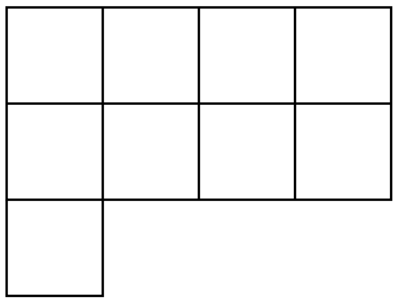
partition λ of $n \geq 0$:

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

Young diagram of $\lambda = (4, 4, 1)$:



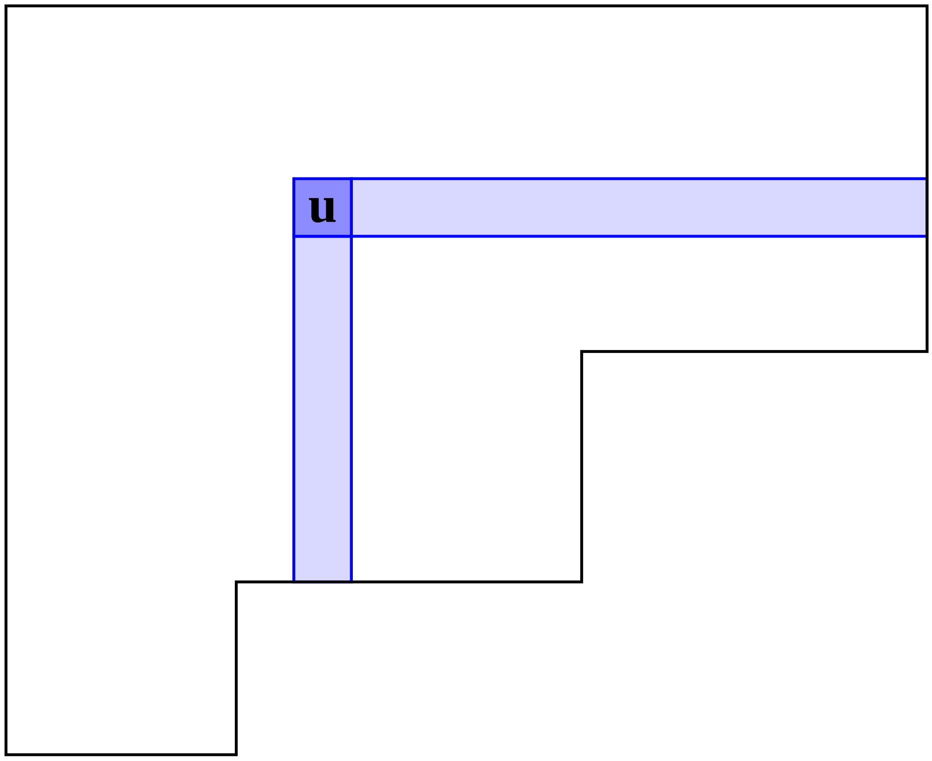
standard Young tableau (**SYT**) of shape $(4, 4, 1)$:

$$\begin{array}{c|c|c|c|c|}
 & \multicolumn{4}{c}{<} \\ \cline{2-5}
 & 1 & 3 & 4 & 7 \\ \cline{2-5}
\wedge & 2 & 6 & 8 & 9 \\ \cline{2-5}
 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}
\end{array}$$

$\boldsymbol{f^{\lambda}} = \#$ of SYT of shape λ

1 2 3 4 5 6	1 2 3 5 4 6	1 2 3 6 4 5
1 2 4 5 3 6	1 2 4 6 3 5	1 2 5 6 3 4
1 3 4 5 2 6	1 3 4 6 2 5	1 3 5 6 2 4

$$f^{4,2} = 9$$



$\boldsymbol{H(u)}$: **hook** at (or of) u

$\boldsymbol{h(u)} = \#H(u)$: **hook length** at u

8	7	5	4	1
6	5	3	2	
5	4	2	1	
2	1			

Frame-Robinson-Thrall hook length formula (1954):

