Two Enumerative Tidbits

Richard P. Stanley

M.I.T.
The first tidbit

The Smith normal form of some matrices connected with Young diagrams
\( \lambda \) is a partition of \( n \):

\[
\lambda = (\lambda_1, \lambda_2, \ldots), \, \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \, \sum \lambda_i = n
\]
\[ \lambda \text{ is a partition of } n: \]

\[ \lambda = (\lambda_1, \lambda_2, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = n \]

**Example.** \( \lambda = (5, 3, 3, 1) = (5, 3, 3, 1, 0, 0, \ldots) \).

**Young diagram:**

![Young diagram](image)
Extended Young diagrams

\( \lambda \): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\[
\begin{array}{cccc}
\square & \square & \square & \\
\square & & & \\
\end{array}
\]

(3,1)
Extended Young diagrams

\(\lambda\): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\(\lambda^*\): \(\lambda\) extended by a border strip along its entire boundary
Extended Young diagrams

$\lambda$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram

$\lambda^*$: $\lambda$ extended by a border strip along its entire boundary

(3,1)  

(3,1)* = (4,4,2)
Insert 1 into each square of $\lambda^*/\lambda$.

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$(3,1)^* = (4,4,2)$
Let \( t \in \lambda \). Let \( M_t \) be the largest square of \( \lambda^* \) with \( t \) as the upper left-hand corner.
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Let $t \in \lambda$. Let $M_t$ be the largest square of $\lambda^*$ with $t$ as the upper left-hand corner.
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Determinantal algorithm

Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{pmatrix}
1 & 1 \\
\cdot & 2 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

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Determinantal algorithm

Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\begin{bmatrix}
9 & 5 & 2 & 1 \\
3 & 2 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
Easy to see: the numbers $n_t$ are well-defined and unique.
Uniqueness

Easy to see: the numbers $n_t$ are well-defined and unique.

Why? Expand $\det M_t$ by the first row. The coefficient of $n_t$ is 1 by induction.
If $t \in \lambda$, let $\lambda(t)$ consist of all squares of $\lambda$ to the southeast of $t$. 
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$\lambda = (4,4,3)$

$\lambda(t)$
If $t \in \lambda$, let $\lambda(t)$ consist of all squares of $\lambda$ to the southeast of $t$.

\[ \lambda = (4,4,3) \]

\[ \lambda(t) = (3,2) \]
\[ u_\lambda = \# \{ \mu : \mu \subseteq \lambda \} \]
\[ u_\lambda = \# \{ \mu : \mu \subseteq \lambda \} \]

**Example.** \( u_{(2,1)} = 5: \)

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]
\[ u_\lambda = \# \{ \mu : \mu \subseteq \lambda \} \]

**Example.** \[ u_{(2,1)} = 5: \]

\[
\begin{array}{cccc}
\square & \square & \square & \varnothing \\
\square & \square & \square & \square \\
\end{array}
\]

There is a determinantal formula for \[ u_\lambda, \] due essentially to **MacMahon** and later **Kreweras** (not needed here).
Berlekamp (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.

Carlitz-Scoville-Roselle theorem

- **Berlekamp** (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.

- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of $n_t$ (over $\mathbb{Z}$).

**Theorem.** $n_t = u \lambda(t)$. 
Berlekamp (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.


**Theorem.** $n_t = u_\lambda(t)$.

**Proofs.** 1. Induction (row and column operations).
2. Nonintersecting lattice paths.
An example

```
7  3  2  1
2  1  1  1
1  1
```
An example
Smith normal form

\[ A: \text{n} \times \text{n} \text{ matrix over commutative ring } R \text{ (with 1)} \]

Suppose there exist \( P, Q \in \text{GL}(n, R) \) such that

\[ PAQ = B = \text{diag}(d_1d_2 \cdots d_n, d_1d_2 \cdots d_{n-1}, \ldots, d_1), \]

where \( d_i \in R \). We then call \( B \) a Smith normal form (SNF) of \( A \).
Smith normal form

\( A: n \times n \) matrix over commutative ring \( R \) (with 1)

Suppose there exist \( P, Q \in \text{GL}(n, R) \) such that

\[ PAQ = B = \text{diag}(d_1 d_2 \cdots d_n, d_1 d_2 \cdots d_{n-1}, \ldots, d_1), \]

where \( d_i \in R \). We then call \( B \) a Smith normal form (SNF) of \( A \).

