Reduced Decompositions

Richard P. Stanley

M.I.T.

Definitions

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reduced decomposition $(a_1, \ldots, a_p)$ of $w \in S_n$:

$$w = s_{a_1} \cdots s_{a_p},$$

where p is **minimal**, i.e.,

$$p = \ell(w) = \#\{(i,j) \,:\, i < j,\ w(i) > w(j)\}.$$

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p is the **number of inversions** $\mathbf{inv}(\boldsymbol{w})$ or **length** $\boldsymbol{\ell}(\boldsymbol{w})$ of w.

An example

$$1\mathbf{23}4 \xrightarrow{\boldsymbol{s_2}} 13\mathbf{24} \xrightarrow{\boldsymbol{s_3}} 1\mathbf{34}2 \xrightarrow{\boldsymbol{s_2}} \mathbf{14}32 \xrightarrow{\boldsymbol{s_1}} 4132$$

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$\boldsymbol{R(w)}$: set of reduced decompositions of w

$$(2,3,2,1) \in R(4132)$$

Tits' theorem

Theorem. *If* $a = (a_1, a_2, \ldots, a_p) \in R(w)$ *then all reduced decompositions of* w *can be obtained from* a *by applying*

$$\begin{aligned} s_i s_j &= s_j s_i, \quad |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

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E.g., $(2,3,2,1) \in R(4132) \Rightarrow$

$$R(4132) = \{(2,3,2,1), (3,2,3,1), (3,2,1,3)\}.$$

$\boldsymbol{r(w)}$

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$$\begin{aligned} R(4132) &= \{(2,3,2,1),(3,2,3,1),(3,2,1,3)\} \\ &\Rightarrow r(4132) = 3. \end{aligned}$$

The main tool

Let $\boldsymbol{w} \in S_n$ and $\boldsymbol{p} = \ell(w)$. Define

$$\boldsymbol{G_w} = \sum_{(a_1,\ldots,a_p)\in R(w)} \sum_{\substack{1\leq i_1\leq\cdots\leq i_p \\ i_j<i_{j+1} \text{ if } a_j<a_{j+1}}} x_{i_1}\cdots x_{i_p},$$

a power series in $x_1, x_2, \ldots$, homogeneous of degree p.

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Example. $w = 321 \in S_3$, so $R(w) = \{121, 212\}$.

$$G_{321} = \sum_{1\le i<j\le k} x_i x_j x_k + \sum_{1\le i\le j<k} x_i x_j x_k.$$

Symmetry of G_w

Theorem (**Billey-Jockusch-S**, **Fomin-S**, **Jia-Miller**, $\ldots$). G_w *is a symmetric function of* $x_1, x_2, \ldots$.

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Schur functions

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ be a **partition** of p (denoted $\lambda \vdash p$), i.e.,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = p.$$

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$\boldsymbol{s_\lambda}$: the **Schur function** indexed by λ.

Fact: the Schur functions s_λ for $\lambda \vdash p$ form a $\mathbb{Z}$-basis for all symmetric functions in $x_1, x_2, \ldots$ over $\mathbb{Z}$ that are homogeneous of degree p.

The case $p = 3$

Example. $s_3 = \sum_{i \le j \le k} x_i x_j x_k$

$$\begin{aligned} s_{21} &= \sum_{1 \le i < j \le k} x_i x_j x_k + \sum_{1 \le i \le j < k} x_i x_j x_k \\ s_{111} &= \sum_{i < j < k} x_i x_j x_k \end{aligned}$$

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Thus every G_w can be uniquely written

$$G_w = \sum_{\lambda \vdash p} \boldsymbol{\alpha_{w\lambda}} s_\lambda.$$

The case $w = 321$

Recall that

$$\begin{aligned} G_{321} &= \sum_{1\le i<j\le k} x_i x_j x_k + \sum_{1\le i\le j<k} x_i x_j x_k \\ s_{21} &= \sum_{1\le i<j\le k} x_i x_j x_k + \sum_{1\le i\le j<k} x_i x_j x_k, \end{aligned}$$

so

$$G_{321} = s_{21}.$$

Coefficient of $x_1 \cdots x_p$

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$[\boldsymbol{x_1 \cdots x_p}]F$: coefficient of $x_1 \cdots x_p$ in F

$$\Rightarrow r(w) = [x_1 \cdots x_p]G_w,$$

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What is $[x_1 \cdots x_p] s_\lambda$?

