Some Recent Work on Special Polytopes

Richard P. Stanley

M.I.T.

Descent polytopes

Denis Chebikin, Ph.D. thesis, M.I.T., 2008, and **Richard Ehrenborg**

$S \subseteq [n-1] = \{1, 2, \ldots, n-1\}$

Descent polytope $\mathbf{DP}_S \subset \mathbb{R}^n$:

$$0 \leq x_i \leq 1$$

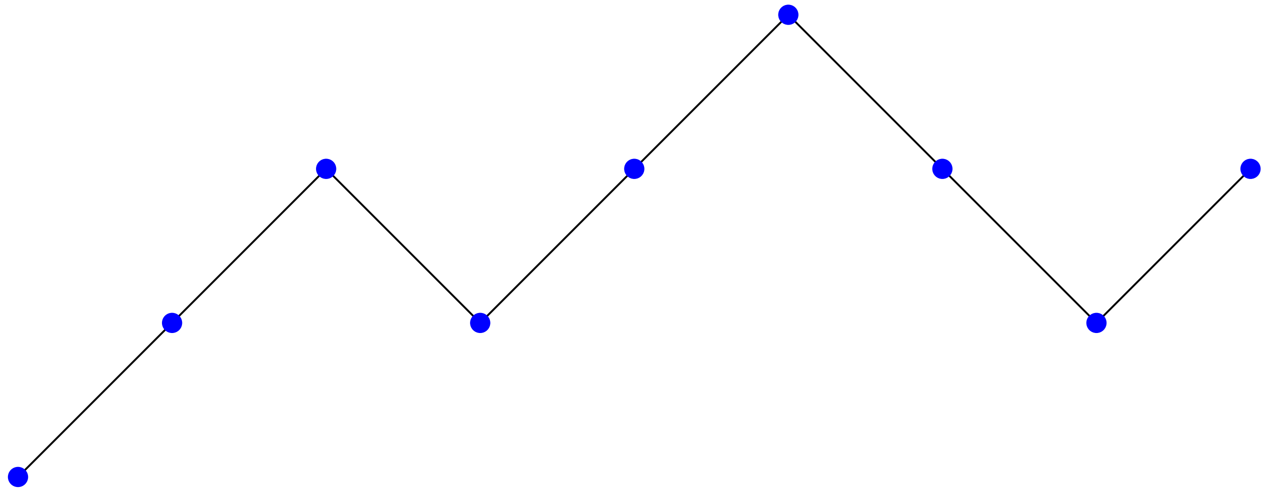
$$x_i \geq x_{i+1} \ \text{ if } i \in S$$

$$x_i \leq x_{i+1} \ \text{ if } i \notin S$$

Same as **order polytope** $\mathcal{O}(Z_S)$ of **zigzag poset** Z_S.

Example of zigzag poset

$n = 9, \quad S = \{3, 6, 7\}$



$\mathcal{O}(Z_S) = \{$order-preserving maps $f : Z_S \to [0, 1]\}$

Combinatorics of DP_S

Volume and Ehrhart polynomial of DP_S follows from theory of P-partitions. In particular, let $w = a_1 \cdots a_n \in \mathfrak{S}_n$ and define

$$\boldsymbol{D(w)} = \{i \,:\, a_i > a_{i+1}\} \subseteq [n-1],$$

the **descent set** of w. Define

$$\beta_n(S) = \#\{w \in \mathfrak{S}_n \,:\, D(w) = S\}.$$

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Theorem. $\mathrm{vol}(\mathrm{DP}_S) = \dfrac{\beta_n(S)}{n!}$

The f-vector of DP_S

$(\boldsymbol{f_0, f_1, \ldots, f_{n-1}})$: f-vector of DP_S, i.e., f_i is the number of i-dimensional faces. Set $\boldsymbol{f_n} = 1$.

Define the **f-polynomial** $\boldsymbol{F_S(t)} = \sum_{i=0}^{n} f_i t^i$.

$\boldsymbol{x, y}$: noncommuting variables

For $S \subseteq [n-1]$ define $\boldsymbol{v_S} = v_1 \cdots v_{n-1}$, where

$$\boldsymbol{v_i} = \begin{cases} x, & \text{if } i \notin S \\ y, & \text{if } i \in S. \end{cases}$$

A generating function

$$\boldsymbol{\Phi_n(x,y)} := \sum_{S\subseteq[n-1]} F_S(t)v_S$$

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$$\begin{aligned}\Phi(x,y) &= \sum_{n\geq 1}\Phi_n(x,y)\\ &= (2+t)+(3+3t+t^2)(x+y)+\cdots.\end{aligned}$$

E.g., $n=2$, $S=\emptyset$: $0\leq x_1\leq x_2\leq 1$, a triangle, so coefficient of x is $3+3t+t^2$.

