Pitman-Stanley polytope)
**Theorem.** Let $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. Then

$$n! \, V(P_n) = \sum_{\text{parking functions}} x_{i_1} \cdots x_{i_n}.$$
Theorem. Let $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. Then

$$n! \, V(P_n) = \sum_{\text{parking functions}} x_{i_1} \cdots x_{i_n}.$$ 

Note. If each $x_i > 0$, then $P_n$ has the combinatorial type of an $n$-cube.
The case $n = 2$

$x + y = a + b$

$x = a$
A **noncrossing partition** of \{1, 2, \ldots, n\} is a partition \{B_1, \ldots, B_k\} of \{1, \ldots, n\} such that

\[ a < b < c < d, \quad a, c \in B_i, \quad b, d \in B_j \Rightarrow i = j. \]

\((B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \text{ if } i \neq j, \quad \bigcup B_i = \{1, \ldots, n\})\)
Number of noncrossing partitions
Theorem (H. W. Becker, 1948–49). The number of noncrossing partitions of \{1, \ldots, n\} is the Catalan number

\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]
Catalan numbers

214 combinatorial interpretations:
Maximal chains of noncrossing partitions

A maximal chain $m$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \pi_1, \pi_2, \ldots, \pi_n$$

of noncrossing partitions of $\{1, \ldots, n+1\}$ such that $\pi_i$ is obtained from $\pi_{i-1}$ by merging two blocks into one. (Hence $\pi_i$ has exactly $n + 1 - i$ blocks.)
A **maximal chain** \( m \) of noncrossing partitions of \( \{1, \ldots, n+1\} \) is a sequence

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\pi_0, \pi_1, \pi_2, \ldots, \pi_n
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of noncrossing partitions of \( \{1, \ldots, n+1\} \) such that \( \pi_i \) is obtained from \( \pi_{i-1} \) by merging two blocks into one. (Hence \( \pi_i \) has exactly \( n + 1 - i \) blocks.)

\[
125-34 \quad 12345
\]
A maximal chain labeling

Define:

\[ \min B = \text{least element of } B \]

\[ j < B : j < k \quad \forall k \in B. \]

Suppose \( \pi_i \) is obtained from \( \pi_{i-1} \) by merging together blocks \( B \) and \( B' \), with \( \min B < \min B' \). Define

\[ \Lambda_i(m) = \max\{j \in B : j < B'\} \]

\[ \Lambda(m) = (\Lambda_1(m), \ldots, \Lambda_n(m)). \]
A maximal chain labeling

Define:

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\Lambda_i(m) = \max\{j \in B : j < B'\}
\]

\[
\Lambda(m) = (\Lambda_1(m), \ldots, \Lambda_n(m)).
\]

For above example:

\[
1\,2\,3\,4\,5 \quad 1\,2\,5\,3\,4 \quad 1\,2\,5\,3\,4
\]

\[
125\,3\,4 \quad 12\,3\,4\,5
\]

we have

\[
\Lambda(m) = (2, 3, 1, 2).
\]
Theorem. Λ is a bijection between the maximal chains of noncrossing partitions of \{1, \ldots, n + 1\} and parking functions of length \(n\).
Labelings and parking functions

**Theorem.** \( \Lambda \) is a bijection between the maximal chains of noncrossing partitions of \( \{1, \ldots, n + 1\} \) and parking functions of length \( n \).

**Corollary (Kreweras, 1972)** The number of maximal chains of noncrossing partitions of \( \{1, \ldots, n + 1\} \) is

\[
(n + 1)^{n-1}.
\]
The symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length $n$ by permuting coordinates.
Sample properties

- Multiplicity of trivial representation (number of orbits)
  \[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

  \( n = 3 : \) 111 211 221 311 321
Sample properties

- Multiplicity of trivial representation (number of orbits)
  \[ C_n = \frac{1}{n+1} \binom{2n}{n} \]
  
  \[ n = 3 : \quad 111 \quad 211 \quad 221 \quad 311 \quad 321 \]

- Number of elements of \( P_n \) fixed by \( w \in S_n \) (character value at \( w \)):
  \[ \#\text{Fix}(w) = (n + 1)^{\#\text{cycles of } w} - 1 \]
Sample properties

- Multiplicity of trivial representation (number of orbits)
  \[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

  \[ n = 3 : \quad 111 \ 211 \ 221 \ 311 \ 321 \]

- Number of elements of \( \mathcal{P}_n \) fixed by \( w \in S_n \) (character value at \( w \)):
  \[ \#\text{Fix}(w) = (n + 1)(\# \text{cycles of } w) - 1 \]

- Multiplicity of the irreducible representation indexed by \( \lambda \vdash n \):
  \[ \frac{1}{n+1}s_{\lambda}(1^{n+1}) \]
The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$R^{\mathfrak{S}_n} = \{ f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n \}.$$
The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$R^{\mathfrak{S}_n} = \{ f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n \}.$$  

Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n],$$

where

$$e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$
The coinvariant algebra

\( R^\mathfrak{S}_n^+ \): symmetric functions with 0 constant term

\((\text{irrelevant ideal of } R^\mathfrak{S}_n)\)

\[ D \coloneqq R / \left( R^\mathfrak{S}_n^+ \right) = R / (e_1, \ldots, e_n). \]

Then \( \dim D = n! \), and \( \mathfrak{S}_n \) acts on \( D \) according to the regular representation.
Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$ 

As before, let

$$R^{\mathfrak{S}_n} = \{ f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n \}$$

$$D = R / \left( R^{\mathfrak{S}_n} \right).$$
Haiman’s theorem

Theorem (Haiman, 1994, 2001). \( \dim D = (n + 1)^{n-1} \), and the action of \( \mathfrak{S}_n \) on \( D \) is isomorphic to the action on \( \mathcal{P}_n \), tensored with the sign representation.
Haiman’s theorem

Theorem (Haiman, 1994, 2001). \( \dim D = (n + 1)^{n-1} \), and the action of \( \mathfrak{S}_n \) on \( D \) is isomorphic to the action on \( \mathcal{P}_n \), tensored with the sign representation.

Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.
The last slide
The last slide

Darn!
That's the end...