Two Poset Polytopes

Slides available at:

`www-math.mit.edu/~rstan/transparencies/polytopes.pdf`

The order polytope

P: p-element poset, say $P = \{t_1, \dots, t_p\}$

Definition. The **order polytope** $\mathcal{O}(P) \subset \mathbb{R}^p$ is defined by

$$\mathcal{O}(P) = \{(x_1, \dots, x_p) \in \mathbb{R}^p : 0 \le x_i \le 1,\ t_i \le_P t_j \Rightarrow x_i \le x_j\}.$$

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Example. (a) If P is an antichain, then $\mathcal{O}(P)$ is the p-dimensional unit cube $[0,1]^p$.

(b) If P is a chain $t_1 < \cdots < t_p$, then $\mathcal{O}(P)$ is the p-dimensional simplex $0 \le x_1 \le \cdots \le x_p \le 1$.

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$\dim \mathcal{O}(P) = p$, since if $t_{i_1}, \dots, t_{i_p}$ is a linear extension of P, then $\mathcal{O}(P)$ contains the p-dimensional simplex $0 \le x_{i_1} \le \cdots \le x_{i_p} \le 1$ (of volume $1/p!$).

Convex polytope

convex hull $\mathbf{conv}(X)$ of a subset X of $\mathbb{R}^p$: intersection of all convex sets containing X

half-space in $\mathbb{R}^p$: a subset H of $\mathbb{R}^p$ of the form $\{x \in \mathbb{R}^p : x \cdot v \leq \alpha\}$ for some fixed $0 \neq v \in \mathbb{R}^p$ and $\alpha \in \mathbb{R}$.

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Such a set $\mathcal{P}$ is a **convex polytope**.

Vertices

A **vertex** v of a convex polytope (or even convex set) $\mathcal{P}$ is a point in $\mathcal{P}$ such that

$$v = \lambda x + (1 - \lambda) y, \;\; x, y \in \mathcal{P}, \;\; 0 \leq \lambda \leq 1 \Rightarrow x = y.$$

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Equivalently, v is a vertex of $\mathcal{P}$ if and only if there exists a half-space H such that $\{v\} = \mathcal{P} \cap H$.

Order ideals

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Let $S \subseteq P = \{t_1, \ldots, t_p\}$. The **characteristic vector** χ_S of S is defined by $\chi_S = (x_1, x_2, \ldots, x_p)$ such that

$$x_i = \begin{cases} 0, & t_i \notin S \\ 1, & t_i \in S. \end{cases}$$

Write $\overline{\chi_S} = (1 - x_1, 1 - x_2, \ldots, 1 - x_p)$.

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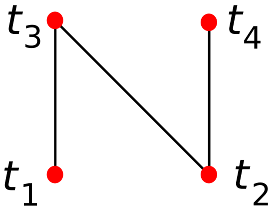
Write $\overline{\chi_S} = (1 - x_1, 1 - x_2, \ldots, 1 - x_p)$.

Theorem. *The vertices of $\mathcal{O}(P)$ are the sets $\overline{\chi}_I$, where I is an order ideal of P.*

An example



(1,1,1,1)
(0,1,1,1)
(1,0,1,1)
(0,0,1,1)
(1,0,1,0)
(0,0,0,1)
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Proof. Clearly $\overline{\chi}_I \in \mathcal{O}(P)$, and $\overline{\chi}_I$ is a vertex (e.g., since it is a vertex of the binary unit cube containing $\mathcal{O}(P)$). Also, any binary vector in $\mathcal{O}(P)$ has the form $\overline{\chi}_I$.

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Assume $v \in \mathcal{O}(P)$ is a vertex and $v \neq \overline{\chi}_I$ for some I.

Idea: show $v = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$ and $x, y \in \mathcal{O}(P)$.

