Some aspects of \((r, k)\)-parking functions

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(r, k)-parking function: a sequence $(a_1, \ldots, a_n)$ of positive integers whose decreasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies

$$b_i \leq k + (i - 1)r.$$ 

$\text{PF}_{n}^{(r,k)}$: set of $(r, k)$-parking functions of length $n$

$r = k = 1$ (so $b_i \leq i$): ordinary parking function
Basic definition

*(r, k)*-parking function: a sequence \((a_1, \ldots, a_n)\) of positive integers whose decreasing rearrangement \(b_1 \leq \cdots \leq b_n\) satisfies

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\(\text{PF}_{n}^{(r,k)}\): set of \((r, k)\)-parking functions of length \(n\)

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Example. \((8, 4, 8, 2)\) is not a \((2, 3)\)-parking function, since \((2, 4, 8, 8) \nleq (3, 5, 7, 9)\) (termwise).
Cars $C_1, \ldots, C_{rn}$ need to park in spaces $1, 2, \ldots, rn + k - 1$.

**Preference vector** $\alpha = (a_1, \ldots, a_n)$, $1 \leq a_i \leq rn + k - 1$, where cars $C_{r(i-1)+1}, \ldots, C_{ri}$ all prefer $a_i$.

Cars go one at a time to their preferred space and then park in first available space.

**Easy:** all cars can park if and only if $\alpha$ is an $(r, k)$-parking function.
Theorem (Steck 1969, essentially).

\[ \#\text{PF}_{n}^{(r,k)} = k(rn + k)^{n-1} \]
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\[ \#PF_{n}^{(r,k)} = k(rn + k)^{n-1} \]

Proof. Completely analogous to Pollak’s proof for \( r = k = 1 \).
$\mathfrak{S}_n$ acts on $\text{PF}^{(r,k)}_n$ by permuting coordinates. Let $F^{(r,k)}_n$ denote the Frobenius characteristic of this action.
\( \mathfrak{S}_n \) acts on \( \mathbb{P}F_n^{(r,k)} \) by permuting coordinates. Let \( F_n^{(r,k)} \) denote the Frobenius characteristic of this action.

Equivalently,

\[
F_n^{(r,k)} = \sum_{\beta} h_{m_1(\beta)} h_{m_2(\beta)} \cdots ,
\]

where \( \beta \) runs over all \textbf{weakly increasing} \((r, k)\)-parking functions, and \( m_i(\beta) \) is the number of \( i \)'s in \( \beta \).
An example

Let $r = 1, k = 2, n = 3$. The weakly increasing $(1, 2)$-parking functions $(a, b, c)$ of length three, i.e, $(a, b, c) \leq (2, 3, 4)$:

- 111 112 113 114 122 123 124
- 133 134 222 223 224 233 234

Hence

$$F_3^{(2,1)} = 2h_3 + 8h_2h_1 + 4h_1^3.$$
Basis expansions

$F_{n}^{(r,k)}$ has “nice” expansions in terms of the six classical bases $m, p, h, e, s, f$.

E.g.,

$$F_{n}^{(r,k)} = \frac{k}{rn + k} \sum_{\lambda \vdash n} \left( d_1(\lambda), \ldots, d_n(\lambda), rn + k - \ell(\lambda) \right) h_{\lambda}$$

$$= k \sum_{\lambda \vdash n} z_{\lambda}^{-1} (rn + k)^{\ell(\lambda) - 1} p_{\lambda},$$

where $d_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$. 
A generating function

\[ P^{(r,k)}(t) := \sum_{n \geq 0} F^{(r,k)}_n t^n \]
A generating function

\[ \mathcal{P}^{(r,k)}(t) := \sum_{n \geq 0} F_n^{(r,k)} t^n \]

Theorem. For \( n \geq 1 \), we have

\[ \mathcal{P}^{(r,k)}(t) = \left( \mathcal{P}^{r,1}(t) \right)^k . \]
A generating function

\[ P^{(r,k)}(t) := \sum_{n \geq 0} F_n^{(r,k)} t^n \]

**Theorem.** For \( n \geq 1 \), we have

\[ P^{(r,k)}(t) = (P^{r,1}(t))^k. \]

**Proof:** simple factorization argument.
Negative exponents

What about $\left( P^{r,1}(t) \right)^k$ for $k < 0$?

Simplest case: $r = 1$ and $k = -1$. 
Motivation

Let

\[ A(t) = \sum_{n \geq 0} a_n t^n \]

\[ B(t) = \sum_{n \geq 0} b_n t^n \]

\[ = \frac{1}{1 - A(t)} = \sum_{k \geq 0} A(t)^k. \]

Thus \( a_n \) counts “prime” objects and \( b_n \) all objects.
Note. \( B(t) = \frac{1}{1-A(t)} \iff A(t) = 1 - \frac{1}{B(t)}. \)
Note. $B(t) = \frac{1}{1-A(t)} \iff A(t) = 1 - \frac{1}{B(t)}$.

