Smith Normal Form and Combinatorics

Richard P. Stanley
Outline

Part I

- basics
- random matrices
Outline

Part I

- basics
- random matrices

Part II: symmetric functions

- \( \frac{\partial}{\partial p_1} p_1 \) (operator)
- Jacobi-Trudi specializations
A: \( n \times n \) matrix over commutative ring \( R \) (with 1)

Suppose there exist \( P, Q \in \text{GL}(n, R) \) such that

\[ PAQ := B = \text{diag}(d_1, d_1d_2, \ldots d_1d_2 \cdots d_n), \]

where \( d_i \in R \). We then call \( B \) a Smith normal form (SNF) of \( A \).
**Smith normal form**

\[ A : n \times n \] matrix over commutative ring \( R \) (with 1)

Suppose there exist \( P, Q \in \text{GL}(n, R) \) such that

\[
P A Q := B = \text{diag}(d_1, d_1 d_2, \ldots d_1 d_2 \cdots d_n),
\]

where \( d_i \in R \). We then call \( B \) a **Smith normal form (SNF)** of \( A \).

**NOTE.** (1) Can extend to \( m \times n \).

(2) unit \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.

Thus SNF is a refinement of \( \det \).
Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a unit in $\mathbb{R}$. 
Row and column operations

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- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a unit in $\mathbb{R}$.

Over a field, SNF is row reduced echelon form (with all unit entries equal to 1).
PIR: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.

If $R$ is a PIR then $A$ has a unique SNF up to units.
Existence of SNF

**PIR**: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.  

If $R$ is a PIR then $A$ has a unique SNF up to units.  

Otherwise $A$ “typically” does not have a SNF but may have one in special cases.
Not known in general for which rings $R$ does every matrix over $R$ have an SNF.
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Necessary condition: $R$ is a Bézout ring, i.e., every finitely generated ideal is principal.

Example. ring of entire functions and ring of all algebraic integers (not PIR’s)
Not known in general for which rings $R$ does every matrix over $R$ have an SNF.

Necessary condition: $R$ is a Bézout ring, i.e., every finitely generated ideal is principal.

**Example.** ring of entire functions and ring of all algebraic integers (not PIR’s)

**Open:** every matrix over a Bézout domain has an SNF.
$\mathbb{R}$: a PID

$A$: an $n \times n$ matrix over $\mathbb{R}$ with rows

$v_1, \ldots, v_n \in \mathbb{R}^n$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
**Algebraic interpretation of SNF**

\( R \): a PID

\( A \): an \( n \times n \) matrix over \( R \) with rows \( v_1, \ldots, v_n \in R^n \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( A \)

**Theorem.**

\[
R^n / (v_1, \ldots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).
\]
$\mathbb{R}$: a PID

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$v_1, \ldots, v_n \in \mathbb{R}^n$

diag($e_1, e_2, \ldots, e_n$): SNF of $A$

**Theorem.**

$$R^n/(v_1, \ldots, v_n) \cong (\mathbb{R}/e_1 \mathbb{R}) \oplus \cdots \oplus (\mathbb{R}/e_n \mathbb{R}).$$

$R^n/(v_1, \ldots, v_n)$: *(Kasteleyn)* cokernel of $A$
An explicit formula for SNF

$\mathbb{R}$: a PID

$A$: an $n \times n$ matrix over $\mathbb{R}$ with $\det(A) \neq 0$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
A PID\( R \): an \( n \times n \) matrix over \( R \) with \( \det(A) \neq 0 \)
diag\((e_1, e_2, \ldots, e_n)\): SNF of \( A \)

**Theorem.** \( e_1 e_2 \cdots e_i \) is the gcd of all \( i \times i \) minors of \( A \).

**Minor:** determinant of a square submatrix.

**Special case:** \( e_1 \) is the gcd of all entries of \( A \).
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}$$
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Matrix-tree theorem $\implies \det(A) = 16$, the number of spanning trees of $K_4$. 
An example

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\end{bmatrix}$$

Matrix-tree theorem $\implies \det(A) = 16$, the number of spanning trees of $K_4$.

