Some Recent Work on Special Polytopes

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Descent polytopes


$S \subseteq [n - 1] = \{1, 2, \ldots, n - 1\}$

Descent polytope $\text{DP}_S \subset \mathbb{R}^n$:

$0 \leq x_i \leq 1$

$x_i \geq x_{i+1}$ if $i \in S$

$x_i \leq x_{i+1}$ if $i \notin S$

Same as order polytope $O(Z_S)$ of zigzag poset $Z_S$. 

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Example of zigzag poset

\[ n = 9, \quad S = \{3, 6, 7\} \]

\[ \mathcal{O}(Z_S) = \{\text{order-preserving maps } f : Z_S \to [0, 1]\} \]
Volume and Ehrhart polynomial of $\text{DP}_S$ follows from theory of $P$-partitions. In particular, let $w = a_1 \cdots a_n \in \mathfrak{S}_n$ and define

$$D(w) = \{ i : a_i > a_{i+1} \} \subseteq [n-1],$$

the **descent set** of $w$. Define

$$\beta_n(S) = \# \{ w \in \mathfrak{S}_n : D(w) = S \}.$$
Combinatorics of $\text{DP}_S$

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$$\beta_n(S) = \#\{w \in \mathfrak{S}_n : D(w) = S\}.$$

**Theorem.** $\text{vol}(\text{DP}_S) = \frac{\beta_n(S)}{n!}$
The $f$-vector of $\text{DP}_S$:

$(f_0, f_1, \ldots, f_{n-1})$: $f$-vector of $\text{DP}_S$, i.e., $f_i$ is the number of $i$-dimensional faces. Set $f_n = 1$.

Define the $f$-polynomial $F_S(t) = \sum_{i=0}^{n} f_i t^i$.

$x, y$: noncommuting variables

For $S \subseteq [n - 1]$ define $v_S = v_1 \cdots v_{n-1}$, where

$$v_i = \begin{cases} 
  x, & \text{if } i \not\in S \\
  y, & \text{if } i \in S.
\end{cases}$$
A generating function

\[ \Phi_n(x, y) := \sum_{S \subseteq [n-1]} F_S(t) v_S \]
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\[ \Phi(x, y) = \sum_{n \geq 1} \Phi_n(x, y) \]

\[ = (2 + t) + (3 + 3t + t^2)(x + y) + \cdots. \]

E.g., \( n = 2, S = \emptyset \): \( 0 \leq x_1 \leq x_2 \leq 1 \), a triangle, so coefficient of \( x \) is \( 3 + 3t + t^2 \).
Theorem. \( \Phi(x, y) = \)

\[
\left( 1 + \frac{t + 1}{1 - (t + 1) \left((1 - y)^{-1}x + (1 - x)^{-1}y\right)} \right) \cdot \frac{1}{1 - x - y}.
\]
Let $T = \{a_0 < \cdots < a_k\} \subseteq [0, n - 1]$. Define

$$\alpha_S(T) = \# \{ F_0 \subset F_1 \subset \cdots \subset F_k : \dim F_i = a_i \}.$$

Call $\alpha_S$ the flag $f$-vector of $\text{DP}_S$.
Let $T = \{a_0 < \cdots < a_k\} \subseteq [0, n - 1]$. Define

$$\alpha_S(T) = \# \{ F_0 \subset F_1 \subset \cdots \subset F_k : \dim F_i = a_i \}.$$

Call $\alpha_S$ the **flag $f$-vector** of $DP_S$.

**Open.** Is there a “nice” generating function for $\alpha_S(T)$’s (or equivalently, the flag $h$-vector of $cd$-index) generalizing Chebikin’s theorem?
Origin (Postnikov & RS): Let

\[ M_n = x_{12} x_{23} \cdots x_{n-1,n}. \]

Continually apply

\[ x_{ij} x_{jk} \rightarrow x_{ik} (x_{ij} + x_{jk}), \]

ending with \( P_n(x_{ij}) \).
Example.

\[ x_{12} x_{23} x_{34} \rightarrow x_{13} x_{12} x_{34} + x_{13} x_{23} x_{34} \]
\[ \rightarrow x_{14} x_{13} x_{12} + x_{14} x_{34} x_{12} \]
\[ + x_{14} x_{13} x_{23} + x_{14} x_{34} x_{23} \]
\[ \rightarrow x_{14} x_{13} x_{12} + x_{14} x_{34} x_{12} \]
\[ + x_{14} x_{13} x_{23} + x_{14} x_{24} x_{23} + x_{14} x_{24} x_{34} \]
\[ = P_3(x_{ij}). \]
The polynomials $P_n(x_{ij})$ depend on the sequence of operations. However:

Theorem. $P_n(1;1;\ldots;1) = C_n = \frac{1}{n+1}$, a Catalan number.
Invariance of $P_n(x_{ij})$

The polynomials $P_n(x_{ij})$ depend on the sequence of operations. However:

**Theorem.** $P_n(1, 1, \ldots, 1) = C_n = \frac{1}{n+1} \binom{2n}{n}$, a Catalan number.
Full root polytopes

$e_i$: $i$th unit vector in $\mathbb{R}^{n+1}$

$A_n^+$: the positive roots

\[ \{ e_i - e_j : 1 \leq i < j \leq n + 1 \} \]

full root polytope $\mathcal{P}(A_n^+)$: convex hull of $A_n^+$ and the origin in $\mathbb{R}^{n+1}$ (Gelfand-Graev-Postnikov)
**Root polytopes**

$T$: a tree on the vertex set $[n + 1]$

**root polytope** $\mathcal{P}(T)$ (of type $A_n$): intersection of $\mathcal{P}(A_n^+)$ with the cone generated by $e_i - e_j$, where $ij \in E(T)$, $i < j$
A graph $G$ on $[n + 1]$ is **noncrossing** if $\forall$ vertices $i < j < k < l$ such that $ik \in E(G)$ and $jl \in E(G)$.

$G$ is **alternating** if $\forall$ $i < j < k$ such that $ij \in E(G)$ and $jk \in E(G)$.
A graph $G$ on $[n + 1]$ is \textbf{noncrossing} if $\forall$ vertices $i < j < k < l$ such that $ik \in E(G)$ and $jl \in E(G)$. $G$ is \textbf{alternating} if $\forall$ $i < j < k$ such that $ij \in E(G)$ and $jk \in E(G)$. 

\begin{figure}
\centering
\begin{tikzpicture}
    \node [vertex] (1) at (0,0) {1};
    \node [vertex] (2) at (1,0) {2};
    \node [vertex] (3) at (2,0) {3};
    \node [vertex] (4) at (3,0) {4};
    \node [vertex] (5) at (4,0) {5};
    \node [vertex] (6) at (5,0) {6};
    \node [vertex] (7) at (6,0) {7};
    \node [vertex] (8) at (7,0) {8};

    \draw [thick] (1) .. controls (0.5,1) and (1.5,1) .. (2);
    \draw [thick] (2) .. controls (2.5,1) and (3.5,1) .. (3);
    \draw [thick] (3) .. controls (4.5,1) and (5.5,1) .. (4);
    \draw [thick] (4) .. controls (5.5,1) and (6.5,1) .. (5);
    \draw [thick] (5) .. controls (6.5,1) and (7.5,1) .. (6);
    \draw [thick] (6) .. controls (7.5,1) and (8.5,1) .. (7);
    \draw [thick] (7) .. controls (8.5,1) and (9.5,1) .. (8);
\end{tikzpicture}
\end{figure}
Some notation

$\overline{G}$: graph with vertex set $[n + 1]$ and edge set

$\{ij : \exists \ i i_1, i_1 i_2, \ldots, i_k j \in E(G), \ i < i_1 < \cdots < i_k < j\}$,

the **transitive closure** of $G$
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$\{ij : \exists i i_1, i_1 i_2, \ldots, i_k j \in E(G), i < i_1 < \cdots < i_k < j\}$,

the **transitive closure** of $G$

$T$: a noncrossing tree on $[n + 1]$

$T_1, \ldots, T_k$: noncrossing, alternating spanning trees of $\overline{T}$
**Theorem.** The root polytopes $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are $n$-simplices with disjoint interior and union $\mathcal{P}(T)$. Moreover, 

$$\text{vol } \mathcal{P}(T) = \frac{f_T}{n!},$$

where $f_T$ is the number of noncrossing alternating spanning trees of $\overline{T}$.
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where $f_T$ is the number of noncrossing alternating spanning trees of $T$.

(several generalizations)
Example

\[ \text{vol } \mathcal{P}(T) = \frac{2}{3!} \]
Yang-Baxter algebras

Proof of theorem: $B(A_n)$: quasi-classical Yang-Baxter algebra (Anatol Kirillov). It is an associative algebra over $\mathbb{Q}[\beta]$ ($\beta$ a central indeterminate) generated by

$$\{x_{ij} : 1 \leq i < j \leq n + 1\},$$

with relations

$$x_{ij} x_{jk} = x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik},$$
$$x_{ij} x_{kl} = x_{kl} x_{ij}, \text{ if } i, j, k, l \text{ are distinct.}$$
$S(A_n)$: subdivision algebra (Meszaros). It is made commutative, i.e.,