\textbf{Note.}

\[ \text{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n. \]

Thus SNF is a refinement of \( \det(A) \).
Existence of SNF

If $R$ is a PID, such as $\mathbb{Z}$ or $K[x]$ ($K = \text{field}$), then $A$ has a unique SNF up to units.
Existence of SNF

If \( R \) is a PID, such as \( \mathbb{Z} \) or \( K[x] \) (\( K = \) field), then \( A \) has a unique SNF up to units.

Otherwise \( A \) “typically” does not have a SNF but may have one in special cases.
**Algebraic interpretation of SNF**

\( R \): a PID

\( A \): an \( n \times n \) matrix over \( R \) with \( \det(A) \neq 0 \) and rows \( v_1, \ldots, v_n \in R^n \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( A \)
\textbf{Algebraic interpretation of SNF}

\(R\): a PID

\(A\): an \(n \times n\) matrix over \(R\) with \(\det(A) \neq 0\) and rows \(v_1, \ldots, v_n \in R^n\)

\(\text{diag}(e_1, e_2, \ldots, e_n)\): SNF of \(A\)

\textbf{Theorem.}

\[R^n/(v_1, \ldots, v_n) \cong (R/e_1 R) \oplus \cdots \oplus (R/e_n R).\]
An explicit formula for SNF

$\mathcal{R}$: a PID

$A$: an $n \times n$ matrix over $\mathcal{R}$ with $\det(A) \neq 0$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
An explicit formula for SNF

$R$: a PID

$A$: an $n \times n$ matrix over $R$ with $\det(A) \neq 0$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$

**Theorem.** $e_{n-i+1}e_{n-i+2} \cdots e_n$ is the gcd of all $i \times i$ minors of $A$.

**minor**: determinant of a square submatrix.

**Special case**: $e_n$ is the gcd of all entries of $A$. 
Many indeterminates

For each square \((i, j) \in \lambda\), associate an indeterminate \(x_{ij}\) (matrix coordinates).
Many indeterminates

For each square \((i, j) \in \lambda\), associate an indeterminate \(x_{ij}\) (matrix coordinates).

\[
\begin{array}{ccc}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & \\
\end{array}
\]
A refinement of $u_\lambda$

\[ u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda / \mu} x_{i,j} \]
A refinement of $u_{\lambda}$

$$u_{\lambda}(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{i,j}$$

$$\prod_{(i,j) \in \lambda/\mu} x_{i,j} = cde$$
**An example**

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\[ \begin{array}{ccc}
abcde + bcde + bce + cde & bce + ce + c + e + 1 & c + 1 \\
+ ce + de + c + e + 1 & e + 1 & 1 \\
d + e + 1 & 1 & 1 \\
\end{array} \]
\[ A_t = \prod_{(i,j) \in \lambda(t)} x_{ij} \]
\[ A_t = \prod_{(i,j) \in \lambda(t)} x_{ij} \]
\[ A_t = \prod_{(i,j) \in \lambda(t)} x_{ij} \]

\[ A_t = bcdeghiklmo \]
Theorem. Let $t = (i, j)$. Then $M_t$ has SNF

$$\text{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).$$
Theorem. Let \( t = (i, j) \). Then \( M_t \) has SNF

\[
\text{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).
\]

Proof. 1. Explicit row and column operations putting \( M_t \) into SNF.
2. (C. Bessenrodt) Induction.
### An example

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An example

\[
\begin{array}{ccc}
    a & b & c \\
    d & e \\
\end{array}
\]

\[
\begin{array}{ccc}
    abcde+bcde+bce+cde +ce+de+c+e+1 & bce+ce+c +e+1 & c+1 \\
    de+e+1 & e+1 & 1 \\
    1 & 1 & 1 \\
\end{array}
\]

\[
\text{SNF} = \text{diag}(abcde, e, 1)
\]
Let $\lambda$ be the staircase $\delta_n = (n - 1, n - 2, \ldots, 1)$. Set each $x_{ij} = q$. 
A special case

Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. Set each $x_{ij} = q$. 
Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. Set each $x_{ij} = q$.

