POLYNOMIALS WITH REAL ZEROS

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Rolle's theorem. *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b) = 0$, then there exists $a < c < b$ such that $f'(c) = 0$.*

Corollary. *If $P(x) \in \mathbb{R}[x]$ and every zero of $P(x)$ is real, then every zero of $P'(x)$ is real.*

Let $P(x) =$

$$a_n x^n + \cdots + \binom{n}{2} a_2 x^2 + \binom{n}{1} a_1 x + a_0 \in \mathbb{R}[x].$$

Theorem (Newton). *If all zeros of $P(x)$ are real, then*

$$a_i^2 \geq a_{i-1} a_{i+1}, \quad 1 \leq i \leq n-1.$$

Proof. $P^{(n-i-1)}(x)$ has real zeros

$\Rightarrow Q(x) := x^{i+1} P^{(n-i-1)}(1/x)$ has real zeros

$$\Rightarrow Q^{(i-1)}(x) \text{ has real zeros.}$$

But $Q^{(i-1)}(x) = \frac{n!}{2}\left(a_{i+1} + 2a_i x + a_{i-1} x^2\right)$

$$\Rightarrow a_i^2 \geq a_{i-1} a_{i+1}. \quad \square$$

Let $P(x) = \sum a_i x^i$ have only non-positive real zeros. Let

$$i = \textbf{mode}(\boldsymbol{P}) \text{ if } a_i = \max a_j.$$

(If $a_i = a_{i+1} = \max a_j$, let $\mathrm{mode}(P) = i + \frac{1}{2}$.)

Theorem (J. N. Darroch, 1964):

$$\left| \frac{P'(1)}{P(1)} - \mathrm{mode}(P) \right| < 1.$$

Example.

Hermite polynomials:

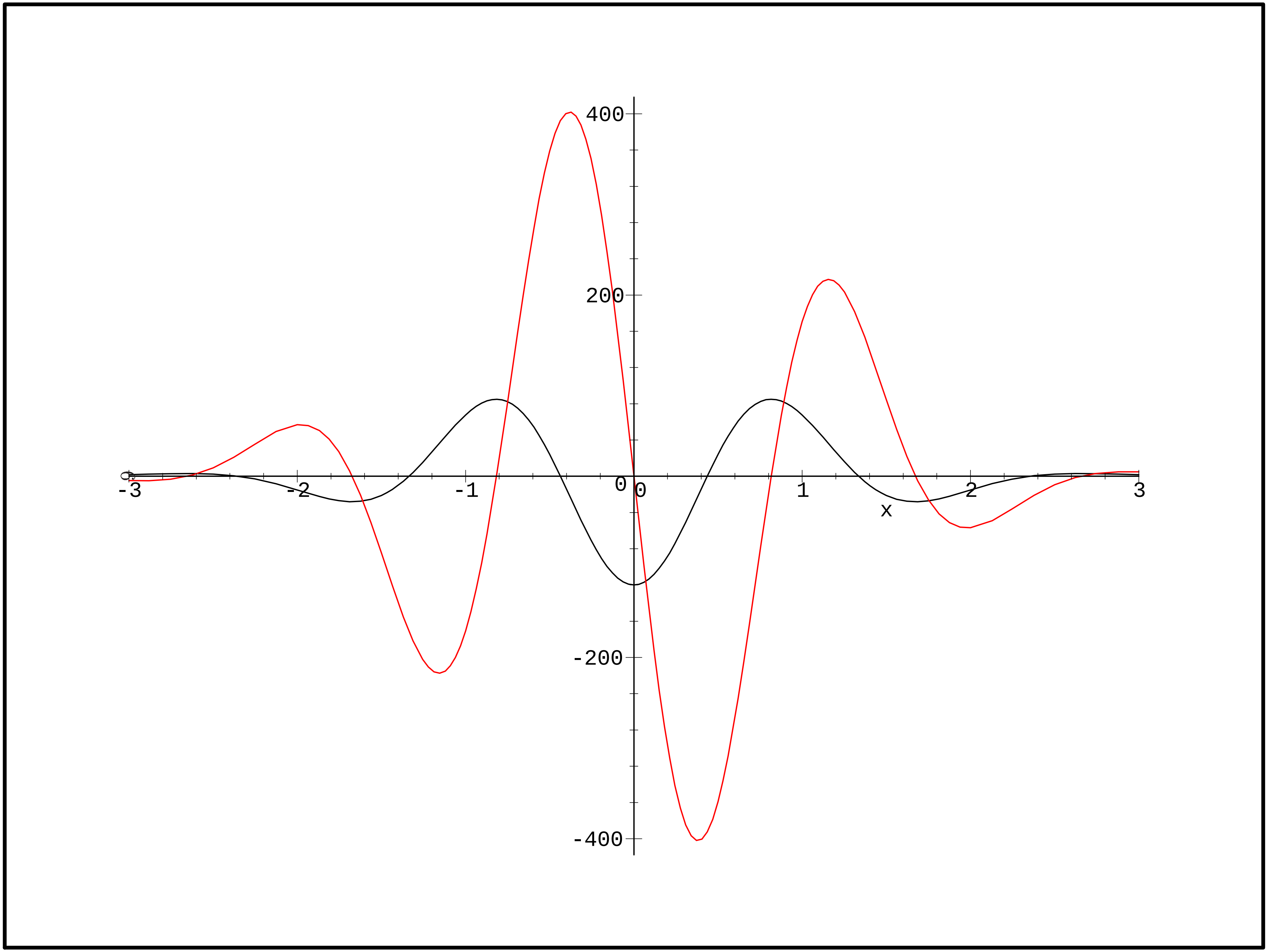
$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!\,(2x)^{n-2k}}{k!\,(n-2k)!}$$