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Let $v = (v_1, \ldots, v_p)$. Choose v_i so $0 < v_i < 1$ (exists since the only binary vectors in $\mathcal{O}(P)$ have the form $\overline{\chi}_I$). Choose $\epsilon > 0$ sufficiently small. Let $\boldsymbol{v^-}$ (respectively, $\boldsymbol{v^+}$) be obtained from v by subtracting (respectively, adding) ϵ to each entry equal to v_i. Then $v^-, v^+ \in \mathcal{O}(P)$ and

$$v = \frac{1}{2}(v^- + v^+). \quad \square$$

Two remarks

Note. Can also prove the previous theorem by showing directly that every $v \in \mathcal{O}(P)$ is a convex combination of the $\overline{\chi}_I$'s (not difficult).

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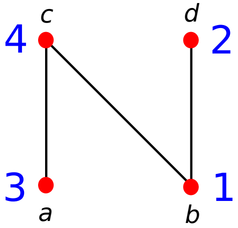
Note. Entire facial structure of $\mathcal{O}(P)$ can be described, but omitted here.

Linear extensions

$\mathcal{L}(P)$: set of linear extensions $\sigma\colon P \to \{1, \ldots, p\}$, i.e., σ is bijective and order-preserving.

$e(P) \coloneqq \#\mathcal{L}(P)$

An example (from before)



$a\ b\ c\ d$
$b\ a\ c\ d$
$a\ b\ d\ c$
$b\ a\ d\ c$
$b\ d\ a\ c$

Decomposition of $\mathcal{O}(P)$

Let $\sigma \in \mathcal{L}(P)$, regarded as the permutation $t_{i_1}, t_{i_2}, \ldots, t_{i_p}$ of P (so $\sigma(t_{i_j}) = j$). Define

$$\mathcal{O}(\sigma) = \{(x_1, \ldots, x_n) \in \mathbb{R}^p : 0 \le x_{i_1} \le x_{i_2} \le \cdots \le x_{i_p} \le 1\},$$

a **simplex** of volume $1/p!$.

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Corollary. *The volume* $\mathrm{vol}(\mathcal{O}(P))$ *of* $\mathcal{O}(P)$ *is equal to* $e(P)/p!$.

Proof of Corollary. $\mathcal{O}(P)$ is the union of $e(p)$ simplices $\mathcal{O}(\sigma)$ with disjoint interiors and volume $1/p!$ each. $\square$

Proof of decomposition theorem

The interior of the simplex $0 \leq x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_p} \leq 1$ is given by $0 < x_{i_1} < x_{i_2} < \cdots < x_{i_p} < 1$. Thus the interiors of the $\mathcal{O}(\sigma)$'s are disjoint, since a set of distinct real numbers has a unique ordering with respect to $<$. And clearly $\mathcal{O}(\sigma) \subseteq \mathcal{O}(P)$.

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Now suppose that $(x_1, \ldots, x_p) \in \mathcal{O}(P)$. Define $i_1, \ldots, i_p$ (not necessarily unique) by

$$x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_p}$$

and

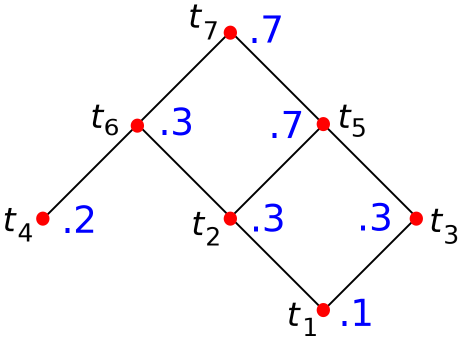
$$t_{i_j} < t_{i_k} \text{ and } x_{i_j} = x_{i_k} \Rightarrow j < k.$$

Then $t_{i_1}, t_{i_2}, \ldots, t_{i_p}$ is a linear extension σ of P, and $(x_1, \ldots, x_p) \in \mathcal{O}(P)$. $\square$

An aside

Note. This decomposition of $\mathcal{O}(P)$ is actually a **triangulation**, i.e., the intersection of any two of the simplices is a common face (possibly empty) of both. (Easy to see.)