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}$$

$$\text{hook lengths}: \qquad \begin{array}{l} 5\,4\,2\,1 \\ 2\,1 \end{array}$$

$$f^{4,2} = \frac{6!}{5\cdot 4\cdot 2\cdot 2\cdot 1\cdot 1} = 9$$

"nice" bijective proof by Novelli-Pak-Stoyanovskii (1997)

Robinson-Schensted-Knuth (RSK) algorithm: $w \xrightarrow{\text{rsk}} (P, Q)$, where $w \in \mathfrak{S}_n$ and P, Q are SYT of same shape $\lambda \vdash n$

Note. Schensted = Ea Ea

(ea.ea.home.mindspring.com)

$$w = 4273615$$

$$\begin{array}{l} \mathbf{4} \end{array} \qquad \begin{array}{l} 1 \end{array}$$

$$\begin{array}{l} \mathbf{2} \\ \mathbf{4} \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array}$$

$$\begin{array}{ll} 2 & \mathbf{7} \\ 4 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array}$$

$$\begin{array}{ll} 2 & \mathbf{3} \\ 4 & \mathbf{7} \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & 4 \end{array}$$

$$\begin{array}{lll} 2 & 3 & \mathbf{6} \\ 4 & 7 & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

$$\begin{array}{lll} \mathbf{1} & 3 & 6 \\ \mathbf{2} & 7 & \\ \mathbf{4} & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & & \end{array}$$

$$\begin{array}{lll} 1 & 3 & \mathbf{5} \\ 2 & \mathbf{6} & \\ 4 & \mathbf{7} & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array}$$

An element j bumps the smallest $i > j$.

First symmetry property.

$$w = 4273615 \xrightarrow{\text{rsk}} \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{array} \quad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array}$$

$$w^{-1} = 6241753 \xrightarrow{\text{rsk}} \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array} \quad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{array}$$

Theorem (Schützenberger) *If* $w \xrightarrow{\text{rsk}} (P, Q)$, *then*

$$w^{-1} \xrightarrow{\text{rsk}} (Q, P).$$

Corollary. *Let* $t(n)$ *denote the number of SYT with* n *squares. Then*

$$t(n) = \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1\}.$$

$$w = 3\mathbf{1}8\mathbf{4}9\mathbf{67}25$$

$$\boldsymbol{is}(w) := \text{length of longest increasing subsequence of } w \in \mathfrak{S}_n$$

$$\text{is}(318496725) = 4$$

$$\boldsymbol{ds}(w) := \text{length of longest decreasing subsequence of } w \in \mathfrak{S}_n$$

$$\text{ds}(31\mathbf{84}967\mathbf{2}5) = 3$$

$$318496725 \xrightarrow{\text{rsk}} \begin{array}{llll} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & \\ 8 & 9 & & \end{array} \qquad \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & \\ 8 & 9 & & \end{array}$$

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$,

$$\text{shape}(P) = (\lambda_1, \lambda_2, \ldots).$$

Then

$$\text{is}(w) = \lambda_1.$$

Proof (sketch). Let the first row be

$$b_1, b_2, \ldots, b_k.$$

Straightforward to show by induction on n that b_i is the rightmost element j of w for which the longest increasing subsequence of w ending at j has length i. $\square$

Second symmetry property.

$$w = a_1 a_2 \cdots a_n, \quad \overline{w} := a_n \cdots a_2 a_1$$

Theorem (Schensted). *If* $w \xrightarrow{\text{rsk}} (P, Q)$, *then*

$$\overline{w} \xrightarrow{\text{rsk}} (P^t, \text{evac}(Q)^t).$$

Corollary. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, $\text{shape}(P) = (\lambda_1, \lambda_2, \ldots)$. *Then*

$$\text{ds}(w) = \lambda_1' = \ell(\lambda).$$

Corollary (Erdős-Szekeres). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either*