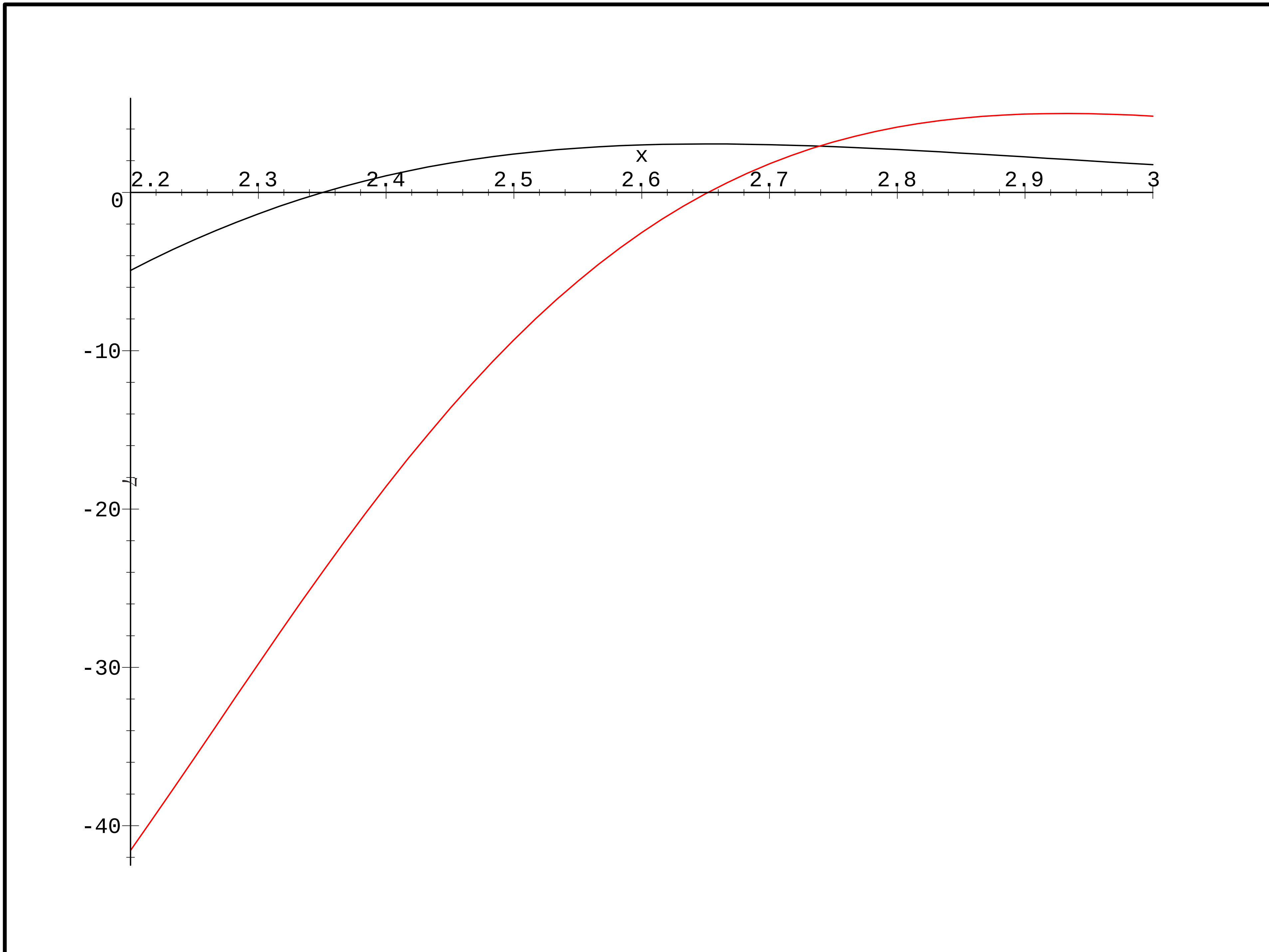
$$H_n(x) = -e^{x^2} \frac{d}{dx} \left(e^{-x^2} H_{n-1}(x) \right)$$