$u_{\delta_{n-1}}(x)|_{x_{ij}=q}$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known $q$-analogue $C_n(q)$ of the Catalan number $C_n$. 

\[ u_{\delta_{n-1}}(x)|_{x_{ij}=q} \]
A \(q\)-Catalan example

\[ C_3(q) = q^3 + q^2 + 2q + 1 \]
A $q$-Catalan example

\[ C_3(q) = q^3 + q^2 + 2q + 1 \]

\[
\begin{vmatrix}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1 \\
\end{vmatrix} \overset{\text{SNF}}{\sim} \text{diag}(q^6, q, 1)\]
A $q$-Catalan example

\[ C_3(q) = q^3 + q^2 + 2q + 1 \]

\[
\begin{vmatrix}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1
\end{vmatrix} \overset{\text{SNF}}{\sim} \text{diag}(q^6, q, 1)
\]
- $q$-Catalan determinant previously known
- SNF is new
$q$-Catalan determinant previously known

SNF is new

END OF FIRST TIDBIT
The second tidbit

A distributive lattice associated with three-term arithmetic progressions
New York Times Numberplay blog (March 25, 2013): Let $S \subset \mathbb{Z}$, $\#S = 8$. Can you two-color $S$ such that there is no monochromatic three-term arithmetic progression?
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bad: $1, 2, 3, 4, 5, 6, 7, 8$
Numberplay blog problem

New York Times Numberplay blog (March 25, 2013): Let $S \subseteq \mathbb{Z}$, $\#S = 8$. Can you two-color $S$ such that there is no monochromatic three-term arithmetic progression?

**bad:** 1, 2, 3, 4, 5, 6, 7, 8

1, 4, 7 is a monochromatic 3-term progression
New York Times Numberplay blog (March 25, 2013): Let $S \subset \mathbb{Z}$, $\#S = 8$. Can you two-color $S$ such that there is no monochromatic three-term arithmetic progression?

bad: $1, 2, 3, 4, 5, 6, 7, 8$

$1, 4, 7$ is a monochromatic 3-term progression

good: $1, 2, 3, 4, 5, 6, 7, 8$. 
New York Times Numberplay blog (March 25, 2013): Let $S \subset \mathbb{Z}$, $\#S = 8$. Can you two-color $S$ such that there is no monochromatic three-term arithmetic progression?

bad: 1, 2, 3, 4, 5, 6, 7, 8

1, 4, 7 is a monochromatic 3-term progression

good: 1, 2, 3, 4, 5, 6, 7, 8.

Finally proved by Noam Elkies.
Elkies’ proof is related to the following question:

Let \( 1 \leq i < j < k \leq n \) and \( 1 \leq a < b < c \leq n \).

\( \{i, j, k\} \) and \( \{a, b, c\} \) are **compatible** if there exist integers \( x_1 < x_2 < \cdots < x_n \) such that \( x_i, x_j, x_k \) is an arithmetic progression and \( x_a, x_b, x_c \) is an arithmetic progression.
Example. \{1, 2, 3\} and \{1, 2, 4\} are not compatible. Similarly 124 and 134 are not compatible.
**Example.** \{1, 2, 3\} and \{1, 2, 4\} are *not* compatible. Similarly 124 and 134 are *not* compatible.

123 and 134 *are* compatible, e.g.,

\[(x_1, x_2, x_3, x_4) = (1, 2, 3, 5).\]
What subsets $S \subseteq \binom{[n]}{3}$ have the property that any two elements of $S$ are compatible?
Elkies’ question

What subsets \( S \subseteq \binom{[n]}{3} \) have the property that any two elements of \( S \) are compatible?

**Example.** When \( n = 4 \) there are eight such subsets \( S \):

\[
\emptyset, \{123\}, \{124\}, \{134\}, \{234\}, \\
\{123, 134\}, \{123, 234\}, \{124, 234\}.
\]

*Not* \( \{123, 124\} \), for instance.
Elkies’ question

What subsets $S \subseteq \binom{[n]}{3}$ have the property that any two elements of $S$ are compatible?

**Example.** When $n = 4$ there are eight such subsets $S$:

$$\emptyset, \{123\}, \{124\}, \{134\}, \{234\}, \{123, 134\}, \{123, 234\}, \{124, 234\}.$$

*Not* $\{123, 124\}$, for instance.