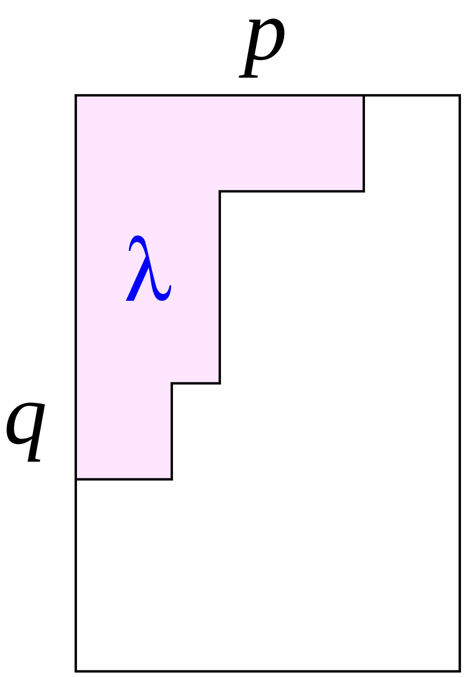
$$\mathrm{is}(w) > p \quad \text{or} \quad \mathrm{ds}(w) > q.$$

Proof. Let

$$w \xrightarrow{\text{rsk}} (P, Q), \quad \mathrm{shape}(P) = \lambda.$$

$$\begin{aligned} \mathrm{is}(w) \le p &\Rightarrow \lambda_1 \le p \\ \mathrm{ds}(w) \le q &\Rightarrow \lambda_1' \le q \end{aligned}$$

$$\Rightarrow |\lambda| \le pq.$$



Corollary. *Let $p \le q$ (say). Then*

$$\#\{w \in \mathfrak{S}_{pq} \ : \ \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}$$
$$= \left(f^{q\times p}\right)^2,$$

where $f^{q\times p} =$

$$\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}.$$

7	6	5	4	3
6	5	4	3	2
5	4	3	2	1

A **reverse semistandard tableau** T of shape $(5,4,3)$:

$$\begin{array}{lllll} 6 & 6 & 4 & 2 & 2 \\ 4 & 4 & 3 & 1 & \\ 2 & 1 & 1 & & \end{array}$$

(weakly decreasing in rows, strictly decreasing in columns)

$$A = (a_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$w_A = \begin{pmatrix} 3 & 3 & 2 & 2 & 2 & 1 & 1 \\ 3 & 1 & 3 & 2 & 2 & 4 & 4 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{3} \end{array} \qquad \begin{array}{l} 3 \end{array}$$

$$\begin{array}{l} 3\,\mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3 \end{array}$$

$$\begin{array}{l} 3\,\mathbf{3} \\ \mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3 \\ 2 \end{array}$$

$$\begin{array}{l} 3\,3\,\mathbf{2} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2 \\ 2 \end{array}$$

$$\begin{array}{l} 3\,3\,2\,\mathbf{2} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{4}\,3\,2\,2 \\ \mathbf{3} \\ \mathbf{1} \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2 \\ 1 \end{array}$$

$$\begin{array}{l} 4\,\mathbf{4}\,2\,2 \\ 3\,\mathbf{3} \\ 1 \end{array} \qquad \begin{array}{l} 3\,3\,2\,2 \\ 2\,1 \\ 1 \end{array}$$

- a_{ij} columns of w_A are equal to $\begin{matrix} i \\ j \end{matrix}$.
- Top row of w_A is weakly decreasing.
- Bottom row is weakly decreasing under equal elements of the top row.
- An element j bumps the largest element $i < j$.

Lemma (simple). Let $A \xrightarrow{\text{rsk}} (P, Q)$. Equal elements of Q are inserted left-to-right, allowing the construction of the inverse map $(P, Q) \to A$.

Note.

j appears in P: $\sum_i a_{ij}$ times (jth column sum)

i appears in Q: $\sum_j a_{ij}$ times (ith row sum)

$$\sum(\text{entries of } P) = \sum_{i,j} j a_{ij}$$

$$\sum(\text{entries of } Q) = \sum_{i,j} i a_{ij}$$

$$\#(\text{entries of } P \text{ or } Q) = \sum_{i,j} a_{ij}$$

Plane partition of $n \geq 0$:

$$\pi = (\pi_{ij})_{i,j\geq 1}$$

$$\pi_{ij} \geq \pi_{i+1,j}, \quad \pi_{ij} \geq \pi_{i,j+1}, \quad \sum \pi_{ij} = n$$