By induction, $H_{n-1}(x)$ has $n-1$ real zeros. Since

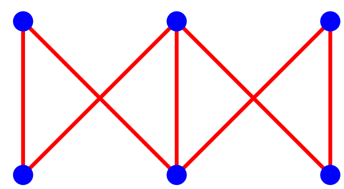
$$e^{-x^2} H_{n-1}(x) \to 0 \text{ as } x \to \pm\infty,$$

it follow that $H_n(x)$ has n real zeros interlaced by the zeros of $H_{n-1}(x)$.


400
200
-200
-400
-3
-2
-1
0
0
1
2
3
x


2.2
2.3
2.4
2.5
2.6
2.7
2.8
2.9
3
x
0
-10
-20
-30
-40

Example (Heilmann-Lieb, 1972). Let G be a graph with t_i i-sets of edges with no vertex in common (**matching** of size i). Then $\sum_i t_i x^i$ has only real zeros.



$$3x^3 + 11x^2 + 7x + 1$$

Let

$$\boldsymbol{T(x)} = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x].$$

Set $a_k = 0$ for $k < 0$ or $k > n$. Define

$$\boldsymbol{A_T} = \left[a_{j-i}\right]_{i,j\geq 1},$$

an infinite **Toeplitz matrix**.

Theorem (Aissen-Schoenberg-Whitney, 1952) *TFAE:*

- *Every minor of A_T is ≥ 0, i.e., A_T is* **totally nonnegative**.
- *Every zero of $T(x)$ is real and ≤ 0.*

Gives **infinitely** many conditions, even for $ax^2 + bx + c$.