What about SNF?
An example (continued)

\[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & -1 \\
8 & -4 & 3
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & 0 \\
8 & -4 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & -1 \\
0 & 4 & 0 \\
4 & -4 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
0 & 4 & 0 \\
4 & 0 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
4 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Reduced Laplacian matrix of $K_n$:

\[ L_0(K_n) = nI_{n-1} - J_{n-1} \]

\[ \det L_0(K_n) = n^{n-2} \]
Reduced Laplacian matrix of $K_n$

$$L_0(K_n) = nI_{n-1} - J_{n-1}$$

$$\det L_0(K_n) = n^{n-2}$$

**Trick:** $2 \times 2$ submatrices (up to row and column permutations):

$$\begin{bmatrix} n - 1 & -1 \\ -1 & n - 1 \end{bmatrix}, \quad \begin{bmatrix} n - 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i|e_{i+1}$, we get the SNF diag$(1, n, n, \ldots, n)$. 

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Laplacian matrices of general graphs

SNF of the Laplacian matrix of a graph: very interesting

cannections with sandpile models, chip firing, abelian avalanches, etc.
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Laplacian matrices of general graphs

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no time for further details 😞
SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.
$\text{Mat}_k(n)$: all $n \times n \mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \text{Mat}_k(n)$ and $\text{SNF}(M) = (e_1, \ldots, e_n)$, then $e_1 = d$. 
Is the question interesting?

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**Recall:** \( e_1 = \text{gcd} \) of \( 1 \times 1 \) minors (entries) of \( M \)
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Recall: \( e_1 = \gcd \) of \( 1 \times 1 \) minors (entries) of \( M \)

Theorem. \( \lim_{k \to \infty} p_k(n, d) = \frac{1}{d^{n^2}} \zeta(n^2) \)
Specifying some $e_i$

with Yinghui Wang
Specifying some $e_i$

with Yinghui Wang (王颖慧)
Two general results.

Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{P}$, $\alpha_i | \alpha_{i+1}$.

$\mu_k(n)$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_i = \alpha_i$ for $1 \leq \alpha_i \leq n - 1$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$

Then $\mu(n)$ exists, and $0 < \mu(n) < 1$. 

with Yinghui Wang (王颖慧)
Let $\alpha_n \in \mathbb{P}$.

$\nu_k(n)$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

Then

$$\lim_{k \to \infty} \nu_k(n) = 0.$$
Sample result

$$\mu_k(n)$$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$
\[
\mu(n) = 2^{-n^2} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i} \right) \\
\cdot \frac{3}{2} \cdot 3^{-(n-1)^2} \left( 1 - 3^{(n-1)^2} \right) \left( 1 - 3^{-n} \right)^2 \\
\cdot \prod_{p>3} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i} \right).
\]
$\kappa(n)$: probability that an $n \times n \mathbb{Z}$-matrix has SNF $\text{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$. 
Cyclic cokernel

\( \kappa(n) \): probability that an \( n \times n \) \( \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \).

Theorem. \( \kappa(n) = \frac{\prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right)}{\zeta(2) \zeta(3) \cdots} \)
Cyclic cokernel

\( \kappa(n) \): probability that an \( n \times n \) \( \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \).

Theorem. \( \kappa(n) = \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right) \frac{\zeta(2) \zeta(3) \cdots}{\zeta(6) \prod_{j \geq 4} \zeta(j)} \)

Corollary. \( \lim_{n \to \infty} \kappa(n) = \frac{1}{\zeta(6) \prod_{j \geq 4} \zeta(j)} \approx 0.846936 \cdots \).
Small number of generators

$g$: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

previous slide: $\text{Prob}(g = 1) = 0.846936 \cdots$
Small number of generators

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$\text{Prob}(g \leq 2) = 0.99462688 \cdots$
$g$: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

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$\text{Prob}(g \leq 3) = 0.99995329 \cdots$
Small number of generators

$g$: number of generators of cokernel (number of entries of SNF ≠ 1) as $n \to \infty$

previous slide: Prob$(g = 1) = 0.846936 \cdots$

Prob$(g \leq 2) = 0.99462688 \cdots$

Prob$(g \leq 3) = 0.99995329 \cdots$

Theorem. Prob$(g \leq \ell) =

1 - (3.46275 \cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$
Number of generators $g$ as $n \to \infty$

Previous slide: $\Pr(g = 1) = 0.846936 \ldots$

$\Pr(g \leq 2) = 0.99462688 \ldots$

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Theorem. $\Pr(g \leq \ell) = 1 - (3.46275 \ldots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$
\[3.46275 \cdots = \frac{1}{\prod_{j \geq 1} \left( 1 - \frac{1}{2^j} \right)}\]
Universality

What other probability distributions on $n \times n$ integer matrices give the same conclusions?
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What other probability distributions on $n \times n$ integer matrices give the same conclusions?