$$\boldsymbol{a(n)} = \#\text{ plane partitions of } n$$

$$\begin{array}{llllll} 5 & 5 & 3 & 2 & 2 & 1 \\ 5 & 3 & 3 & 1 & 1 & \\ 5 & 2 & 2 & 1 & 1 & \\ 2 & 2 & 1 & & & \end{array}$$

$$3 \qquad 21 \qquad 111 \qquad \begin{array}{l} 2 \\ 1 \end{array} \qquad \begin{array}{l} 1 \\ 1 \\ 1 \end{array} \qquad \begin{array}{ll} 1 & 1 \\ 1 & \end{array}$$

$$a(3) = 6$$

MacMahon:

$$\sum_{n\geq 0} a(n)x^n = \prod_{i\geq 1}(1-x^i)^{-i}.$$

Compare Euler's theorem for $p(n)$, the number of partitions (or one-row plane partitions) of n:

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1}(1-x^i)^{-1}$$

(much easier).

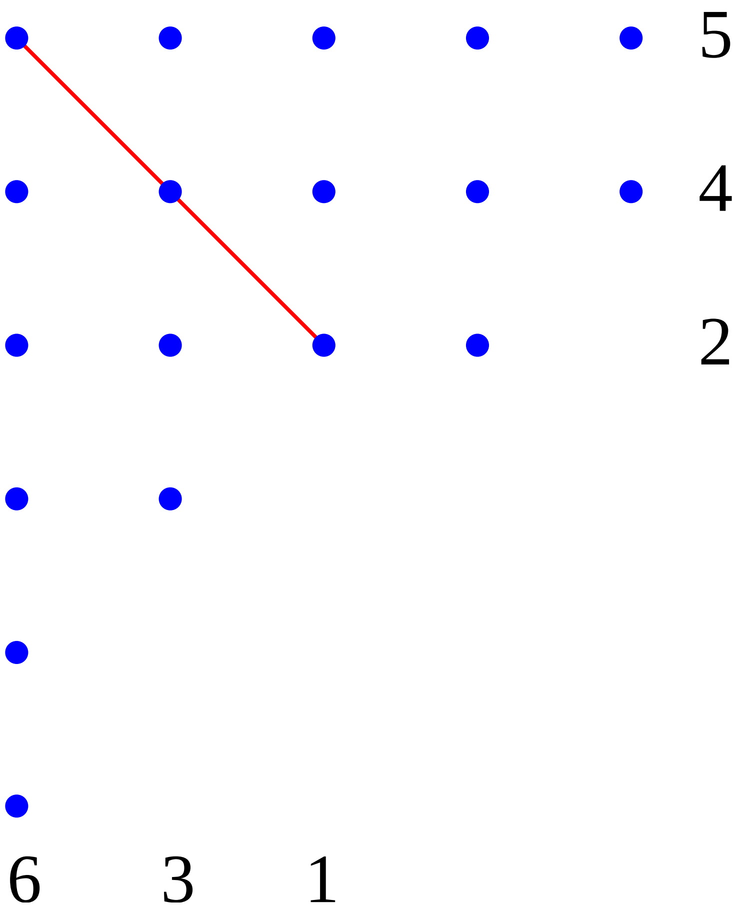
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{rsk}} \begin{array}{lllll} 3 & 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & & \\ 1 & & & & \end{array} \quad \begin{array}{lllll} 4 & 4 & 3 & 2 & 1 \\ 3 & 2 & 2 & & \\ 1 & & & & \end{array}$$

$$\xrightarrow{\text{merge}} \begin{array}{lllll} 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 1 & \\ 3 & 3 & 2 & & \\ 2 & 1 & & & \end{array} = \pi_A$$

Merge column-by-column.

merge of $(5, 4, 2)$ and $(6, 3, 1)$ is $(5, 5, 4, 2, 1, 1)$:



$$
\begin{aligned}
|\pi_A| &= \sum(\text{entries of } P) \\
&\qquad + \sum(\text{entries of } Q) \\
&\qquad -\#(\text{entries of } P \text{ or } Q) \\
&= \sum_{i,j}(i+j-1)a_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \sum_{\pi} x^{|\pi|} &= \sum_{A} x^{\sum(i+j-1)a_{ij}} \\
&= \prod_{i,j\geq 1}\left(\sum_{a_{ij}\geq 0} x^{(i+j-1)a_{ij}}\right) \\
&= \prod_{i,j\geq 1}\left(1-x^{i+j-1}\right)^{-1} \\
&= \prod_{i\geq 1}(1-x^i)^{-i}
\end{aligned}
$$