Chebikin-Ehrenborg theorem

Theorem. $\Phi(x, y) =$

$$\left(1 + \frac{t+1}{1 - (t+1)\left((1-y)^{-1}x + (1-x)^{-1}y\right)}\right)$$

$$\cdot \frac{1}{1 - x - y}.$$

The flag f-vector of DP_S

Let $\boldsymbol{T} = \{a_0 < \cdots < a_k\} \subseteq [0, n-1]$. Define

$$\boldsymbol{\alpha_S}(T) = \#\{F_0 \subset F_1 \subset \cdots \subset F_k \ : \ \dim F_i = a_i\}.$$

Call α_S the **flag f-vector** of DP_S.

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Call α_S the **flag f-vector** of DP_S.

Open. Is there a “nice” generating function for $\alpha_S(T)$’s (or equivalently, the flag h-vector of cd-index) generalizing Chebikin’s theorem?

Root polytopes, subdivision algebras

Karola Meszaros, graduate student, M.I.T.

Origin (Postnikov & RS): Let

$$\boldsymbol{M_n} = x_{12}x_{23}\cdots x_{n-1,n}.$$

Continually apply

$$x_{ij}x_{jk} \rightarrow x_{ik}(x_{ij} + x_{jk}),$$

ending with $P_n(x_{ij})$.

An example

Example.

$$\begin{aligned}
\boldsymbol{x_{12}x_{23}}x_{34} &\to \boldsymbol{x_{13}}x_{12}\boldsymbol{x_{34}} + \boldsymbol{x_{13}}x_{23}\boldsymbol{x_{34}} \\
&\to x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\quad +x_{14}x_{13}x_{23} + x_{14}\boldsymbol{x_{34}x_{23}} \\
&\to x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\quad +x_{14}x_{13}x_{23} + x_{14}x_{24}x_{23} + x_{14}x_{24}x_{34} \\
&= P_3(x_{ij}).
\end{aligned}$$

Invariance of $P_n(x_{ij})$

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Theorem. $P_n(1, 1, \ldots, 1) = C_n = \frac{1}{n+1}\binom{2n}{n}$, a **Catalan number**.

Full root polytopes

e_i: ith unit vector in $\mathbb{R}^{n+1}$

A_n^+: the positive roots

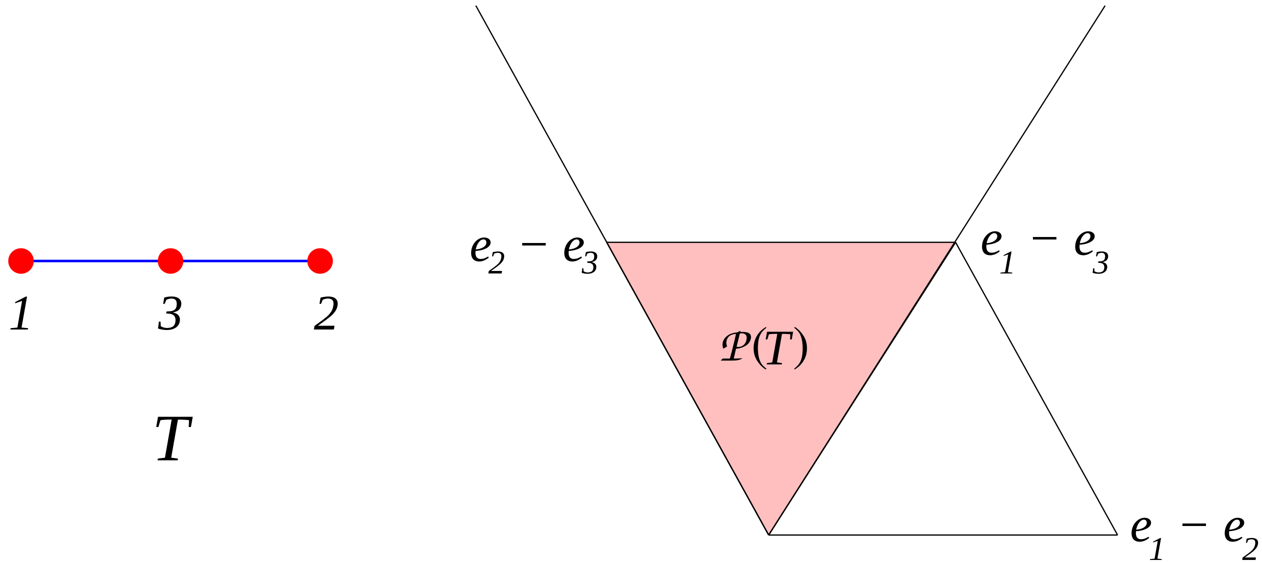
$$\{e_i - e_j \ : \ 1 \leq i < j \leq n+1\}$$

full root polytope $\mathcal{P}(A_n^+)$: convex hull of A_n^+ and the origin in $\mathbb{R}^{n+1}$ (**Gelfand-Graev-Postnikov**)

Root polytopes

T: a tree on the vertex set $[n+1]$

root polytope $\mathcal{P}(T)$ (of type A_n): intersection of $\mathcal{P}(A_n^+)$ with the cone generated by $e_i - e_j$, where $ij \in E(T)$, $i < j$



Noncrossing alternating trees

A graph G on $[n+1]$ is **noncrossing** if $\nexists$ vertices $i < j < k < l$ such that $ik \in E(G)$ and $jl \in E(G)$.