Standard Young tableaux

standard Young tableau **(SYT)** of shape 4421:

$$\begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 7 & 9 & 11 \\ 4 & 10 & & \\ 8 & & & \end{array}$$

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***f*^λ**: number of SYT of shape λ

E.g., $f^{32} = 5$: $\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array}$ $\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}$ $\begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}$ $\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \end{array}$ $\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}$

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Facts:

- $\exists$ simple formula for f^λ **(hook length formula)**
- If $\lambda \vdash p$ then $[x_1 \cdots x_p]s_\lambda = f^\lambda$.

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Recall:

$$r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda}[x_1 \cdots x_p]s_\lambda$$

Thus

$$r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda} f^\lambda.$$

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$$9\mathbf{5}7\mathbf{28}14\mathbf{6}3 \ \ \text{not vexillary}$$

Vexillary asymptotics

$\boldsymbol{v(n)}$ = number of vexillary $w \in S_n$

Theorem (A. Regev, J. West).

$$
\begin{aligned}
v(n) &\sim \frac{81}{16}\sqrt{3}\pi\frac{9^n}{n^4} \\
&= 2.791102533\cdots\frac{9^n}{n^4}.
\end{aligned}
$$

$\boldsymbol{\lambda(w)}$

$$\boldsymbol{w} = a_1 \cdots a_n \in S_n$$

$$\boldsymbol{c_i} = \#\{j \,:\, i < j \le n,\ a_i > a_j\},\ \ 1 \le i \le n-1$$

$\boldsymbol{\lambda(w)}$: partition whose parts are the c_i's (sorted into decreasing order).

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Example. $w = 5361472 \in S_7$,

$$\begin{aligned} (c_1, \ldots, c_6) &= (4,2,3,0,1,1,0) \\ \Rightarrow \lambda(w) &= 43211 \end{aligned}$$

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Clearly $\lambda(w) \vdash p = \ell(w)$.

Vexillary theorem

Theorem. *We have $G_w = s_\lambda$ for some λ if and only if w is vexillary. In this case $\lambda = \lambda(w)$, so* $r(w) = f^{\lambda(w)}$.

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Example. $w = 5361472 \in S_7$ is vexillary, and $\lambda(w) = 43211$. Hence

$$G_w = s_{43211}, \ \ r(w) = f^{43211} = 2310.$$

Example. $\boldsymbol{w_0} = n, n-1, \ldots, 1 \in S_n$ is vexillary, and $\lambda(w_0) = (n-1, n-2, \ldots, 1)$. Hence

$$r(w_0) = f^{(n-1,n-2,\ldots,1)} = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-1)^1}.$$

w_0

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n	3	4	5	6	7
$r(w_0)$	2	16	768	292864	1100742656

Combinatorial interpretation of $\alpha_{w\lambda}$

Recall:

$$\begin{aligned} G_w &= \sum_{\lambda \vdash p} \alpha_{w\lambda} s_\lambda \\ \Rightarrow r(w) &= \sum_{\lambda \vdash p} \alpha_{w\lambda} f^\lambda \end{aligned}$$

What can we say about $\alpha_{w\lambda}$?

Semistandard tableaux

A **semistandard (Young) tableau** (**SSYT**) T of shape $\lambda = (4, 3, 3, 1, 1)$:

$$
\begin{array}{cc}
 & \leqslant \\
\wedge & \begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 4 \\
\hline
2 & 3 & 3 & \multicolumn{1}{|c}{} \\
\cline{1-3}
4 & 4 & 6 & \multicolumn{1}{|c}{} \\
\cline{1-3}
5 & \multicolumn{3}{|c}{} \\
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 & 7 & \multicolumn{3}{c}{} \\
\cline{2-2}
\end{array}
$$

Reading word of T: 421133264457

Fomin-Greene theorem

Theorem (S. Fomin and C. Greene). *Let $w \in S_n$, $\ell(w) = p$, and $\lambda \vdash p$. The coefficient $\alpha_{w\lambda}$ is equal to the number of SSYT of shape λ whose row reading word is a reduced decomposition of w.*