Let $M_n$ be the collection of all such $S \subseteq \binom{[n]}{3}$, so for instance $\#M_4 = 8$. 
Example. For $n = 5$ one example is

$$S = \{123, 234, 345, 135\} \in M_5,$$

achieved by $1 < 2 < 3 < 4 < 5$. 
Conjecture of Elkies

Conjecture. \[ \#M_n = 2^{\binom{n-1}{2}}. \]
Conjecture of Elkies

Conjecture. \( \#M_n = 2^{(n-1)/2} \).

Proof (with Fu Liu).
Conjecture of Elkies

**Conjecture.** \( \#M_n = 2^{\binom{n-1}{2}} \).

**Proof** (with Fu Liu 刘拂).
Jim Propp: Let $Q_n$ be the subposet of $[n] \times [n] \times [n]$ (ordered componentwise) defined by

$$Q_n = \{(i, j, k) : i + j < n + 1 < j + k\}.$$

**antichain**: a subset $A$ of a poset such that if $x, y \in A$ and $x \leq y$, then $x = y$

There is a simple bijection from the antichains of $Q_n$ to $M_n$ induced by $(i, j, k) \mapsto (i, n + 1 - j, k)$. 
The case $n = 4$

antichains:

$\emptyset, \{123\}, \{124\}, \{134\}, \{234\},$

$\{123, 134\}, \{123, 234\}, \{124, 234\}.$
**order ideal**: a subset $I$ of a poset such that if $y \in I$ and $x \leq y$, then $x \in I$

There is a bijection between antichains $A$ of a poset $P$ and order ideals $I$ of $P$, namely, $A$ is the set of maximal elements of $I$. 
**Order ideals**

**order ideal**: a subset $I$ of a poset such that if $y \in I$ and $x \leq y$, then $x \in I$

There is a bijection between antichains $A$ of a poset $P$ and order ideals $I$ of $P$, namely, $A$ is the set of maximal elements of $I$.

$J(P)$: set of order ideals of $P$, ordered by inclusion (a distributive lattice)
join-irreducible of a finite lattice $L$: an element $y$ such that exactly one element $x$ is maximal with respect to $x < y$ (i.e., $y$ covers $x$)

**Theorem** (FTFDL). If $L$ is a finite distributive lattice with the subposet $P$ of join-irreducibles, then $L \cong J(P)$. 
**Join-irreducibles**

**join-irreducible** of a finite lattice $L$: an element $y$ such that exactly one element $x$ is maximal with respect to $x < y$ (i.e., $y$ covers $x$)

**Theorem** (FTFDL). *If* $L$ *is a finite distributive lattice with the subposet $P$ of join-irreducibles, then* $L \cong J(P)$.

Thus regard $J(P)$ as the **definition** of a finite distributive lattice.
Two distributive lattices $L$ and $L'$ are isomorphic if and only if their posets $P$ and $P'$ of join-irreducibles are isomorphic.

$L$ and $L'$ may be large and complicated, but $P$ and $P'$ will be much smaller and (hopefully) more tractable.
The case $n = 4$

$P = Q_4$

$J(P) = M_4$
Recall: there is a simple bijection from the antichains of $Q_n$ to $M_n$ induced by $(i, j, k) \mapsto (i, n + 1 - j, k)$.

Also a simple bijection from antichains of a finite poset to order ideals.
A partial order on $M_n$

**Recall:** there is a simple bijection from the antichains of $Q_n$ to $M_n$ induced by

$$(i, j, k) \mapsto (i, n + 1 - j, k).$$

Also a simple bijection from antichains of a finite poset to order ideals.

Hence we get a bijection $J(Q_n) \rightarrow M_n$ that induces a distributive lattice structure on $M_n$.
$T$: semistandard Young tableau of shape of shape $\delta_{n-1} = (n - 2, n - 3, \ldots, 1)$, maximum part $\leq n - 1$

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Semistandard tableaux

\( T \): semistandard Young tableau of shape of shape \( \delta_{n-1} = (n - 2, n - 3, \ldots, 1) \), maximum part \( \leq n - 1 \)

```
1 1 2 5
2 3 3
4 4
5
```

\( L_n \): poset of all such \( T \), ordered componentwise (a distributive lattice)
$L_4$ and $M_4$ compared

$M_4$

$L_4$
Theorem. \( L_n \cong M_n \ (\cong J(Q_n)) \).
Theorem. \( L_n \cong M_n \cong J(Q_n) \).