G is **alternating** if $\nexists$ $i < j < k$ such that $ij \in E(G)$ and $jk \in E(G)$.

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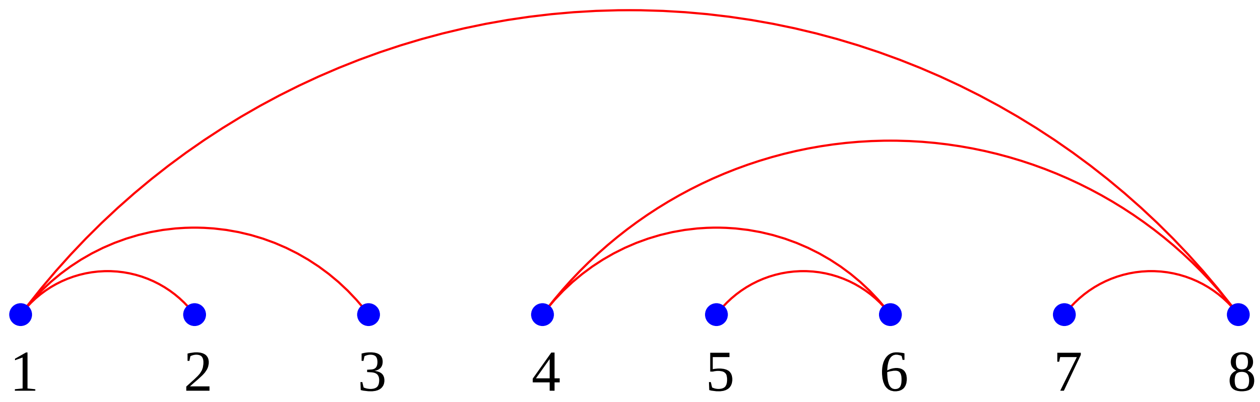

Some notation

$\overline{G}$: graph with vertex set $[n+1]$ and edge set

$$\{ij\ :\ \exists\, ii_1, i_1i_2, \ldots, i_kj \in E(G),\ i < i_1 < \cdots < i_k < j\},$$

the **transitive closure** of G

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T: a noncrossing tree on $[n+1]$

$T_1, \ldots, T_k$: noncrossing, alternating spanning trees of $\overline{T}$

Volume of $\mathcal{P}(T)$

Theorem. *The root polytopes* $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ *are* n*-simplices with disjoint interior and union* $\mathcal{P}(T)$*. Moreover,*

$$\operatorname{vol} \mathcal{P}(T) = \frac{f_T}{n!},$$

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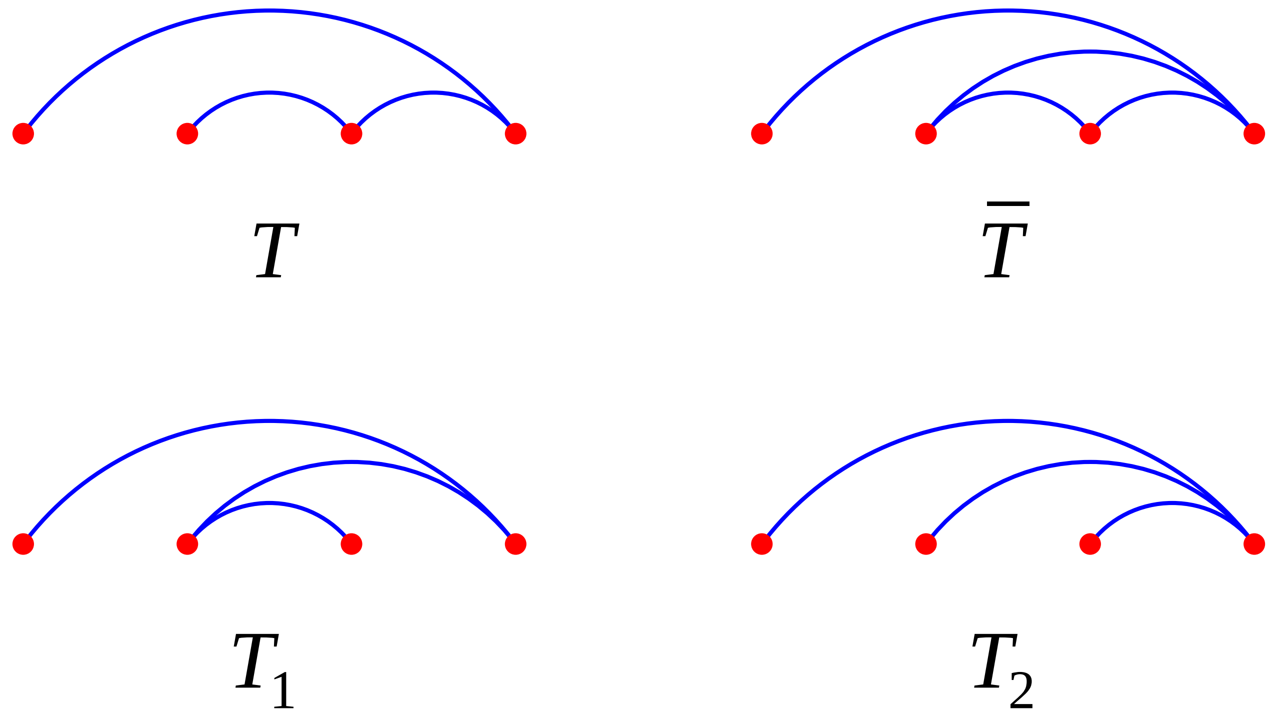
(several generalizations)