An example



Three "compatible" linear extensions:

$$t_1,\ t_4,\ \boldsymbol{t_2},\ \boldsymbol{t_3},\ \boldsymbol{t_6},\ t_5,\ t_7$$

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Why volume $1/p!$?

The simplices $0 \leq x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_p} \leq 1$ all have the same volume (for any permutation $i_1, \ldots, i_p$ of $1, \ldots, p$), since they differ only by a permutation of coordinates.

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Let P be a p-element **antichain**. Thus $\mathcal{L}(P) = \mathfrak{S}_p$ (all permutations of $1, \ldots, p$). Moreover, $\mathcal{O}(P)$ is a unit cube so has volume 1. By the decomposition theorem, it is a union of $p!$ simplices $0 \le x_{i_1} \le \cdots \le x_{i_p} \le 1$, all with the same volume. Thus the volume of each simplex is $1/p!$.

Ehrhart theory

$\mathcal{P}$: a d-dimensional convex polytope in $\mathbb{R}^p$ with integer vertices.

$$\boldsymbol{n}\mathcal{P} = \{nx : x \in \mathcal{P}\}, \quad n \geq 1$$

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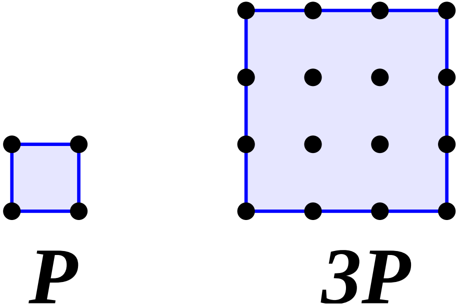
For $n \geq 1$, define

$$\begin{aligned} i(\mathcal{P}, n) &= \#\left(n\mathcal{P} \bigcap \mathbb{Z}^p\right) \\ \bar{i}(\mathcal{P}, n) &= \#\left(n\,\mathrm{int}(\mathcal{P}) \bigcap \mathbb{Z}^p\right). \end{aligned}$$

$i(\mathcal{P}, n)$ is the **Ehrhart polynomial** of $\mathcal{P}$.

An example



$$
\begin{aligned}
i(\mathcal{P}, n) &= (n+1)^2 \\
\bar{i}(\mathcal{P}, n) &= (n-1)^2 = i(\mathcal{P}, -n)
\end{aligned}
$$

Main results of Ehrhart theory

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Proofs: see e.g. EC1, §4.6.2, or Beck-Robins, *Computing the Continuous Discretely*.

The order polynomial

P: p-element poset

For $n \geq 1$, define the **order polynomial** $\Omega_P(n)$ of P by

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$$\begin{aligned}
\Omega_P(1) &= 1 \\
\Omega_P(2) &= \#\text{ order ideals of } P \\
&= \#\text{ vertices of } \mathcal{O}(P) \\
\Omega(p\text{-chain}, n) &= \left(\!\!\binom{n}{p}\!\!\right) = \binom{n+p-1}{p} \\
\Omega(p\text{-antichain}, n) &= n^p
\end{aligned}$$

Strict order polynomial

For $n \geq 1$, define the **strict** order polynomial $\overline{\Omega}_P(n)$ of P by

$$\overline{\Omega}_P(n) = \#\left\{f\colon P \to \{1,\ldots,n\} \;\middle|\; s <_P t \Rightarrow f(s) <_{\mathbb{Z}} f(t)\right\}.$$

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$$\begin{aligned}
\overline{\Omega}_P(1) &= 0 \text{ unless } P \text{ is an antichain} \\
\Omega(p\text{-chain}, n) &= \binom{n}{p} \\
\Omega(p\text{-antichain}, n) &= n^p
\end{aligned}$$

Polynomiality

Theorem. *$\Omega_P(n)$ is a polynomial in n of degree p and leading coefficient $e(P)/p!$. (Thus $\Omega_P(n)$ determines $e(P)$.)*

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To obtain $f : P \to \{1, \ldots, n\}$ order-preserving, choose $1 \leq s \leq p$, then choose an s-element subset $\boldsymbol{S}$ of $\{1, \ldots, n\}$ is $\binom{n}{s}$ ways, and finally choose a surjective order-preserving map $P \to S$ in e_s ways.