Example of Fomin-Greene theorem

Example. $w = 4152736 \in S_7$

$$\begin{array}{lll} 1\,2\,3 & 1\,2\,3 & 1\,2\,3\,6 \\ 3\,4 & 3\,4\,6 & 3\,4 \\ 5\,6 & 5 & 5 \end{array}$$

$$3214365, 3216435, 6321435 \in R(w)$$

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$$3214365, 3216435, 6321435 \in R(w)$$

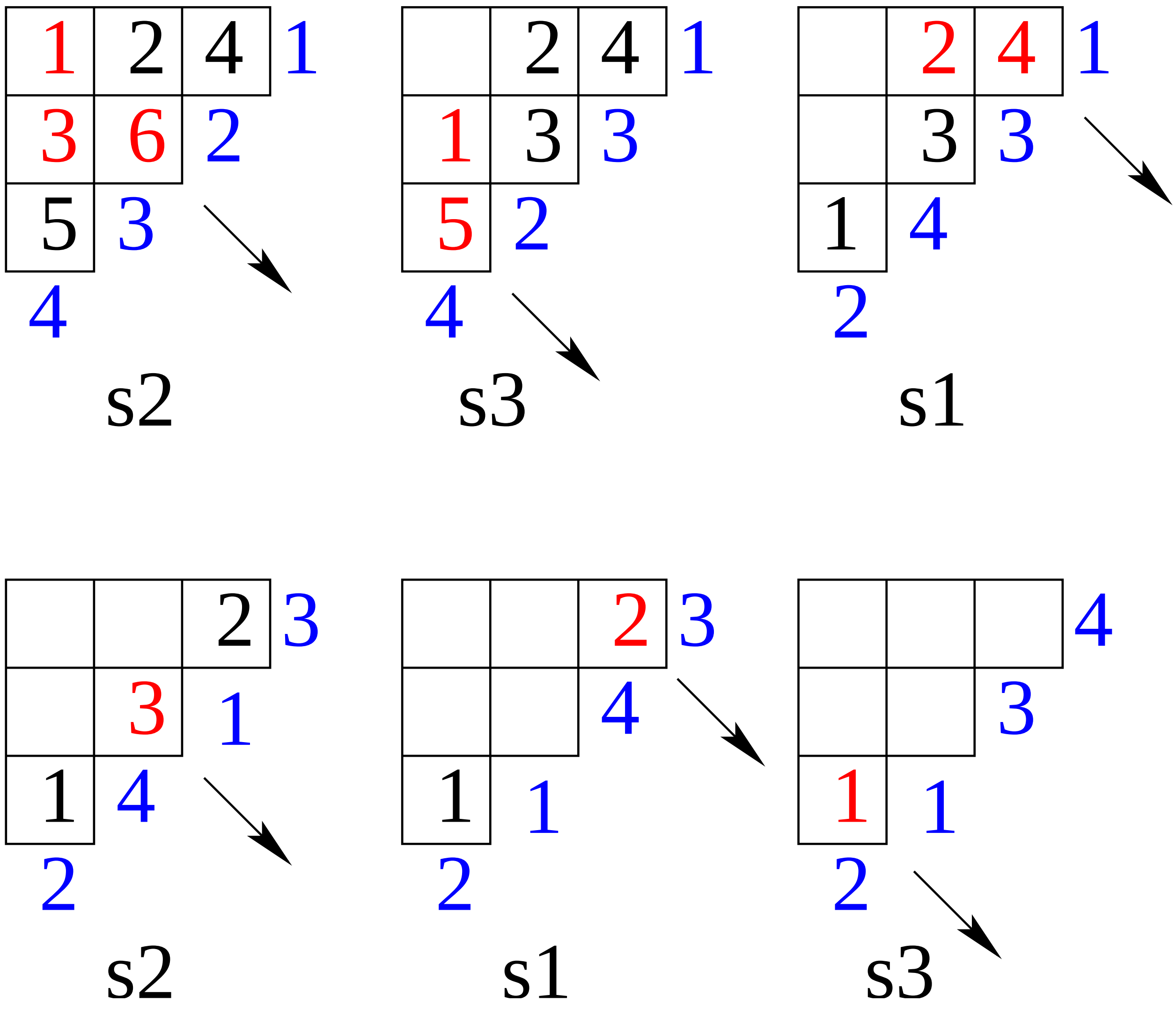
$$\Rightarrow r(w) = f^{322} + f^{331} + f^{421} = 21 + 21 + 35 = 77.$$

Recall: $\boldsymbol{w_0} = n, n-1, \ldots, 1 \in S_n$,

$$r(w_0) = f^{n-1,n-2,\ldots,1}.$$

Is there a bijective proof?