$$x_{ij} x_{kl} = x_{kl} x_{ij} \text{ for all } i, j, k, l.$$
Reduction rule

Treat the first relation as a **reduction rule**:

\[ x_{ij} x_{jk} \rightarrow x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik}. \]
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\[ x_{ij} x_{jk} \rightarrow x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik}. \]
A **reduced form** of the monomial $m$ in $\mathcal{B}(A_n)$ or $S(A_n)$ is a polynomial obtained from $m$ by applying successive reductions until no longer possible.
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For $S(A_n)$ and $\beta = 0$, same as reduction of Postnikov and RS.
A reduction redux

\[ x_{12}x_{23}x_{34} \rightarrow x_{13}x_{12}x_{34} + x_{13}x_{23}x_{34} \]

\[ \rightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \]

\[ \quad + x_{14}x_{13}x_{23} + x_{14}x_{34}x_{23} \]

\[ \rightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \]

\[ \quad + x_{14}x_{13}x_{23} + x_{14}x_{24}x_{23} + x_{14}x_{24}x_{34} \]

\[ = P_3(x_{ij}). \]
Reduced form of a graph monomial

\( G \): graph on vertex set \([n + 1]\)

\[ m_G = \prod_{ij \in E(G)} x_{ij} \in S(A_n) \]
Reduced form of a graph monomial

**Theorem.** Let $T$ be a noncrossing tree on $[n + 1]$ and $P_T$ a reduced form of $m_G$. Then

$$P_T(x_{ij} = 1, \beta = 0) = f_T,$$

the number of noncrossing alternating spanning trees of $\overline{T}$.
Relation to root polytopes

The monomials appearing in the reduced form $P_T(x_{ij}, \beta = 0)$ correspond to the facets in a triangulation of $P(A_n)$. 
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\[ x_{12}x_{23} \rightarrow x_{12}x_{13} + x_{23}x_{13} \]
Interior faces of $\mathcal{P}(A_n)$

The interior faces (not necessarily facets) of $\mathcal{P}(A_n)$ correspond to the terms in the reduced form of $P_T(x_{ij}, \beta)$.
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$$x_{12}x_{23} \rightarrow x_{12}x_{13} + x_{23}x_{13} + \beta x_{13}$$

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Proof uses noncommutative Gröbner bases.
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Similar results to $S(A_n)$ for a combinatorial interpretation of the monomials appearing in a reduced form.
Noncommutative version

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Proof uses noncommutative Gröbner bases.

Similar results to $S(A_n)$ for a combinatorial interpretation of the monomials appearing in a reduced form.

Many generalizations . . .
Matching polytopes

Ricky Liu, graduate student, M.I.T.

\( G = (V, E) \): a graph; \( n = \# E \)

\( M_G \): matching polytope of \( G \), i.e.,

\[
M_G = \left\{ w : E \rightarrow \mathbb{R}_{\geq 0} \mid \forall v \in V \sum_{e \in \text{out}(v)} w(e) \leq 1 \right\} \subseteq \mathbb{R}^n.
\]
**Vertices of** $M_G$

**Matching** $M$: a set of vertex-disjoint edges. If $L \subseteq E$, define $\chi_L \in M_G$ by

$$
\chi_L(e) = \begin{cases} 
1, & e \in L \\
0, & e \notin L.
\end{cases}
$$

**Note.** $M_G$ has integer vertices if and only if $G$ is bipartite. In that case, the vertices are $\chi_M$, where $M$ is a matching of $G$. 

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Corollary. $G$ bipartite $\Rightarrow$

$$V(G) := n! \cdot \text{vol}(M_G) \in \mathbb{Z}$$
\( H = \text{graph, } u, v \in V(H), u \neq v \)

\( G \): adjoin pendant edges \( uu' \), \( vv' \) (so \( u' \), \( v' \) are endpoints)

\( G_1 \): adjoin pendant edge \( uu' \) and an edge \( uv \)

\( G_2 \): adjoin pendant edge \( vv' \) and an edge \( uv \)
Leaf recurrence

\[ f(G) = f(G_1) + f(G_2) \]

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Leaf recurrence

\[ f : \mathcal{F} \rightarrow \mathbb{R} \] satisfies the leaf recurrence if

\[ f(G) = f(G_1) + f(G_2). \]
Volume of $M_G$

Theorem. There is a unique $f : \mathcal{F} \rightarrow \mathbb{R}$:

For the star $T = K_n; 1$, we have $f(T) = 1$.

If $G_1$ and $G_2$ are disjoint, $\#V(G_1) = m$, and $\#V(G_2) = n - m$, then $f(G_1 + G_2) = n - m f(G_1) f(G_2)$.