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$$\Rightarrow \Omega_P(n) = \sum_{s=1}^{p} e_s \binom{n}{s}.$$

Now $\binom{n}{s}$ is a polynomial in n of degree s and leading coefficient $1/s!$. Moreover, $e_p = e(P)$ (clear). The proof follows. $\square$

Polynomiality (cont.)

Similarly:

Theorem. *$\overline{\Omega}_P(n)$ is a polynomial in n of degree p and leading coefficient $e(P)/p!$.*

Ehrhart polynomial of $\mathcal{O}(P)$

integer points in $n\mathcal{O}(P)$: integer solutions $(a_1, \ldots, a_p)$ to

$$0 \le x_i \le n, \quad t_i \le_P t_j \Rightarrow x_i \le_{\mathbb{R}} x_j$$

Define $f(t_i) = a_i$. Thus

$$f : P \to \{0, 1, \ldots, n\}, \quad t_i \le_P t_j \Rightarrow a_i \le_{\mathbb{Z}} a_j,$$

so $\boxed{i(\mathcal{O}(P), n) = \Omega_P(n+1)}$ (since $\#\{0, 1, \ldots, n\} = n+1$).

Order polynomial reciprocity

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points $(x_1, \ldots, x_p)$ in $n \operatorname{int}(\mathcal{O}(P))$:

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Thus $\bar{i}(\mathcal{O}(P), n)$ is the number of **strict** order-preserving maps $P \to \{1, \ldots, n-1\}$. Since $\Omega_P(n) = i(\mathcal{O}(P), n-1)$ and $\bar{i}(\mathcal{O}(P), n) = (-1)^p i(\mathcal{O}(P), -n)$, we get:

Corollary (reciprocity for order polynomials).
$\overline{\Omega}_P(n) = (-1)^p \Omega(-n)$.

Simple application

Corollary. *Let ℓ be the length (one less than the number of elements) of the longest chain of P. Then*

$$\Omega_P(0) = \Omega_P(-1) = \cdots = \Omega_P(-\ell) = 0.$$

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$$\Omega_P(0) = \Omega_P(-1) = \cdots = \Omega_P(-\ell) = 0.$$

Proof. If $s_0 < s_1 < \cdots < s_m$ is a chain of length m and $f \colon P \to \{1, \ldots, n\}$ is strictly order-preserving, then

$$f(s_0) < f(s_1) < \cdots < f(s_m).$$

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Many other interesting results on order polynomials, but no time here!

Mixed volumes

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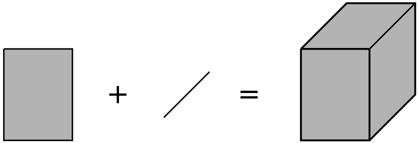
Let $\alpha, \beta \geq 0$. The **Minkowski sum** $\alpha K + \beta L$ of two convex bodies K, L is given by

$$\alpha K + \beta L = \{\alpha x + \beta y \,:\, x \in K,\ y \in L\}.$$

An example



(up to translation)

Minkowski's theorem

$\boldsymbol{K, L}$: convex bodies in $\mathbb{R}^p$

$\boldsymbol{\alpha, \beta} \geq 0$

Theorem (**Minkowski**). *We have*

$$\mathrm{vol}(\alpha K + \beta L) = \sum_{i=0}^{p} \binom{p}{i} V_i(K, L) \alpha^{p-i} \beta^i,$$

for certain real numbers $V_i(K, L) \geq 0$, *called the* **mixed volumes** *of* K *and* L.