Culture: Edrei-Thoma generalization (conjectured by Schoenberg). Let $T(x) = 1 + a_1 x + \cdots \in \mathbb{R}[[x]]$. As before, let

$$\boldsymbol{A_T} = \left[a_{j-i}\right]_{i,j \geq 1}.$$

TFAE:

- Every minor of A_T is nonnegative.
- $T(x) = e^{\gamma x} \dfrac{\prod_i (1 + r_i x)}{\prod_j (1 - s_j x)}$, where

  $$\gamma, r_i, s_j \geq 0, \ \sum r_i + \sum s_j < \infty.$$

$$T(x) = e^{\gamma x} \frac{\prod_i (1 + r_i x)}{\prod_j (1 - s_j x)}$$

Note:

- A_T easily seen to be t.n. for

$$T(x) = 1 + ax,\ a \geq 0, \text{ or } T(x) = \frac{1}{1 - bx},\ b \geq 0.$$

- A, B t.n. $\Rightarrow$ AB t.n. (by Binet-Cauchy)
- $A_{TU} = A_T A_U$
- $e^{\gamma x} = \lim_{n \to \infty} \left(1 + \frac{\gamma x}{n}\right)^n$

Connection with S_∞ (Thoma, Vershik, Kerov, et al). Let $\lambda^n \vdash n$ and

$\tilde{\chi}^{\lambda^n}$ = normalized irred. character of $\mathfrak{S}_n$

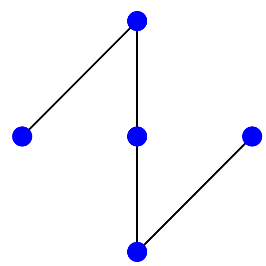
Then $\lim_{n\to\infty} \tilde{\chi}^{\lambda^n}$ exists if and only if

$$\begin{aligned} r_i &= \lim_{n\to\infty} \lambda_i^n / n \\ s_j &= \lim_{n\to\infty} (\lambda^n)'_j / n \end{aligned}$$

exist.

An application of A-S-W:

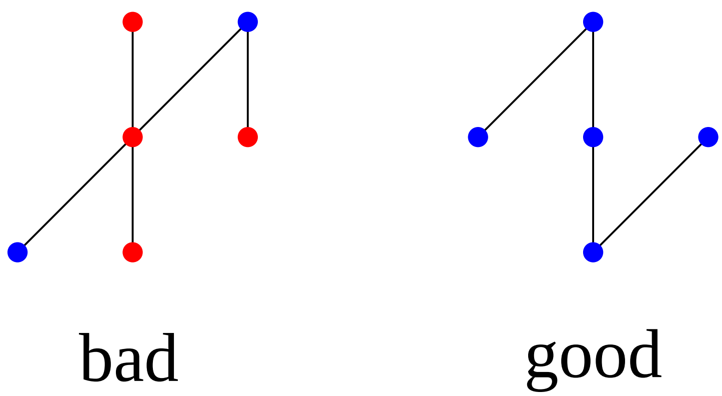
Let P be a finite poset. Let $\boldsymbol{c_i}$ be the number of i-element chains of P.



$c_0 = 1$
$c_1 = 5$
$c_2 = 5$
$c_3 = 1$

Chain polynomial: $\boldsymbol{C_P(x)} = \sum c_i x^i$

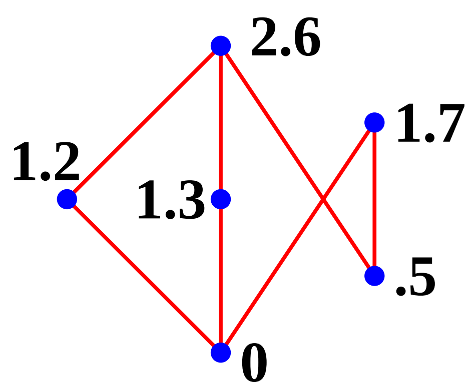
Theorem (Gasharov (essentially), Skandera) *Let P have no induced $\mathbf{3}+\mathbf{1}$. Then $C_P(x)$ has only real zeros.*



Proof of Gasharov based on combinatorial interpretation of minors of A_C.