Edelman-Greene bijection



Inverse bijection

The inverse to the previous bijection is given by a version of RSK algorithm (discussed in first lecture).

Representation theory of S_n

Irreducible representations $\varphi^\lambda \colon S_n \to \mathrm{GL}(N, \mathbb{C})$ are indexed by partitions $\lambda \vdash n$.

$$N = \dim \varphi^\lambda = f^\lambda$$

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Specht module $\boldsymbol{M_\lambda}$: an S_n-module constructed from the (Young) diagram of λ (using row-symmetrizers and column anti-symmetrizers) that affords the representation φ^λ.

Arbitrary diagrams

For **any** diagram D (finite subset of a square grid) we can carry out the Specht module construction, obtaining an S_n-module M_D (in general reducible).

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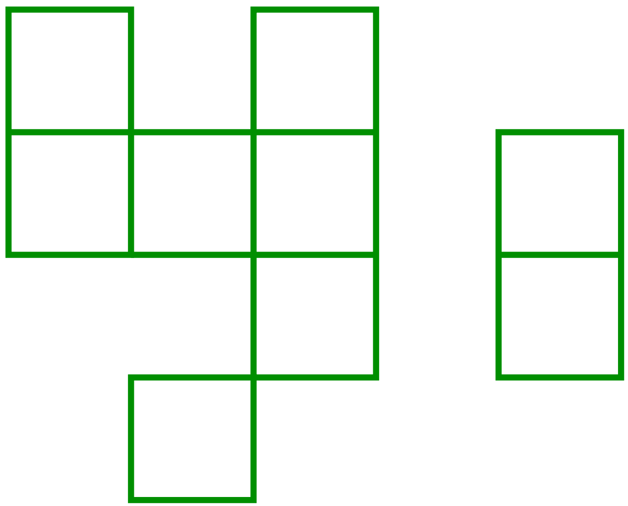

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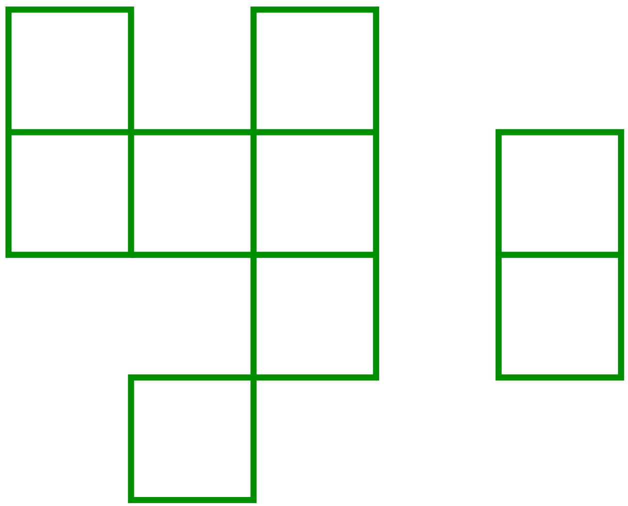

In general, M_D is not well-understood.

Diagram D_w of a permutation w

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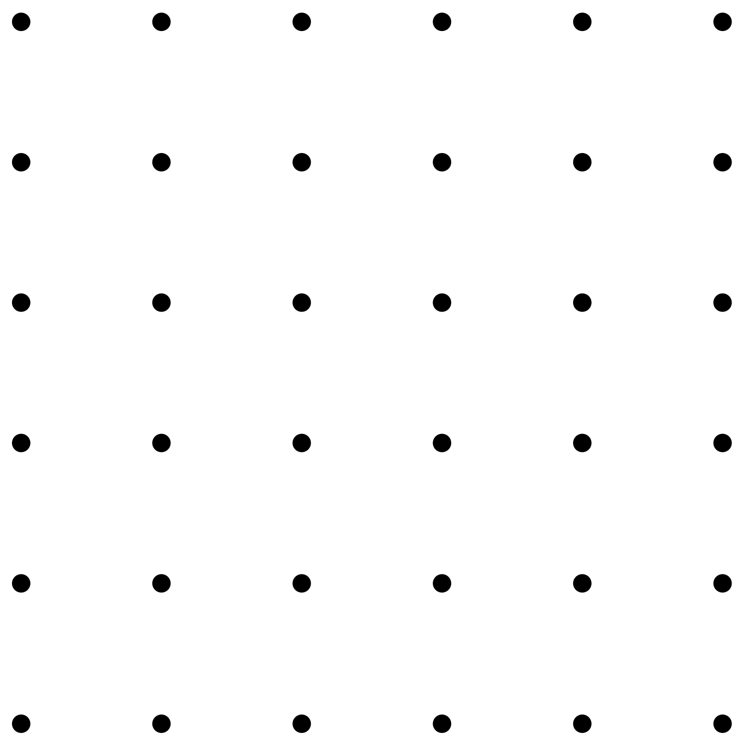