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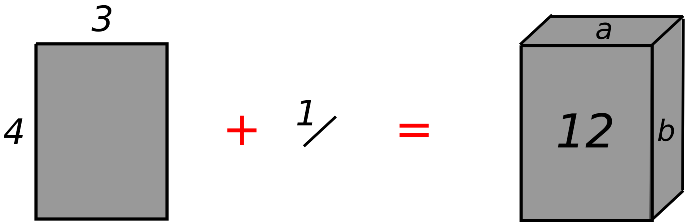
Note. $V_0(K, L) = \mathrm{vol}(K)$ (set $\alpha = 1, \beta = 0$) and $V_p(K, L) = \mathrm{vol}(L)$ (set $\alpha = 0, \beta = 1$).

An example



$$a = 3/\sqrt{2}, \quad b = 4/\sqrt{2}$$

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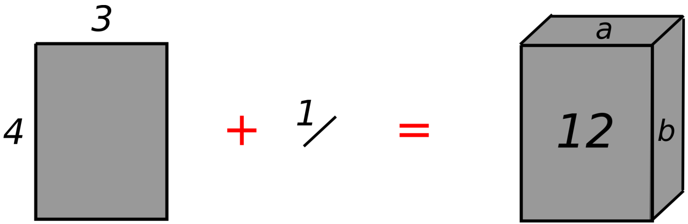

$$a = 3/\sqrt{2}, \quad b = 4/\sqrt{2}$$

$$\mathrm{vol}(\alpha K + \beta L) = 12\alpha^2 + 2\frac{7}{2\sqrt{2}}\alpha\beta$$

$$\Rightarrow V_0(K,L) = 12, \quad V_1(K,L) = \frac{7}{2\sqrt{2}}, \quad V_2(K,L) = 0$$

Unimodality and log-concavity

$a_0, a_1, \dots, a_n$: sequence of **nonnegative** real numbers

unimodal: $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$ for some j

log-concave: $a_i^2 \ge a_{i-1}a_{i+1},\ 1 \le i \le n-1$

no internal zeros: $i < j < k,\ a_i \ne 0, a_k \ne 0 \Rightarrow a_j \ne 0$

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Note. Log-concave and no internal zeros $\Rightarrow$ unimodal.

Proof. Suppose $a_0, a_1, \ldots, a_n$ is not unimodal. Then for some $i < j-1$, we have

$$a_i > a_{i+1} = a_{i+2} = \cdots = a_{j-1} < a_j.$$

If no internal zeros, then $0 = a_{i+1} > 0$. Then $a_{i+1}^2 < a_i a_{i+2}$, so the sequence is not log-concave. $\square$

Height of an element in a linear extension

Let $t \in P$.

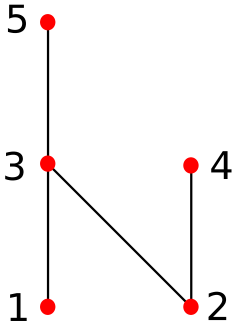
$\sigma = s_1, \ldots, s_p$: linear extension of P

height of t in σ is k, if $s_k = t$. Denoted $\mathrm{ht}_\sigma(t)$.