Example



$$\operatorname{vol}\mathcal{P}(T) = \frac{2}{3!}$$

Yang-Baxter algebras

Proof of theorem:

$\mathcal{B}(A_n)$: **quasi-classical Yang-Baxter algebra** (Anatol Kirillov). It is an associative algebra over $\mathbb{Q}[\beta]$ (β a central indeterminate) generated by

$$\{x_{ij} \,:\, 1 \leq i < j \leq n+1\},$$

with relations

$$\begin{aligned} x_{ij}x_{jk} &= x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik} \\ x_{ij}x_{kl} &= x_{kl}x_{ij}, \text{ if } i,j,k,l \text{ are distinct.} \end{aligned}$$

Subdivision algebra

$\boldsymbol{\mathcal{S}(A_n)}$: **subdivision algebra** (Meszaros). It is $\mathcal{B}(A_n)$ made commutative, i.e.,

$$x_{ij}x_{kl} = x_{kl}x_{ij} \text{ for } \textbf{\textit{all}}\ i, j, k, l.$$

Reduction rule

Treat the first relation as a **reduction rule**:

$$x_{ij}x_{jk} \longrightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}.$$

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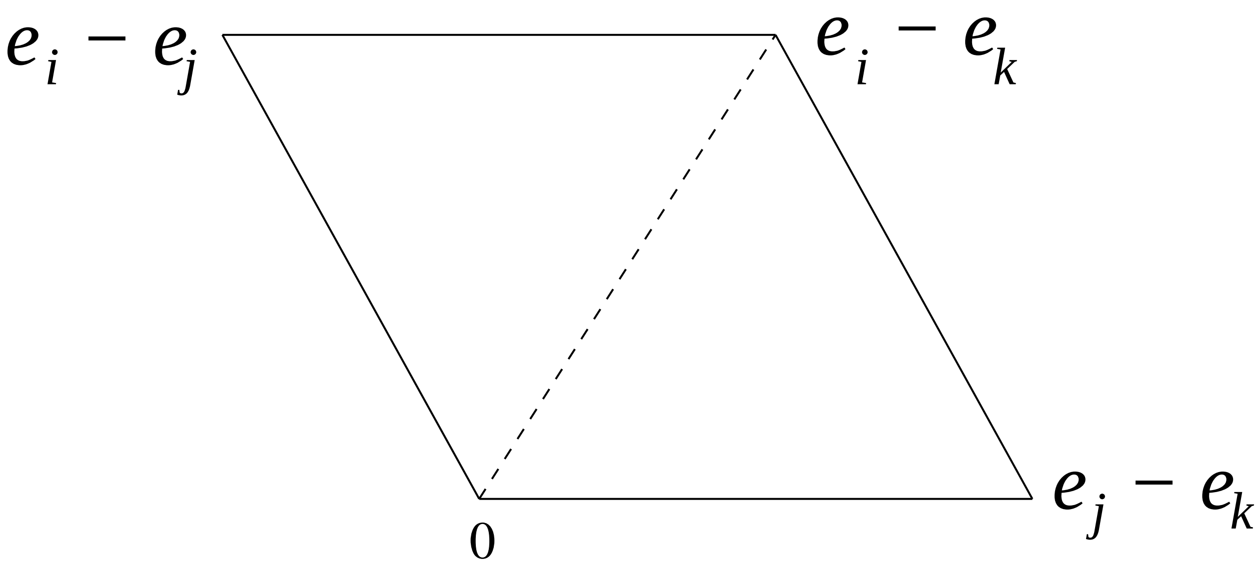

Reduced forms

A **reduced form** of the monomial m in $\mathcal{B}(A_n)$ or $\mathcal{S}(A_n)$ is a polynomial obtained from m by applying successive reductions until no longer possible.

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For $\mathcal{S}(A_n)$ and $\beta = 0$, same as reduction of Postinikov and RS.