Suggests: $1 - \frac{1}{\mathcal{P}(1,1)(t)}$ might be connected with “prime” parking functions.
Definition (I. Gessel). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

**Note.** A sequence $b_1 \leq b_2 \leq \cdots \leq b_n$ is an increasing parking function if and only if $1 \leq b_1 \leq \cdots \leq b_n$ is an increasing prime parking function.
E.g., $n = 4$: increasing prime parking functions are

1111, 1112, 1113, 1122, 1123.
The prime parking function sym. fn.

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$\Rightarrow \mathcal{PF}_{4}^{(1,1)} = h_4 + 2h_3h_1 + h_2^2 + h_2h_1^2$
Factorization of increasing PF’s

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 1 & 3 & 3 & 4 & 4 & 7 & 8 & 8 & 9 & 10 \\
\end{array}
\]
### Factorization of increasing PF’s

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\[\rightarrow (1, 1), \ (1, 1, 2, 2), \ (1), \ (1, 1, 2, 3)\]
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\[\rightarrow (1, 1), \ (1, 1, 2, 2), \ (1), \ (1, 1, 2, 3)\]

**Theorem.** \((P^{(1,1)}(t))^{-1} = 1 - \sum_{n \geq 1} \text{PPF}_n t^n\)
Coefficient of \( t^5 \) in \( -P^{(1,1)}(t)^{-2} \) is

\[
2h_3h_1^2 + 2h_2^2h_1 + 4h_3h_2 + 4h_4h_1 + 2h_5.
\]

Frobenius characteristic of the action of \( S_5 \) on all sequences \((a_1, \ldots, a_5) \in P^{(1,1)} \) whose increasing rearrangement \( b_1 \leq \cdots \leq b_5 \) satisfies either of the conditions

1. \( b_1 = b_2 = 1, b_3 \leq 2, b_4 \leq 3, b_5 \leq 3 \), or
2. \( b_1 = b_2 = b_3 = 2, b_4 \leq 3, b_5 \leq 4 \).
Write $F_n = F_n^{(1,1)}$ (simplest case), with $F_0 = 1$. For $\lambda = (\lambda_1, \lambda_2, \ldots)$ write

$$F_\lambda = F_{\lambda_1} F_{\lambda_2} \cdots.$$

**Easy.** $\{F_\lambda : \lambda \vdash n\}$ is a $\mathbb{Z}$-basis for $\Lambda_n$ (homogeneous symmetric functions of degree $n$ with integer coefficients).
Some problems

- Expand $F_\lambda$ in the classical bases $m, h, e, p, s, f$, and vice versa.

- Formula or combinatorial interpretation of $\langle F_\lambda, F_\mu \rangle$. 
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- Formula or combinatorial interpretation of $\langle F_\lambda, F_\mu \rangle$.

Very little is known.
Theorem.

\[ \langle F_n, F_\lambda \rangle = \frac{1}{n+1} \prod_{i \geq 1} \frac{1}{\lambda_i + 1} \left( \frac{(n + 1)(\lambda_i + 1) + \lambda_i - 1}{\lambda_i} \right) \]
Scalar products

Theorem.

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Corollary.

\[ \langle F_n, F_n \rangle = \frac{1}{n+1} \binom{n(n+3)}{n} \]
Theorem. \[ \langle F_n, F_\lambda \rangle = \frac{1}{n+1} \prod_{i \geq 1} \frac{1}{\lambda_i + 1} \left( \frac{(n+1)(\lambda_i + 1) + \lambda_i - 1}{\lambda_i} \right) \]

Corollary. \[ \langle F_n, F_n \rangle = \frac{1}{n+1} \left( \frac{n(n+3)}{n} \right) \]

In general \( \langle F_\lambda, F_\mu \rangle \) has large prime factors. Is there a combinatorial interpretation?
Theorem.  

\[ \langle F_n, F_\lambda \rangle = \frac{1}{n+1} \prod_{i \geq 1} \frac{1}{\lambda_i + 1} \left( \frac{(n + 1)(\lambda_i + 1) + \lambda_i - 1}{\lambda_i} \right) \]

Corollary.  \[ \langle F_n, F_n \rangle = \frac{1}{n+1} \binom{n(n+3)}{n} \]

In general \[ \langle F_\lambda, F_\mu \rangle \] has large prime factors. Is there a combinatorial interpretation, even for \[ \frac{1}{n+1} \binom{n(n+3)}{n} \]?
Three expansions

\( d_i \): number of parts of \( \lambda \) equal to \( i \)

\[
e_n = \frac{1}{n + 1} \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \binom{n + \ell(\lambda)}{d_1, d_2, \ldots, r n} F_\lambda
\]

\[
p_n = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda) + 1} \binom{n + \ell(\lambda) - 1}{d_1, d_2, \ldots, r n - 1} F_\lambda
\]

\[
h_n = \frac{1}{n - 1} \sum_{\lambda \vdash n} (-1)^{\ell(\lambda) + 1} \binom{n + \ell(\lambda) - 2}{d_1, d_2, \ldots, r n - 2} F_\lambda
\]
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