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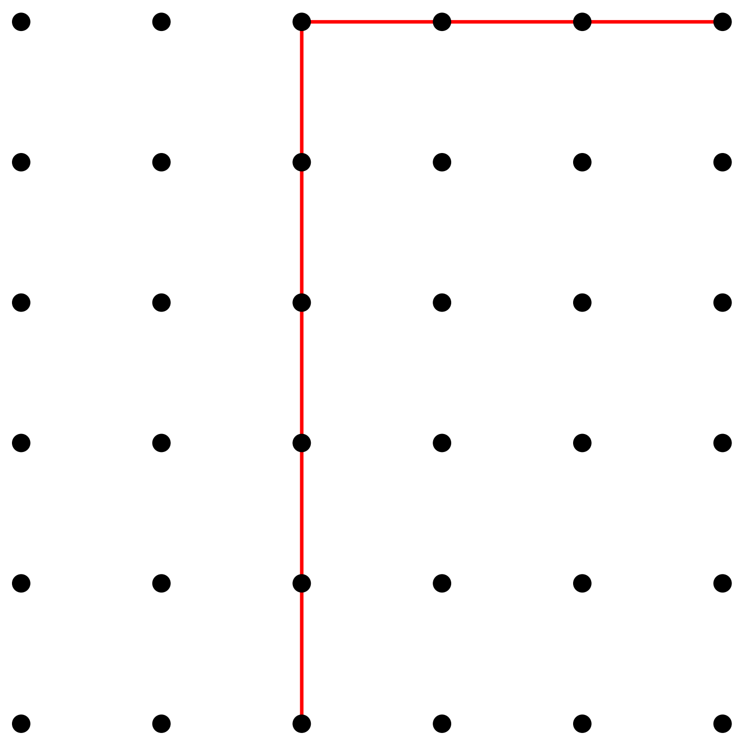

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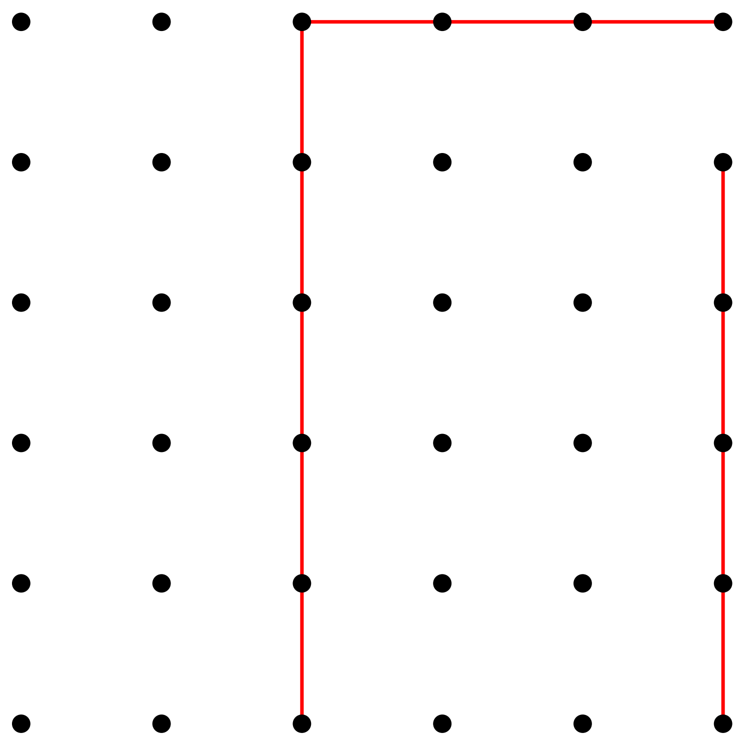