A reduction redux

$$
\begin{aligned}
\boldsymbol{x_{12}}\boldsymbol{x_{23}}x_{34} &\longrightarrow \boldsymbol{x_{13}}x_{12}\boldsymbol{x_{34}} + \boldsymbol{x_{13}}x_{23}\boldsymbol{x_{34}} \\
&\longrightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\qquad + x_{14}x_{13}x_{23} + x_{14}\boldsymbol{x_{34}}\boldsymbol{x_{23}} \\
&\longrightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\qquad + x_{14}x_{13}x_{23} + x_{14}x_{24}x_{23} + x_{14}x_{24}x_{34} \\
&= P_3(x_{ij}).
\end{aligned}
$$

Reduced form of a graph monomial

G: graph on vertex set $[n+1]$

$$m_G = \prod_{ij \in E(G)} x_{ij} \in \mathcal{S}(A_n)$$

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Theorem. *Let T be a noncrossing tree on $[n+1]$ and P_T a reduced form of m_G. Then*

$$P_T(x_{ij} = 1, \beta = 0) = f_T,$$

the number of noncrossing alternating spanning trees of $\overline{T}$.

Relation to root polytopes

The monomials appearing in the reduced form $P_T(x_{ij}, \beta = 0)$ correspond to the facets in a triangulation of $\mathcal{P}(A_n)$.

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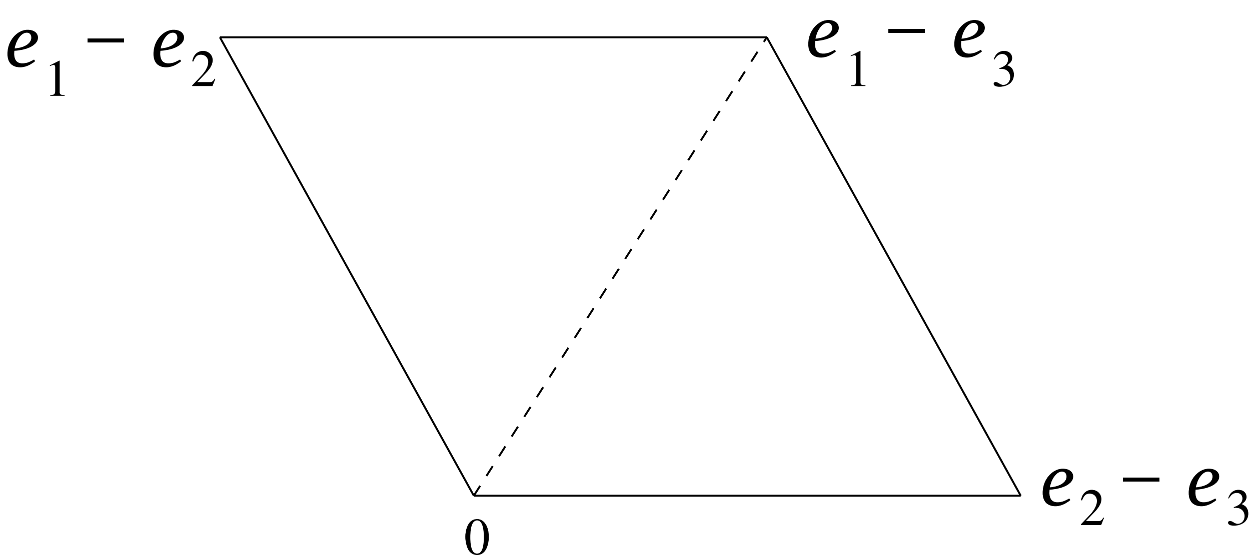
$$x_{12}x_{23} \longrightarrow x_{12}x_{13} + x_{23}x_{13}$$



Interior faces of $\mathcal{P}(A_n)$

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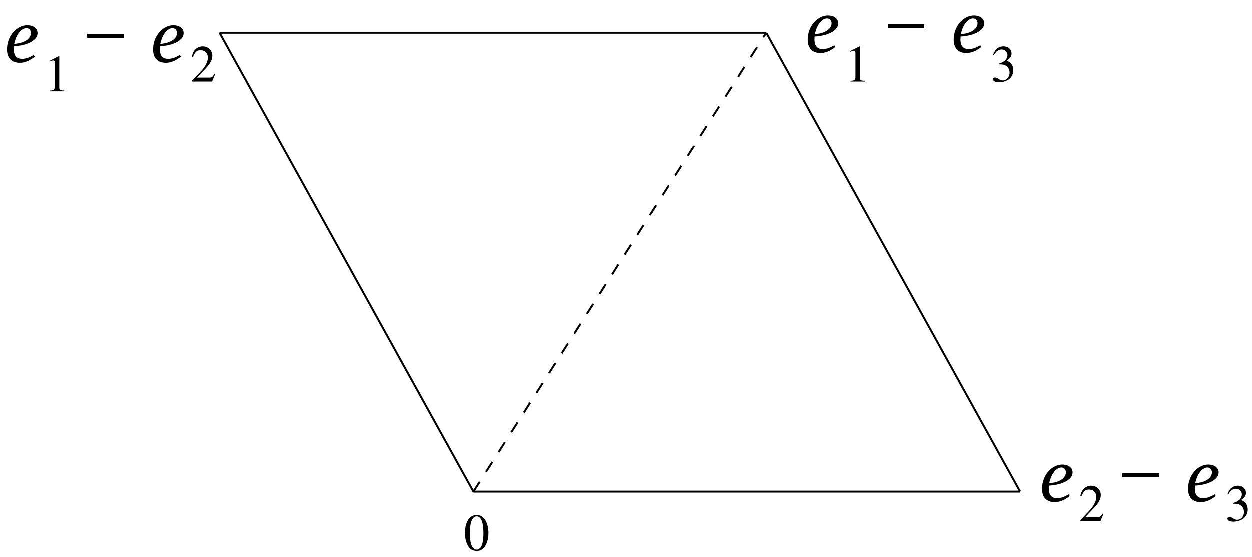
$$x_{12}x_{23} \to x_{12}x_{13} + x_{23}x_{13} + \beta x_{13}$$