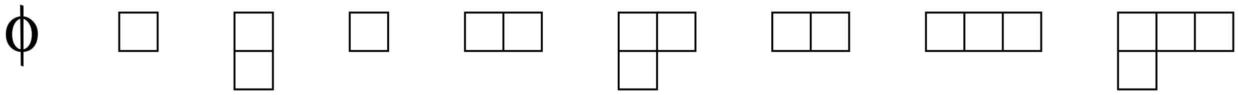
slight modification $\Rightarrow$

$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols}}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} (1 - x^{i+j-1})^{-1}$$

More difficult (MacMahon):

$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols} \\ \max \leq t}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - x^{i+j+k-1}}{1 - x^{i+j+k-2}}$$

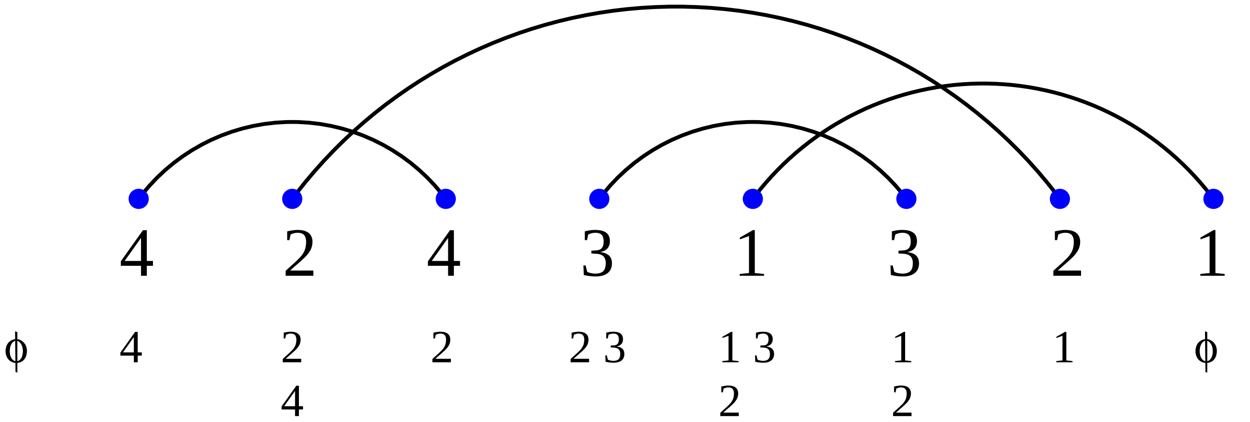
Oscillating tableaux: start with $\emptyset$, add or remove a square at each step.

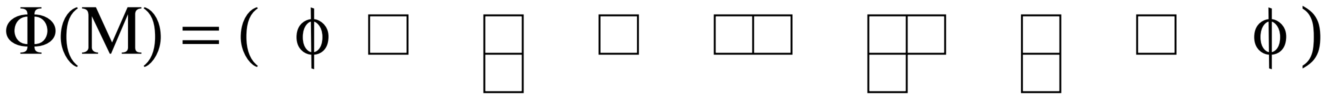


shape $(3,1)$, length 8

$\tilde{f}_n^{\lambda} = \#\{\text{osc. tab. of shape } \lambda, \text{ length } n\}$

M: (complete) **matching** on $1, 2, \ldots, 2n$





Φ is a bijection between matchings on $1, 2, \ldots, 2n$ and oscillating tableaux of length $2n$, shape $\emptyset$.

Hence

$$\begin{aligned} \tilde{f}_{2n}^{\emptyset} &= \sum_{\lambda} \left(\tilde{f}_n^{\lambda}\right)^2 \\ &= (2n-1)!! \\ &:= 1 \cdot 3 \cdot 5 \cdots (2n-1), \end{aligned}$$

the number of matchings on $1, 2, \ldots, 2n$.