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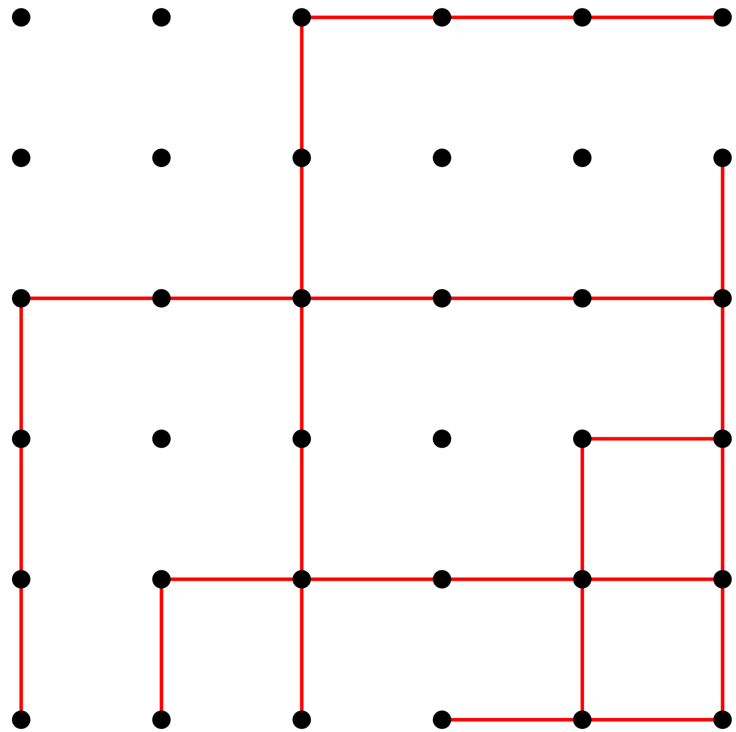

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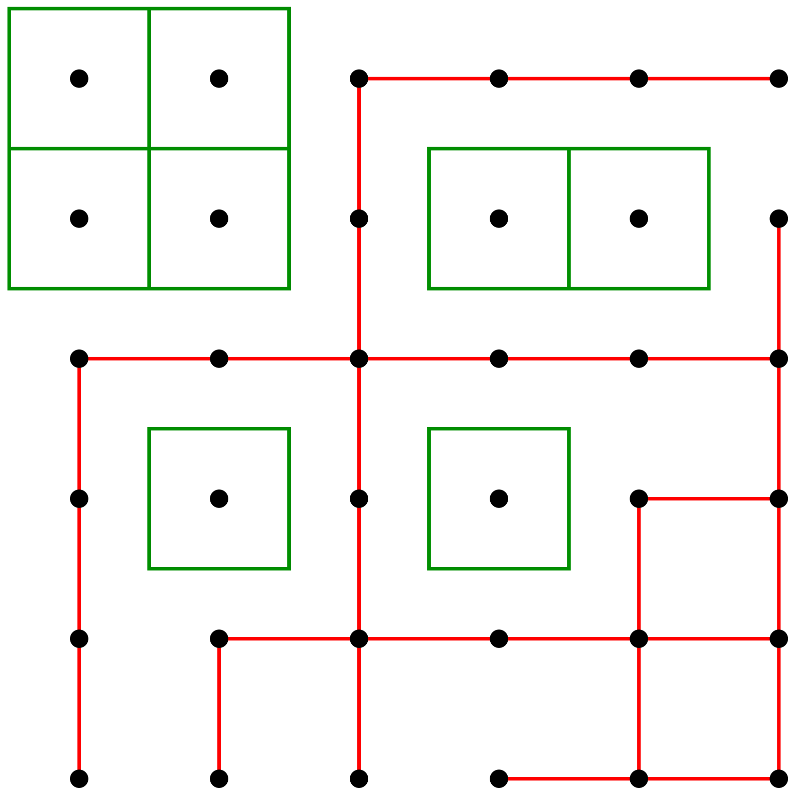