$f$ satisfies the leaf recurrence.

Then $f(G) = V(G)$. 

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Volume of $M_G$

**Theorem.** There is a unique $f : \mathcal{F} \to \mathbb{R}$:

- For the star $T = K_{n,1}$, we have $f(T) = 1$. 

If $G_1$ and $G_2$ are disjoint, $\#V(G_1) = m$, and $\#V(G_2) = n$, then

\[ f(G_1 + G_2) = n \cdot m \cdot f(G_1) \cdot f(G_2) \] 

$satisfies the leaf recurrence. Then $f(G) = V(G)$. 

Some Recent Work on Special Polytopes – p. 31
Volume of $M_G$

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Volume of $M_G$

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- $f$ satisfies the leaf recurrence.

Then $f(G) = V(G)$. 

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Theorem. The previous theorem can be used to compute $V(F)$ for any forest $F$. 
$\mathcal{B}$: the set of unit squares in $\mathbb{R}^2$ with centers $(i, j)$, $i, j \geq 1$. Denote also by $(i, j)$ the unit square with center $(i, j)$. 
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**Diagram $D$**: a finite subset of $\mathcal{B}$
\( \mathcal{B} \): the set of unit squares in \( \mathbb{R}^2 \) with centers \((i, j)\), \(i, j \geq 1\). Denote also by \((i, j)\) the unit square with center \((i, j)\).

**Diagram** \( D \): a finite subset of \( \mathcal{B} \)

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13

31 32 33

16

25

35 36
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Row and column stabilizers

\( D \): diagram with \( n \) boxes, ordered in some way

\( \mathfrak{S}_n \) acts on \( D \)
Row and column stabilizers

$D$: diagram with $n$ boxes, ordered in some way

$\mathfrak{S}_n$ acts on $D$

$R_D$ ($C_D$): subgroup of $\mathfrak{S}_n$ stabilizing each row (column) of $D$

$$R(D) = \sum_{w \in R_D} w, \quad C(D) = \sum_{w \in C_D} \text{sgn}(w)w$$
The Specht module $S^D$ (over $\mathbb{C}$) is the left ideal

$$S^D = \mathbb{C}[\mathfrak{S}_n]C(D)R(D)$$

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$$S^D = \mathbb{C}[\mathfrak{S}_n] C(D) R(D)$$

of $\mathbb{C}[\mathfrak{S}_n]$.

Note. $S^D$ affords a representation of $\mathfrak{S}_n$ by left multiplication.
Note. If $D$ is the (Young) diagram of a partition $\lambda$ of $n$, then $S^D$ is irreducible. Conversely, if $S^D$ is irreducible, then $S^D \cong S^{D'}$ for the diagram $D'$ of some partition.
Note. If \( D \) is the (Young) diagram of a partition \( \lambda \) of \( n \), then \( S^D \) is irreducible. Conversely, if \( S^D \) is irreducible, then \( S^D \cong S^{D'} \) for the diagram \( D' \) of some partition.
Let $V(F) = A \cup B$, so that all edges are between $A$ and $B$. Label the $A$-vertices $1, \ldots, m$ and $B$-vertices $1, \ldots, n$. Let

$$D(F) = \{(i, j) : ij \in E(F), i \in A, j \in B\}.$$
The diagram of a forest \( F \)

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\[
D(F) = \{(i, j) : ij \in E(F), \ i \in A, \ j \in B\}.
\]
The Specht module of $D(F)$

**Note.** $S^{D(F)}$ is independent (up to isomorphism) of the labeling.
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**Theorem.** $\dim S^{D(F)} = V(F)$

**Note for experts.** The diagrams $D(F)$ are **not** %-avoiding diagrams in the sense of Reiner and Shimozono.
Decomposition of $S^D(F)$

How does the Specht module $S^D(F)$ decompose into irreducible representations of $\mathfrak{S}_n$?
How does the Specht module $S^D(F)$ decompose into irreducible representations of $\mathbb{S}_n$?

Recall the leaf recurrence

$$f(G) = f(G_1) + f(G_2)$$

with initial conditions $f(K_{n,1}) = 1$. 
Theorem. For a forest $F$, $f(F)$ is well-defined, and

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In other words, if

$$f(F) = \sum_{\lambda \vdash n} c_\lambda s_\lambda,$$

where $s_\lambda$ is a Schur function, then $c_\lambda$ is the multiplicity of the irreducible representation of $\mathfrak{S}_n$ indexed by $\lambda$ in $S^D(F)$. 
The Ehrhart polynomial of $M_F$? Does it have any representation-theoretic significance?