Let

$$F(x) = x + c_2x^2 + c_3x^3 + \cdots \in \mathbb{C}[[x]].$$

Easy: $F(F(x)) = x \Rightarrow F(x) = x.$

What about $F(-F(-x)) = x$?

Fact #1. *$c_2, c_4, \ldots$ can be arbitrary and uniquely determine $c_3, c_5, \ldots$.*

Example. $c_3 = c_2^2$

$$c_5 = 3c_4c_2 - 2c_2^4$$

$$c_7 = 13c_2^6 - 18c_4c_2^3 + 2c_4^2 + 4c_2c_6$$

1. What are the coefficients?

Fact #2 (EC1, Exercise 1.41). *Given* $F(-F(-x)) = x$, *there is a unique*

$$G(x) = x + b_2x^2 + b_4x^4 + b_6x^6 + \cdots$$

such that $F(x) = G^{\langle -1\rangle}(-G(-x))$.

Example. $b_2 = -\frac{1}{2}c_2$

$$\begin{aligned} b_4 &= \frac{1}{8}(5c_2^3 - 4c_4) \\ b_6 &= -\frac{1}{16}(49c_2^5 - 56c_2^2c_4 + 8c_6) \end{aligned}$$

2. What are the coefficients?

Example.

$$F(x) = \frac{x}{1+2x} \Rightarrow b_{2n} = (-1)^{n-1}E_{2n-1}$$

Fact #3 (Aguiar). *For any* $A(x) = x + a_2x^2 + a_3x^3 + \cdots$ *there are unique*

$$\begin{aligned} G(x) &= x + d_3x^3 + d_5x^5 + \cdots \\ F(x) &= x + c_2x^2 + c_3x^3 + \cdots \end{aligned}$$

such that

$$F(-F(-x)) = x \text{ and } A(x) = G(F(x)).$$

Example. $c_2 = a_2, \quad d_3 = a_3 - a_2^2$

$$\begin{aligned} c_4 &= a_4 - 3a_3a_2 + 3a_2^3 \\ d_5 &= a_5 + 3a_2^2a_3 - 3a_2a_4 - a_2^4 \end{aligned}$$

3. What are the coefficients?