Noncommutative version

In the ring $\mathcal{B}(A_n)$, the reduced form of any monomial m is **unique** (up to commutations).

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Many generalizations . . .

Matching polytopes

Ricky Liu, graduate student, M.I.T.

$G = (V, E)$: a graph; $n = \#E$

M_G: **matching polytope** of G, i.e.,

$$M_G = \left\{ w : E \to \mathbb{R}_{\geq 0} \,\middle|\, \forall\, v \in V \sum_{v \in e} w(e) \leq 1 \right\} \subseteq \mathbb{R}^n.$$

Vertices of M_G

matching M: a set of vertex-disjoint edges

If $L \subseteq E$, define $\chi_L \in M_G$ by

$$\chi_L(e) = \begin{cases} 1, & e \in L \\ 0, & e \notin L. \end{cases}$$

Note. M_G has integer vertices if and only if G is bipartite. In that case, the vertices are χ_M, where M is a matching of G.

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Note. M_G has integer vertices if and only if G is bipartite. In that case, the vertices are $\boldsymbol{\chi_M}$, where M is a matching of G.

Corollary. G bipartite $\Rightarrow$

$$\boldsymbol{V(G)} := n! \cdot \mathrm{vol}(M_G) \in \mathbb{Z}$$

G, G_1, G_2

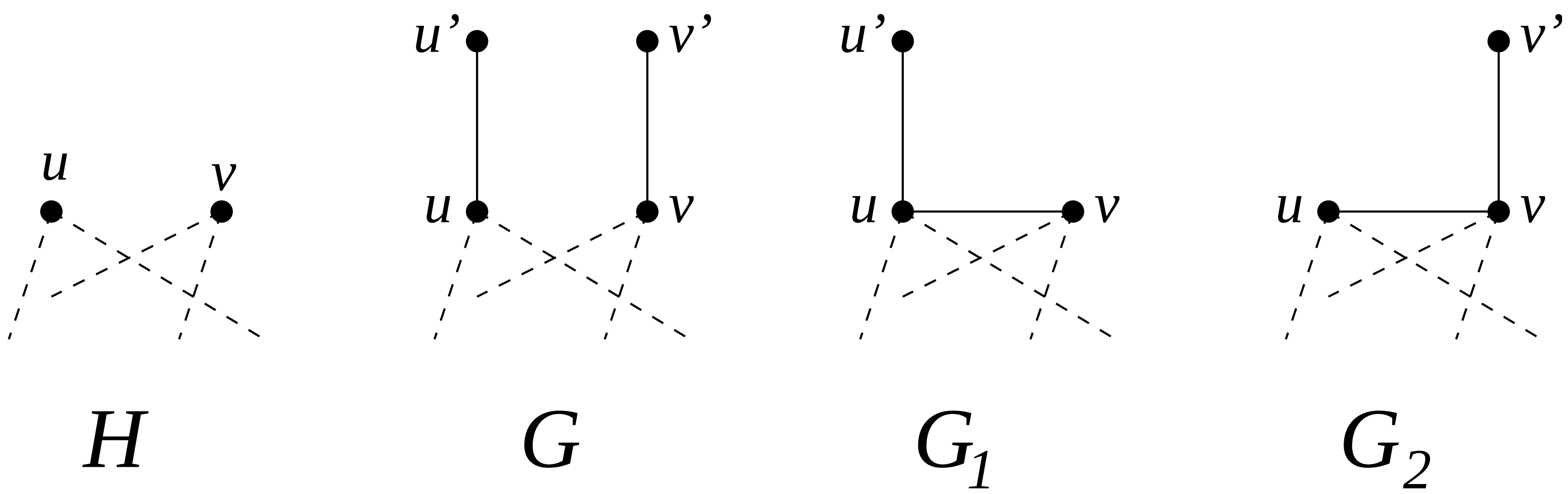
H = graph, $u, v \in V(H)$, $u \neq v$

G: adjoin pendant edges uu', vv' (so u', v' are endpoints)