**Example** (P. Q. Nguyen and I. E. Shparlinski). Fix $k, n$. Choose a subgroup $G$ of $\mathbb{Z}^n$ of index $\leq k$ uniformly.

$\rho_k(n) :$ probability that $G$ is cyclic
Universality

What other probability distributions on $n \times n$ integer matrices give the same conclusions?

Example (P. Q. Nguyen and I. E. Shparlinski). Fix $k, n$. Choose a subgroup $G$ of $\mathbb{Z}^n$ of index $\leq k$ uniformly.

$$\rho_k(n) : \text{probability that } G \text{ is cyclic}$$

$$\lim_{n \to \infty} \lim_{k \to \infty} \rho_k(n) \approx 0.846936 \cdots ,$$

same probability of cyclic cokernel as $k, n \to \infty$ using previous distribution.
Part II: symmetric functions

- $\frac{\partial}{\partial p_1} p_1$ (operator)
- Jacobi-Trudi specializations
A down-up operator

In collaboration with Tommy Wuxing Cai.
A down-up operator

In collaboration with Tommy Wuxing Cai (蔡吴兴).
A down-up operator

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\( \text{Par}(n) \): set of all partitions of \( n \)

E.g., \( \text{Par}(4) = \{ 4, 31, 22, 211, 1111 \} \).
A down-up operator

In collaboration with Tommy Wuxing Cai (蔡吴生).

\[ \text{Par}(n) : \text{set of all partitions of } n \]

E.g., \[ \text{Par}(4) = \{4, 31, 22, 211, 1111\} \].

\[ V_n : \text{real vector space with basis } \text{Par}(n) \]
Define $U = U_n : V_n \rightarrow V_{n+1}$ by

$$U(\lambda) = \sum_{\mu} \mu,$$

where $\mu \in \text{Par}(n + 1)$ and $\mu_i \geq \lambda_i \ \forall i$.

Example.

$$U(42211) = 52211 + 43211 + 42221 + 422111$$
Dually, define $D = D_n : V_n \to V_{n-1}$ by

$$D(\lambda) = \sum_{\nu} \nu,$$

where $\nu \in \text{Par}(n-1)$ and $\nu_i \leq \lambda_i \ \forall i$.

**Example.** $D(42211) = 32211 + 42111 + 4221$
NOTE. Identify $V_n$ with the space $\Lambda^n_{\mathbb{Q}}$ of all homogeneous symmetric functions of degree $n$ over $\mathbb{Q}$, and identify $\lambda \in V_n$ with the Schur function $s_\lambda$. Then

$$U(f) = p_1 f, \quad D(f) = \frac{\partial}{\partial p_1} f.$$
**Symmetric functions**

**Note.** Identify $V_n$ with the space $\Lambda^n_Q$ of all homogeneous symmetric functions of degree $n$ over $\mathbb{Q}$, and identify $\lambda \in V_n$ with the Schur function $s_\lambda$. Then

\[
U(f) = p_1 f, \quad D(f) = \frac{\partial}{\partial p_1} f.
\]

Write

\[
U = U_n : V_n \rightarrow V_{n+1} \\
D = D_{n+1} : v_{n+1} \rightarrow V_n.
\]
Basic commutation relation: \( DU - UD = I \)

Allows computation of eigenvalues of 
\( DU: V_n \rightarrow V_n \).

Or note that the eigenvectors of \( \frac{\partial}{\partial p_1} p_1 \) are the \( p_\lambda \)'s 
\((\lambda \vdash n)\), with eigenvalue \( 1 + m_1(\lambda) \), where \( m_1(\lambda) \) is the number of parts of \( \lambda \) equal to 1.
**Commutation relation**

**Basic commutation relation:**  $DU - UD = I$

Allows computation of eigenvalues of $DU : V_n \rightarrow V_n$.

Or note that the eigenvectors of $\frac{\partial}{\partial p_1} p_1$ are the $p_\lambda$’s ($\lambda \vdash n$), with eigenvalue $1 + m_1(\lambda)$, where $m_1(\lambda)$ is the number of parts of $\lambda$ equal to 1.