Special case: P is a **unit interval order** or **semiorder**, i.e., a set of real numbers with

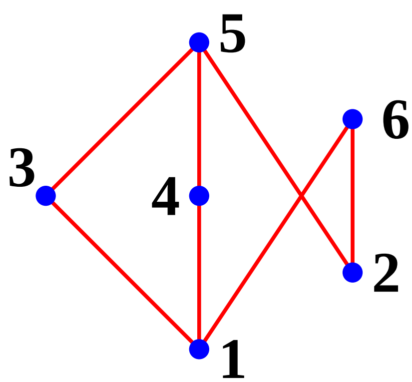
$$u \overset{P}{<} v \iff u \overset{\mathbb{R}}{<} v - 1.$$



Same as no induced $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$.

For any poset, define the **antiadjacency matrix** $\boldsymbol{N_P}$ by

$$(\boldsymbol{N_P})_{\boldsymbol{st}} = \begin{cases} 0, & \text{if } s < t \\ 1, & \text{otherwise.} \end{cases}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Facts.

- $\det(I + xN_P) = C_P(x)$
- P can be ordered so that N_P is totally nonnegative $\Leftrightarrow$ P is a semiorder.
- (Gantmacher-Krein) Eigenvalues of t.n. matrices are real.

Corollary. *If P is a semiorder, then $C_P(x)$ has only real zeros.*

Conjecture (S.-Stembridge) (implies Gasharov-Skandera theorem) Let P be a $(\mathbf{3}+\mathbf{1})$-avoiding poset. Define

$$X_P = \sum_{\substack{f:P\to\mathbb{P} \\ s\|t \Rightarrow f(s)\neq f(t)}} \left(\prod_{t\in P} x_{f(t)} \right),$$

the "chromatic symmetric function" of the incomparability graph of P. Then X_P is an e-positive symmetric function.

Above conjecture, in the special case of semiorders, follows from:

Conjecture (Stembridge) Monomial immanants of Jacobi-Trudi matrices are s-positive.

Rephrasing of A-S-W theorem.

Let $P(x) \in \mathbb{R}[x]$, $P(0) = 1$. Define

$$F_P(\boldsymbol{x}) = P(x_1)P(x_2)\cdots,$$

a symmetric formal series in $\boldsymbol{x} = (x_1, x_2, \ldots)$. TFAE:

- Every zero of $P(x)$ is real and < 0.
- $F_P(\boldsymbol{x})$ is **s-positive**, i.e., a nonnegative linear combination of Schur functions s_λ.
- $F_P(\boldsymbol{x})$ is **e-positive**, i.e., a nonnegative linear combination of elementary symmetric functions e_λ.

Eulerian polynomial:

$$\boldsymbol{A_n(x)} = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)+1},$$

where

$$\mathbf{des(w)} = \#\{i \,:\, w(i) > w(i+1)\}.$$

E.g., des(**4**1**75**236) = 3.

Euler: $\displaystyle\sum_{j\geq 0} j^n x^j = \frac{A_n(x)}{(1-x)^{n+1}}.$

Theorem (Harper). *$A_n(x)$ has only real zeros.*

Example.

$$P(x) = \frac{A_5(x)}{x} = 1+26x+66x^2+26x^3+x^4$$

$$\begin{aligned} F_P \;&=\; 1 + 26s_1 + (66s_2 + 610s_{11}) \\ &\qquad + (26s_3 + 1690s_{21} + 14170s_{111}) + \cdots \\ &=\; 1 + 26e_1 + (544e_2 + 66e_{11}) \\ &\qquad + (12506e_3 + 1638e_{21} + 26e_{111}) + \cdots \end{aligned}$$

Problem. (a) Let $P(x) = A_n(x)/x$. Find a combinatorial interpretation for the coefficients of the expansion of $F_P(\boldsymbol{x})$ in terms of s_λ's or e_λ's, thereby showing they are nonnegative.

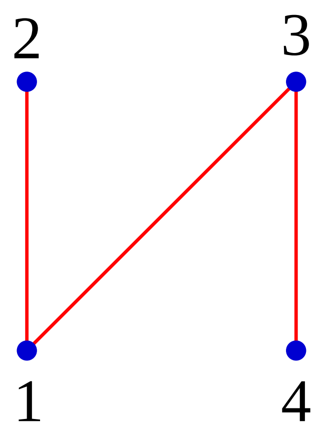
(b) Generalize to other polynomials $P(x)$.