G_1: adjoin pendant edge uu' and an edge uv

G_2: adjoin pendant edge vv' and an edge uv

Leaf recurrence



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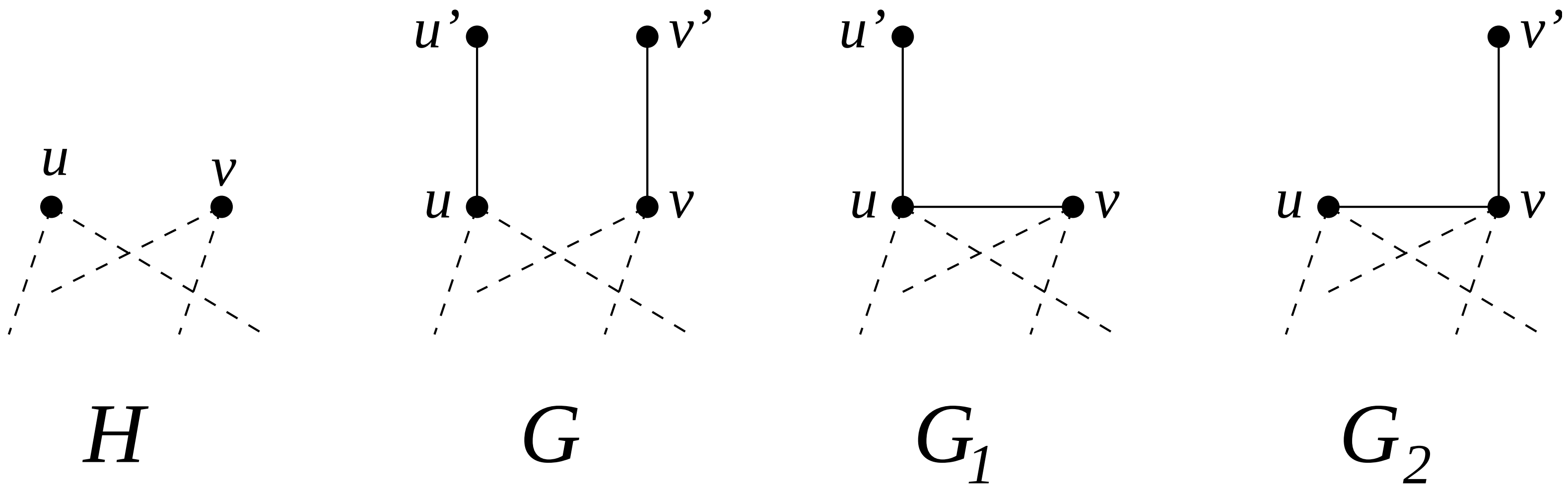

$\mathcal{F}$: set of all forests

$f : \mathcal{F} \to \mathbb{R}$ **satisfies the leaf recurrence** if

$$f(G) = f(G_1) + f(G_2).$$

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$$f(G_1 + G_2) = \binom{n}{m} f(G_1) f(G_2).$$

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- f satisfies the leaf recurrence.

Then $f(G) = V(G)$*.*

Volume of M_G (continued)

Theorem. *The previous theorem can be used to compute $V(F)$ for any forest F.*

Diagrams and tableaux

$\mathcal{B}$: the set of unit squares in $\mathbb{R}^2$ with centers (i, j), $i, j \geq 1$. Denote also by $\boldsymbol{(i, j)}$ the unit square with center (i, j).

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Diagram D: a finite subset of $\mathcal{B}$

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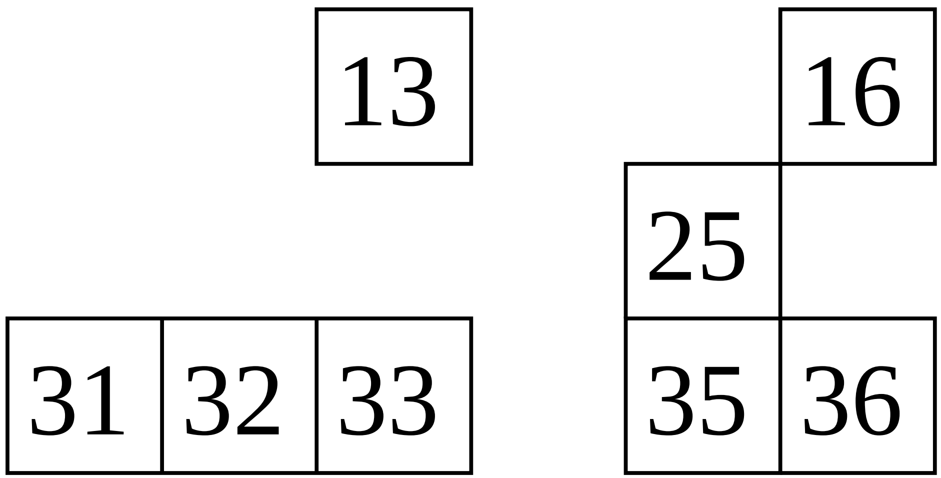

Row and column stabilizers

D: diagram with n boxes, ordered in some way

$\mathfrak{S}_n$ acts on D

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$\mathfrak{S}_n$ acts on D

$\boldsymbol{R_D}$ $(\boldsymbol{C_D})$: subgroup of $\mathfrak{S}_n$ stabilizing each row (column) of D

$$\boldsymbol{R(D)} = \sum_{w \in R_D} w, \qquad \boldsymbol{C(D)} = \sum_{w \in C_D} \mathrm{sgn}(w) w$$

The Specht module S^D

The **Specht module** S^D (over $\mathbb{C}$) is the left ideal

$$\boldsymbol{S^D} = \mathbb{C}[\mathfrak{S}_n]C(D)R(D)$$

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Note. S^D affords a representation of $\mathfrak{S}_n$ by left multiplication.