**NOTE.**

$$\#\{\lambda \vdash n : m_1(\lambda) = i\} = p(n + 1 - i) - p(n - i),$$

where $p(m) = \#\text{Par}(m) = \dim V_m$. 
Eigenvalues of $DU$

Theorem. Let $1 \leq i \leq n + 1$, $i \neq n$. Then $i$ is an eigenvalue of $D_{n+1}U_n$ with multiplicity $p(n + 1 - i) - p(n - i)$. Hence

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i) - p(n-i)}.$$
**Theorem.** Let $1 \leq i \leq n + 1$, $i \neq n$. Then $i$ is an eigenvalue of $D_{n+1}U_n$ with multiplicity $p(n + 1 - i) - p(n - i)$. Hence

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i) - p(n-i)}.$$

What about SNF of the matrix $[D_{n+1}U_n]$ (with respect to the basis $\text{Par}(n)$)?
Conjecture (first form). The diagonal entries of the SNF of $[D_{n+1}U_n]$ are:

- $(n + 1)(n - 1)!$, with multiplicity 1
- $(n - k)!$ with multiplicity $p(k + 1) - 2p(k) + p(k - 1)$, $3 \leq k \leq n - 2$
- 1, with multiplicity $p(n) - p(n - 1) + p(n - 2)$. 
NOTE. \( \{ p^\lambda \}_{\lambda | n} \) is not an integral basis.
$m_1(\lambda)$: number of 1’s in $\lambda$

$\mathcal{M}_1(n)$: multiset of all numbers $m_1(\lambda) + 1$, $\lambda \in \text{Par}(n)$

Let SNF of $[D_{n+1} U_n]$ be $\text{diag}(f_1, f_2, \ldots, f_p(n))$.

**Conjecture** (second form). $f_p(n)$ is the product of the distinct entries of $\mathcal{M}_1(n)$; $f_{p(n)-1}$ is the product of the remaining distinct entries of $\mathcal{M}_1(n)$, etc.
An example:  \( n = 6 \)

\[
\text{Par}(6) = \{6, 51, 42, 33, 411, 321, 222, 3111, 2211, 21111, 111111\}
\]

\[
\mathcal{M}_1(6) = \{1, 2, 1, 1, 3, 2, 1, 4, 3, 5, 7\}
\]

\[
(f_1, \ldots, f_{11}) = (7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, 3 \cdot 2 \cdot 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
\]

\[
= (840, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1)
\]
**Conjecture** (third form). The matrix 
\[ D_{n+1} U_n + x I \] has an SNF over \( \mathbb{Z}[x] \).

Note that \( \mathbb{Z}[x] \) is not a PID.
Theorem. The conjecture of Miller is true.
**Theorem.** The conjecture of Miller is true.

**Proof** (first step). Rather than use the basis \( \{ s_{\lambda} \}_{\lambda \in \text{Par}(n)} \) (Schur functions) for \( \Lambda_{n}^{\mathbb{Q}} \), use the basis \( \{ h_{\lambda} \}_{\lambda \in \text{Par}(n)} \) (complete symmetric functions). Since the two bases differ by a matrix in \( SL(p(n), \mathbb{Z}) \), the SNF’s stay the same.
Conclusion of proof

(second step) Row and column operations.
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Not very insightful.
(second step) Row and column operations.

Not very insightful. 😞
A generalization

\( m_j(\lambda) \): number of \( j \)'s in \( \lambda \)

\( \mathcal{M}_j(n) \): multiset of all numbers \( j(m_j(\lambda) + 1) \), \( \lambda \in \text{Par}(n) \)

\( p_j \): power sum symmetric function \( \sum x_i^j \)

Let SNF of the operator \( f \rightarrow j \frac{\partial}{\partial p_j} p_j f \) with respect to the basis \( \{ s_\lambda \} \) be \( \text{diag}(g_1, g_2, \ldots, g_{p(n)}) \).
A generalization

$m_j(\lambda)$: number of $j$’s in $\lambda$

$M_j(n)$: multiset of all numbers $j(m_j(\lambda) + 1)$, $\lambda \in \text{Par}(n)$

$p_j$: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j \frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be $\text{diag}(g_1, g_2, \ldots, g_{p(n)})$.