Proof. Show that the poset of join-irreducibles of \( L_n \) is isomorphic to \( Q_n \). \( \square \)
Theorem. \( \#L_n = 2^{\binom{n-1}{2}} \) (proving the conjecture of Elkies).
Theorem. \( \#L_n = 2^{\binom{n-1}{2}} \) \textit{(proving the conjecture of Elkies)}.

Proof. \( \#L_n = s \delta_{n-2}(1, 1, \ldots, 1) \). Now use hook-content formula. \( \square \)
**Theorem.** \( \#L_n = 2^{\binom{n-1}{2}} \) (proving the conjecture of Elkies).

**Proof.** \( \#L_n = s_{\delta_{n-2}}(1, 1, \ldots, 1) \). Now use hook-content formula. □

In fact,

\[
s_{\delta_{n-2}}(x_1, \ldots, x_{n-1}) = \prod_{1 \leq i < j \leq n-1} (x_i + x_j).
\]
Maximum size elements of $M_n$

$f(n)$: size of largest element $S$ of $M_n$. 
Maximum size elements of $M_n$

$f(n)$: size of largest element $S$ of $M_n$.

**Example.** Recall

$$M_4 = \{\emptyset, \{123\}, \{124\}, \{134\}, \{234\}, \{123, 134\}, \{123, 234\}, \{124, 234\}\}.$$

Thus $f(4) = 2$. 
$f(n)$: size of largest element $S$ of $M_n$.

**Example.** Recall

$$M_4 = \{\emptyset, \{123\}, \{124\}, \{134\}, \{234\}, \{123, 134\}, \{123, 234\}, \{124, 234\}\}.$$  

Thus $f(4) = 2$.

Since elements of $M_n$ are the antichains of $Q_n$, $f(n)$ is also the maximum size of an antichain of $Q_n$. 
Evaluation of $f(n)$

Easy result (Elkies):

$$f(n) = \begin{cases} 
  m^2, & n = 2m + 1 \\
  m(m - 1), & n = 2m.
\end{cases}$$
Evaluation of $f(n)$

**Easy result (Elkies):**

$$f(n) = \begin{cases} 
  m^2, & n = 2m + 1 \\
  m(m - 1), & n = 2m. 
\end{cases}$$

**Conjecture #2 (Elkies).** Let $g(n)$ be the number of antichains of $Q_n$ of size $f(n)$. (E.g., $g(4) = 3$.) Then

$$g(n) = \begin{cases} 
  2^m(m - 1), & n = 2m + 1 \\
  2^{(m-1)(m-2)}(2^m - 1), & n = 2m. 
\end{cases}$$
**Maximum size antichains**

\( P \): finite poset with largest antichain of size \( m \)

\( J(P) \): lattice of order ideals of \( P \)

\( D(P) := \{ x \in J(P) : x \text{ covers } m \text{ elements} \} \) (in bijection with \( m \)-element antichains of \( P \))
Maximum size antichains

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**Easy theorem** (Dilworth, 1960). \( D(P) \) is a sublattice of \( J(P) \) (and hence is a distributive lattice)
Example: $M_4$

$Q_4$

$M_4 = J(Q_4)$

$D(Q_4) = J(R_4)$

$R_4$
Recall: $g(n)$ is the number of antichains of $Q_n$ of maximum size $f(n)$.

Hence $g(n) = \#D(Q_n)$. The lattice $D(Q_n)$ is difficult to work with directly, but since it is distributive it is determined by its join-irreducibles $R_n$. 
Examples of $R_n$

$R_6$

$R_7 \cong Q_4 + Q_4$
$n = 2m + 1$: $R_n \cong Q_{m+1} + Q_{m+1}$. Hence

$$g(n) = \#J(R_n) = \left(2\binom{m}{2}\right)^2 = 2^m(m-1),$$

proving the Conjecture 2 of Elkies for $n$ odd.
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Thus Conjecture 2 is true for all $n$. 