Let P be a partial ordering of $1, \ldots, n$. Let

$$\boldsymbol{\mathcal{L}_P} = \{w = w_1 \cdots w_n \in \mathfrak{S}_n :$$
$$i \overset{P}{<} j \Rightarrow w^{-1}(i) < w^{-1}(j)$$
$$(\text{i.e., } i \text{ precedes } j \text{ in } w)\}.$$

$$\boldsymbol{W_P(x)} = \sum_{w \in \mathcal{L}_P} x^{\mathrm{des}(w)}.$$

Note. $P = n$-element antichain $\Rightarrow$ $\mathcal{L}_P = \mathfrak{S}_n$ and $W_P(x) = A_n(x)/x$.



w	$\mathrm{des}(w)$
1423	1
4123	1
1432	2
4132	2
1243	1

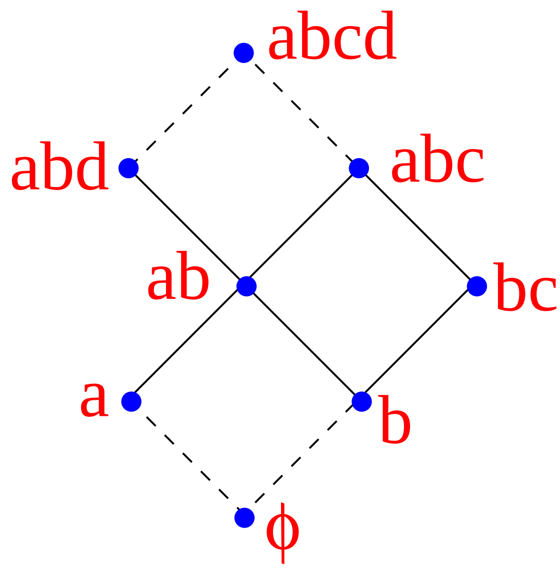
$$W_P(x) = 3x + 2x^2 : \quad \text{all zeros real!}$$

Poset Conjecture (Neggers-S, c. 1970) For any poset P on $1, \ldots, n$, all zeros of $W_P(x)$ are real. (True for $|P| \leq 7$ and naturally labelled P with $|P| = 8$.)

Let Q be a finite poset.

chain polynomial: $C_Q(x) = \sum_{\sigma} x^{\#\sigma}$, where σ ranges over all chains of Q.

Special case (open). Let L be a finite distributive lattice (a collection of sets closed under $\cup$ and $\cap$, ordered by inclusion). Then all zeros of $C_L(x)$ are real.

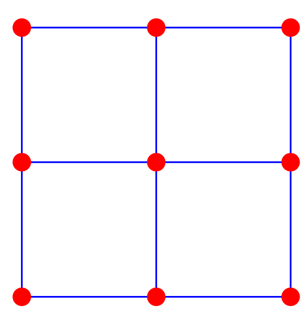


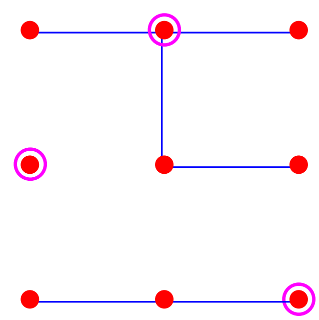
$$C_L(x) = (1 + 6x + 10x^2 + 5x^3)(1 + x)^2$$

Also open: All zeros of $C_L(x)$ are real if L is a finite **modular** lattice.

Example. If A is a (real) symmetric matrix, then every zero of $\det(I + xA)$ is real.