**Theorem (Zipei Nie).** $g_{p(n)}$ is the product of the distinct entries of $M_j(n)$; $g_{p(n)-1}$ is the product of the remaining distinct entries of $M_j(n)$, etc.
Two remarks

The operators $D, U$ and identity $DU - UD = I$ extend to any differential poset $P$. Miller and Reiner have conjectures for the SNF of $DU$. Nie has a conjecture on the structure of $P$ which would prove the Miller-Reiner conjecture.
Two remarks

The operators $D, U$ and identity $DU - UD = I$ extend to any differential poset $P$. Miller and Reiner have conjectures for the SNF of $DU$. Nie has a conjecture on the structure of $P$ which would prove the Miller-Reiner conjecture.

More general operators:

$$\frac{\partial^2}{\partial p_1^2} p_1^2, \quad 2 \frac{\partial}{\partial p_1} \frac{\partial}{\partial p_2} p_2 p_1, \text{ etc.}$$

No conjecture known for SNF.
Jacobi-Trudi identity:

\[ s_\lambda = \det[h_{\lambda_i-i+j}], \]

where \( s_\lambda \) is a **Schur function** and \( h_i \) is a **complete symmetric function**.
Jacobi-Trudi identity:

\[ s_\lambda = \det[h_{\lambda - i + j}], \]

where \( s_\lambda \) is a Schur function and \( h_i \) is a complete symmetric function.

We consider the specialization

\[ x_1 = x_2 = \cdots = x_n = 1, \text{ other } x_i = 0. \]

Then

\[ h_i \rightarrow \binom{n + i - 1}{i}. \]
Specialized Schur function

\[ s_\lambda \rightarrow \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}. \]

c(u): \text{content of the square } u

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 \\
\end{array}
\]
Diagonal hooks $D_1, \ldots, D_m$

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 \\
\end{array}
\]

$\lambda = (5,4,4,2)$
Diagonal hooks $D_1, \ldots, D_m$

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$D_1$
Diagonal hooks $D_1, \ldots, D_m$

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<td>$-1$</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-3$</td>
<td>$-2$</td>
<td></td>
<td></td>
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</tbody>
</table>

$D_2$
## Diagonal hooks $D_1, \ldots, D_m$

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>-1</td>
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<td>1</td>
<td>2</td>
<td></td>
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<tr>
<td>-2</td>
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<td>0</td>
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<tr>
<td>-3</td>
<td>-2</td>
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</tr>
</tbody>
</table>

$D_3$
\[ R = \mathbb{Q}[n] \text{ (a PID)} \]

Let

\[
\text{SNF} \begin{bmatrix}
(n + \lambda_i - i + j - 1) \\
\lambda_i - i + j
\end{bmatrix} = \text{diag}(e_1, \ldots, e_m).
\]

**Theorem.**

\[
e_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}
\]
Idea of proof

\[ f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)} \]

Want to prove \( e_i = f_i \). Note that \( f_1 f_2 \cdots f_i \) is the value of the lower-left \( i \times i \) minor. (Special argument for 0 minors.)
Idea of proof

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Want to prove \( e_i = f_i \). Note that \( f_1 f_2 \cdots f_i \) is the value of the lower-left \( i \times i \) minor. (Special argument for 0 minors.)

Every \( i \times i \) minor is a specialized skew Schur function \( s_{\mu/\nu} \). Let \( s_{\alpha} \) correspond to the lower left \( i \times i \) minor.
Conclusion of proof

Let \( s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho} \). By Littlewood-Richardson rule,

\[
c^{\mu}_{\nu\rho} \neq 0 \implies \alpha \subseteq \rho
\]

\[
\Rightarrow \{\text{contents of } \alpha\} \subseteq \{\text{contents of } \rho\}
\]

(as multisets).
Let $s_{\mu/\nu} = \sum_{\rho} c_{\nu\rho}^{\mu} s_{\rho}$. By Littlewood-Richardson rule,

$$c_{\nu\rho}^{\mu} \neq 0 \Rightarrow \alpha \subseteq \rho$$

$$\Rightarrow \{\text{contents of } \alpha\} \subseteq \{\text{contents of } \rho\}$$

(as multisets).