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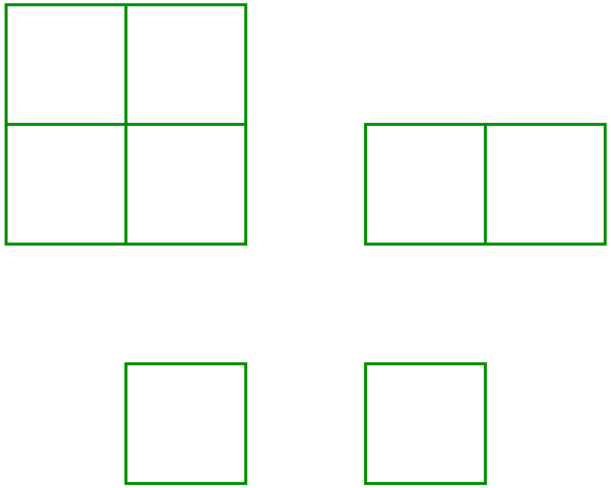

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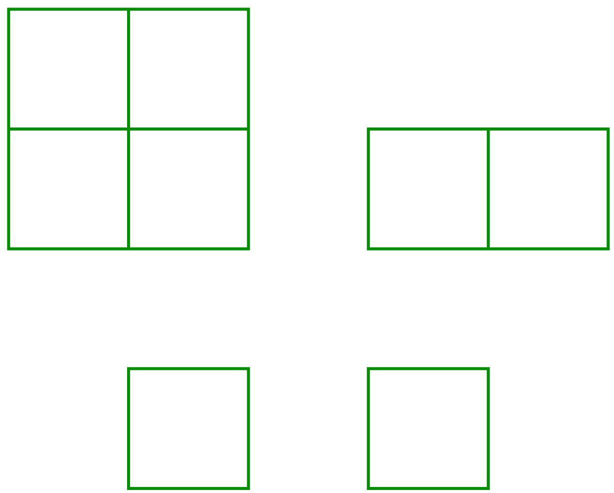

Number of squares of $D_w = \ell(w)$.

The Specht module M_{D_w}

Theorem (**Kraśkiewicz-Pragacz**, 1986, 2004). *Let* $\boldsymbol{w} = S_n$, $\boldsymbol{p} = \ell(w)$. *For* $\lambda \vdash p$, *the multiplicity of* φ^λ *in* M_{D_w} *is* $\alpha_{w\lambda}$.

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Theorem (**Kraśkiewicz-Pragacz**, 1986, 2004). *Let* $\boldsymbol{w} = S_n$*,* $\boldsymbol{p} = \ell(w)$*. For* $\lambda \vdash p$*, the multiplicity of* φ^λ *in* M_{D_w} *is* $\alpha_{w\lambda}$*.*

Since $r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda} f^\lambda$ and $\dim \varphi^\lambda = f^\lambda$, we get:

Corollary. $\dim M_{D_w} = r(w)$

Flag varieties

$\mathbf{Fl}_n$: set of **complete flags**

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

of subspaces in $\mathbb{C}^n$ (so $\dim V_i = i$)

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$\mathbf{Fl}_{\boldsymbol{n}}$: set of **complete flags**

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

of subspaces in $\mathbb{C}^n$ (so $\dim V_i = i$)

$$\mathrm{Fl}_n \cong \mathrm{GL}(n, \mathbb{C})/B,$$

where $\boldsymbol{B}$ is the Borel subgroup of invertible upper triangular matrices.

Cohomology of $\mathbf{Fl}_n$

For each $w \in S_n$ there is a projective subvariety $\boldsymbol{\Omega_w}$ of (complex) dimension $\ell(w)$, the **Schubert variety** corresponding to w, defined by simple geometric conditions.

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Standard result from Schubert calculus: the Schubert cycles σ_w, $w \in S_n$, form a basis of $H^*(\mathrm{Fl}_n;\mathbb{C})$

Schubert polynomials

Schubert polynomial $\mathfrak{S}_w = \mathfrak{S}_w(x_1, \ldots, x_{n-1})$, $w \in S_n$:

$$\mathfrak{S}_w = \sum_{(a_1,\ldots,a_p)\in R(w)} \sum_{\substack{1\leq i_1\leq\cdots\leq i_p \\ i_j<i_{j+1} \text{ if } a_j<a_{j+1} \\ i_j\leq j}} x_{i_1}\cdots x_{i_p}.$$

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Compare

$$G_w = \sum_{(a_1,\ldots,a_p)\in R(w)} \sum_{\substack{1\le i_1\le\cdots\le i_p \\ i_j<i_{j+1} \text{ if } a_j<a_{j+1}}} x_{i_1}\cdots x_{i_p}.$$

Stable Schubert polynomials

G_w is sometimes called a **stable Schubert polynomial** (a certain limit of Schubert polynomials).