Irreducible Specht modules

Note. If D is the (Young) diagram of a partition λ of n, then S^D is irreducible. Conversely, if S^D is irreducible, then $S^D \cong S^{D'}$ for the diagram D' of some partition.

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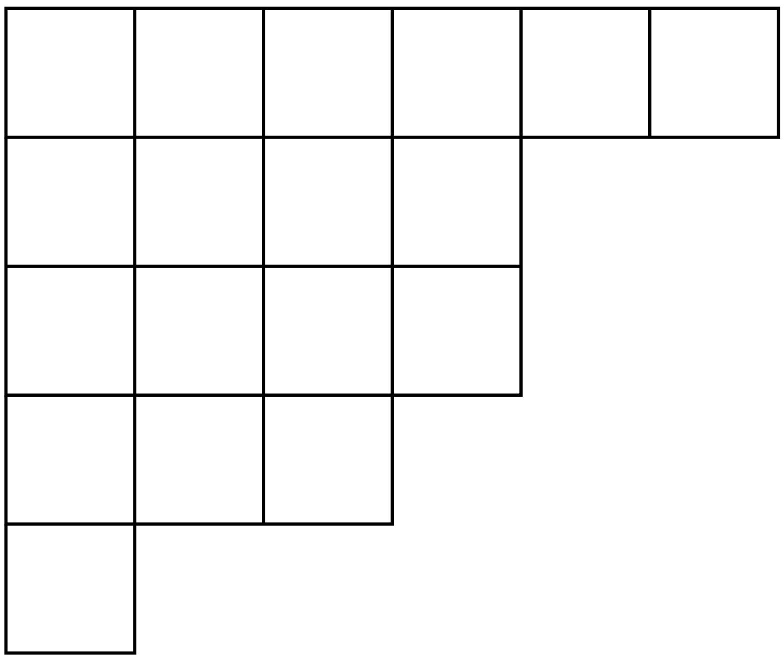

The diagram of a forest F

Let $V(F) = \boldsymbol{A} \cup \boldsymbol{B}$, so that all edges are between A and B. Label the A-vertices $1, \ldots, m$ and B-vertices $1, \ldots, n$. Let

$$\boldsymbol{D(F)} = \{(i,j)\ :\ ij \in E(F),\ i \in A,\ j \in B\}.$$

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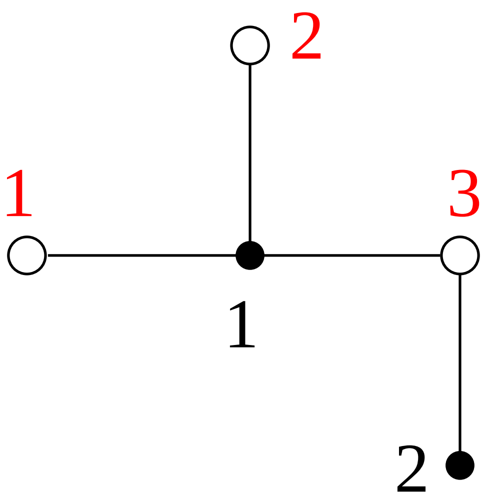

1 1	1 2	1 3
		2 3

The Specht module of $D(F)$

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Note for experts. The diagrams $D(F)$ are **not** %-avoiding diagrams in the sense of Reiner and Shimozono.

Decomposition of $S^{D(F)}$

How does the Specht module $S^{D(F)}$ decompose into irreducible representations of $\mathfrak{S}_n$?

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How does the Specht module $S^{D(F)}$ decompose into irreducible representations of $\mathfrak{S}_n$?

Recall the leaf recurrence

$$f(G) = f(G_1) + f(G_2)$$

with initial conditions $f(K_{n,1}) = 1$.

Decomposition theorem

Theorem. *For a forest F, $f(F)$ is well-defined, and*

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In other words, if

$$f(F) = \sum_{\lambda \vdash n} \boldsymbol{c_\lambda} s_\lambda,$$

where s_λ is a Schur function, then c_λ is the multiplicity of the irreducible representation of $\mathfrak{S}_n$ indexed by λ in $S^{D(F)}$.

The Ehrhart polynomial of M_F

Open. What is the Ehrhart polynomial of M_F? Does it have any representation-theoretic significance?