Hence $f_1 \cdots f_i = \gcd(i \times i \text{ minors}) = e_1 \cdots e_i$. \qed
An example

\[ \lambda = (7, 6, 6, 5, 3), \quad k = 3 \implies \mu = (4, 3, 1) \]
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\[ J T_\lambda = \begin{bmatrix}
    h_7 & h_8 & h_9 & h_{10} & h_{11} \\
    h_5 & h_6 & h_7 & h_8 & h_9 \\
    h_4 & h_5 & h_6 & h_7 & h_8 \\
    h_2 & h_3 & h_4 & h_5 & h_6 \\
    0 & 1 & h_1 & h_2 & h_3
\end{bmatrix} \]
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    h_2 & h_3 & h_4 & h_5 & h_6 \\
    0 & 1 & h_1 & h_2 & h_3
\end{bmatrix}
\]
An example (cont.)

A “random” $3 \times 3$ minor of $JT_\lambda$:

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  h_5 & h_6 & h_7 & h_8 & h_9 \\
  h_4 & h_5 & h_6 & h_7 & h_8 \\
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\end{bmatrix}$$
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$$
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    h_2 & h_3 & h_4 & h_5 & h_6 \\
    0 & 1 & h_1 & h_2 & h_3
\end{bmatrix}
$$

Jacobi-Trudi matrix for $s_{653/21}$
An example (concluded)

Every LR-filling contains 1,1,1,1,2,2,2,3. Thus if \( \langle s_{653/21}, s_\rho \rangle > 0 \), then \( 431 \subseteq \rho \). Therefore

\[
\prod_{u \in 431} (n + c(u)) \mid \prod_{u \in \rho} (n + c(u))
\]

\[
\Rightarrow \prod_{u \in 431} (n + c(u)) \mid s_{653/21}(1^n).
\]
A $q$-analogue

“Natural” $q$-analogue of $f(1^n)$ is $f(1, q, \ldots, q^{n-1})$.

$$h_i(1, q, \ldots, q^{n-1}) = \binom{n + i - 1}{i}_q$$

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^* \prod_{u \in \lambda} \frac{1 - q^{n+c(u)}}{1 - q^{h(u)}}.$$
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Doesn’t work (and SNF is unknown).
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**Doesn’t** work (and SNF is unknown).

Before we had $R = \mathbb{Q}[n]$. Now $R = \mathbb{Q}[q]$. Putting $q = 1$ doesn’t reduce second situation to the first.
Set $y = q^n$. Thus for instance

$$h_3(1, q, \ldots, q^{n-1}) = \frac{(1 - q^{n+2})(1 - q^{n+1})(1 - q^n)}{(1 - q^3)(1 - q^2)(1 - q)} \cdot$$

$$= \frac{(1 - q^2y)(1 - qy)(1 - y)}{(1 - q^3)(1 - q^2)(1 - q)}.$$

Work over the field $\mathbb{Q}(q)[y]$ (a PID).
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$$h_3(1, q, \ldots, q^{n-1}) = \frac{(1 - q^{n+2})(1 - q^{n+1})(1 - q^n)}{(1 - q^3)(1 - q^2)(1 - q)} = \frac{(1 - q^2y)(1 - qy)(1 - y)}{(1 - q^3)(1 - q^2)(1 - q)}.$$

Work over the field $\mathbb{Q}(q)[y]$ (a PID).

Previous proof carries over (using a couple of tricks).
Write

\[(i) = \frac{1 - q^i}{1 - q}.\]

E.g., \((-3) = -q^{-1} - q^{-2} - q^{-3}\) and \((0) = 0\). For \(k \geq 1\) let

\[f(k) = \frac{y(y + (1))(y + (2)) \cdots (y + (k - 1))}{(1)(2) \cdots (k)}.\]

Set \(f(0) = 1\) and \(f(k) = 0\) for \(k < 0\).
Theorem. Define

\[ JT(q)_\lambda = [f(\lambda_i - i + j)]_{i,j=1}^t, \]

where \( \ell(\lambda) \leq t \). Let the SNF of \( JT(q)_\lambda \) over the ring \( \mathbb{Q}(q)[y] \) have main diagonal \( (\gamma_1, \gamma_2, \ldots, \gamma_t) \). Then we can take

\[ \gamma_i = \prod_{u \in D_{t-i+1}} \frac{y + c(u)}{h(u)}. \]