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Example. $\mathfrak{S}_{4213}, \mathfrak{S}_{15324}, \mathfrak{S}_{126435}, \ldots \to G_{4213}$

Ring structure of $H^*(\mathrm{Fl}_n; \mathbb{C})$

$\boldsymbol{R_n} = \mathbb{C}[x_1, x_2, \ldots, x_n]/I_n$, where $\boldsymbol{I_n}$ is generated by the elementary symmetric functions $e_1, \ldots, e_n$.

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Theorem. *There is an algebra isomorphism*

$$\varphi\colon R_n \to H^*(\mathrm{Fl}_n;\mathbb{C}),$$

such that for $w \in S_n$ we have

$$\varphi(\mathfrak{S}_{w_0 w}) = \sigma_w,$$

where $\boldsymbol{w_0} = n, n-1, \ldots, 1$.

A curious identity

Theorem (**Macdonald** 1991, **Fomin-S.** 1994)
Let $w \in S_n$ *and* $\ell(w) = p$*. Then*

$$\sum_{(a_1,a_2,\ldots,a_p)\in R(w)} a_1 a_2 \cdots a_p = p!\,\mathfrak{S}_w(1,1,\ldots,1).$$

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Theorem. $\mathfrak{S}_w(1,1,\ldots,1) = 1$ *if and only if* w *is* ***132-avoiding****, i.e., there does not exist* $i < j < k$ *such that* $a_i < a_k < a_j$.

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Number of 132-avoiding $w \in S_n$: the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

The special case w_0

For $w_0 = n, n-1, \ldots, 1 \in S_n$ we have

$$\sum_{(a_1,a_2,\ldots,a_p)\in R(w_0)} a_1 a_2 \cdots a_p = \binom{n}{2}!.$$

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Example. For $n = 3$ we have

$$R(w_0) = \{(1,2,1),(2,1,2)\}.$$

Thus

$$1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 6 = \binom{3}{2}!.$$


THE END IS NEAR!
SIPRESS
Is that a good thing or a bad thing?

An analogue for any transpositions

$(i, j) \in S_n$: transposition interchanging i and j

For $w \in S_n$, $\ell(w) = p$, define

$$T(w) = \{((i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)) \; : \; w = (i_1, j_1)(i_2, j_2) \cdots (i_p, j_p) \text{ and } \ell((i_1, j_1) \cdots (i_k, j_k)) = k \text{ for all } 1 \le k \le p\}.$$

An example

Let $w = w_0 = 321 \in S_3$.

$$\begin{aligned} 321 &= (1,2)(2,3)(1,2) = (2,3)(1,2)(2,3) \\ &= (1,2)(1,3)(2,3) = (2,3)(1,3)(1,2), \end{aligned}$$

so (abbreviating (i,j) as $\boldsymbol{ij}$)

$$T_{321} = \{(12, 23, 12), (23, 12, 23),$$

$$(12, 13, 23), (23, 13, 12)\}.$$

Theorem of Chevalley–Stembridge

Theorem (**Chevalley** $\sim$1958, **Stembridge** 2002). *For* $w = w_0 \in S_n$ *(so* $p = \binom{n}{2}$*) we have*

$$\sum_{((i_1,j_1),(i_2,j_2),\dots,(i_p,j_p))\in T(w_0)} (j_1-i_1)(j_2-i_2)\cdots(j_p-i_p) = p!.$$

An example (cont.)

Example. Recall

$$T(321) = \{(12, 23, 12), (23, 12, 23), (12, 13, 23), (23, 13, 12)\}.$$

Hence

$$1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 2 \cdot 1 = \binom{3}{2}!.$$

An open problem

$$\sum_{(a_1,a_2,\ldots,a_p)\in R(w_0)} a_1 a_2 \cdots a_p = p!.$$

$$\sum_{((i_1,j_1),(i_2,j_2),\ldots,(i_p,j_p))\in T(w_0)} (j_1 - i_1)(j_2 - i_2)\cdots(j_p - i_p) = p!.$$

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- Is this similarity just a “coincidence”?
- Is there a common generalization?


THIS IS THE END