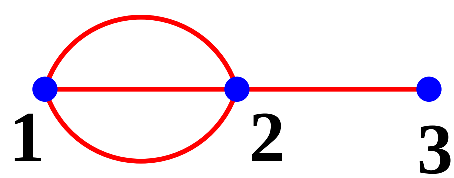
Corollary. *Let G be a graph. Let $a_i(G)$ be the number of* **rooted** *spanning forests with i edges. Then $\sum a_i(G)x^i$ has only real zeros.*





Proof. Define the **Laplacian matrix** $\boldsymbol{L(G)}$, rows and columns indexed by vertex set $V(G)$, by:

$$L(G)_{uv} = -\#(\text{edges between } u \text{ and } v), u \neq v$$
$$L(G)_{uu} = \deg(u).$$



$$L(G) = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\det(I + xL(G)) = 1 + 8x + 9x^2$$

Matrix-Tree Theorem $\Rightarrow$

$$\det(I + xL(G)) = \sum a_i(G)x^i. \quad \square$$

Note. For **unrooted** spanning forests, corresponding result is **false**. I.e, if f_i is the number of i-edge spanning forests of G, then $\sum f_i x^i$ need not have only real zeros. E.g., $G = K_3$, $\sum f_i x^i = 3x^2 + 3x + 1$.

A-S-W gives **infinitely** many inequalities for real zeros. Are there finitely many inequalities?

Example. $x^2 + bx + c$: all zeros real $\Leftrightarrow b^2 \geq 4c$.

Sturm chains. Let $f(x) \in \mathbb{R}[x]$ have positive leading coefficient. Apply Euclidean algorithm to $f(x)$ and $f'(x)$:

$$\begin{aligned}
f(x) &= q_1(x)f'(x) + r_1(x) \\
f'(x) &= q_2(x)r_1(x) + r_2(x) \\
&\cdots \\
r_{k-2}(x) &= q_k(x)r_{k-1}(x) + r_k(x) \\
r_{k-1}(x) &= q_{k+1}(x)r_k(x)
\end{aligned}$$

Theorem. *$f(x)$ has only real zeros $\Leftrightarrow$ $deg(r_i) = \deg(f) - i - 1$ and the leading coefficients of $r_1(x), \ldots, r_k(x)$ have sign sequence* $--++--++\cdots$.

Theorem (source?). *Let*

$$\boldsymbol{V(y_1,\ldots,y_k)} = \prod_{1\le i<j\le k} (y_i - y_j),$$

the **Vandermonde product**. *Let*

$$f(x) = \prod_{i=1}^{n} (x-\theta_i).$$

All zeros of $f(x)$ *are real if and only if*

$$D_k(f) := \sum_{i_1<\cdots<i_k} V(\theta_{i_1},\ldots,\theta_{i_k})^2 \ge 0,$$

$2 \le k \le n.$

$$D_k(f) = \sum_{i_1 < \cdots < i_k} V(\theta_{i_1}, \ldots, \theta_{i_k})^2$$

- $D_k(f)$ is a polynomial in the coefficients of f
- $n - 1$ polynomial inequalities
- $D_n(f) = \mathrm{disc}(f)$
- Condition clearly necessary

Example. $f(x) = x^3 + bx^2 + cx + d$
has real zeros $\Leftrightarrow$

$$\begin{aligned} \mathrm{disc}(f) &\geq 0 \\ b^2 &\geq 3c. \end{aligned}$$

Distribution of real zeros (M. Kac, A. Edelman, *et al.*). Let the coefficients of $a_n x^n + \cdots + a_1 x + a_0$ be independent standard normals.

- Density of expected number of real zeros at $t \in \mathbb{R}$:

$$\rho_n(t) = \frac{1}{\pi}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)t^{2n}}{(t^{2n+2}-1)^2}}.$$

Hence zeros are concentrated near ± 1.

- Expected number of real zeros as $n \to \infty$:

$$E_n = \frac{2}{\pi}\log(n) + C + \frac{2}{n\pi} + O(1/n^2),$$

where

$$C = 0.6257358072\cdots.$$

- Prob(all zeros real) = complicated integral

Suppose the coefficients a_i are independent normals with mean 0 and variance $\binom{n}{i}$. Now

$$E_n = \sqrt{n}.$$