$N_i = N_i(t)$: number of linear extensions σ of P for which $\mathrm{ht}_\sigma(t) = i$. In other words

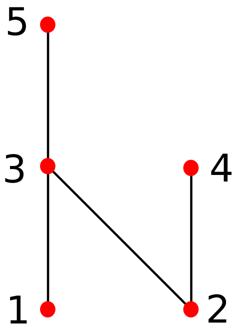
$$N_i = \#\{\sigma \in \mathcal{L}(P) : \sigma(t) = i\}.$$

An example



σ	$\mathrm{ht}(4)$
1 2 3 4 5	4
2 1 3 4 5	4
1 2 4 3 5	3
2 1 4 3 5	3
2 4 1 3 5	2
1 2 3 5 4	5
2 1 3 5 4	5

An example



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1 2 3 4 5	4
2 1 3 4 5	4
1 2 4 3 5	3
2 1 4 3 5	3
2 4 1 3 5	2
1 2 3 5 4	5
2 1 3 5 4	5

$\Rightarrow (N_1, N_2, N_3, N_4, N_5) = (0, 1, 2, 2, 2)$ (log-concave)

The Aleksandrov-Fenchel inequalities

Theorem (**A. D. Aleksandrov**, 1937–38, and **W. Fenchel**, 1936).
For any convex bodies $K, L, \subset \mathbb{R}^p$, we have

$$V_i(K, L)^2 \geq V_{i-1}(K, L) V_{i+1}(K, L), \quad 1 \leq i \leq p-1.$$

. Moreover, the sequence $V_0, V_1, \ldots, V_p$ has no internal zeros.

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Proof is difficult.

Chung-Fishburn-Graham conjecture

Theorem (conjecture of **Fan Chung**, **Peter Fishburn**, and **Ron Graham**, 1980) *For any p-element poset P and $t \in P$, we have*

$$N_i(t)^2 \geq N_{i-1}(t) N_{i+1}(t), \quad 2 \leq i \leq p-1.$$

Moreover, the sequence $N_1, \ldots, N_p$ has no internal zeros (so it is unimodal).

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Moreover, the sequence $N_1, \ldots, N_p$ has no internal zeros (so it is unimodal).

Plan of proof. **No internal zeros**: easy combinatorial argument.

Log-concavity: find convex bodies (actually, polytopes) in $\mathbb{R}^{p-1}$ such that $N_i = V_i(K, L)$ (up to multiplication by a positive constant). (Proof by wishful thinking)

What are K and L?

Let $P = \{t, t_1, t_2, \ldots, t_{p-1}\}$.

$$\boldsymbol{K} = \{(x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1} : 0 \leq x_i \leq 1,\ x_i \leq x_j \text{ if } t_i \leq t_j,\ x_i = 0 \text{ if } t_i < t\}$$

$$\boldsymbol{L} = \{(x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1} : 0 \leq x_i \leq 1,\ x_i \leq x_j \text{ if } t_i \leq t_j,\ x_i = 1 \text{ if } t_i > t\}$$

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Note that $K, L \subset \mathcal{O}(P - t)$.

Claim. $V_i(K, L) = \dfrac{N_{i+1}(t)}{(p-1)!}$

What is $\alpha K + \beta L$?

$$\alpha K + \beta L = \text{set of all } (x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1} \text{ such that :}$$

- $t_i \leq_{P-t} t_j \Rightarrow 0 \leq x_i \leq x_j \leq \alpha + \beta$
- $t_i <_P t \Rightarrow x_i \leq \beta$
- $t_i >_P t \Rightarrow x_i \geq \beta$

Proof that $V_i(K, L) = N_{i+1}(t)$

Recall $P = \{t_1, \ldots, t_{p-1}, t\}$. For $\alpha, \beta \geq 0$ let $\mathcal{P} = \alpha K + \beta L$. For each linear extension $\sigma\colon P \to \{1, \ldots, p\}$, define

$$\Delta_\sigma = \{(x_1, \ldots, x_{p-1}) \in \mathcal{P} :$$

$$\begin{aligned} x_i \leq x_j \quad & \text{if} \quad \sigma(t_i) \leq \sigma(t_j) \\ x_i \leq \beta \quad & \text{if} \quad \sigma(t_i) < \sigma(t) \\ \beta \leq x_i \leq \alpha + \beta \quad & \text{if} \quad \sigma(t_i) > \sigma(t). \end{aligned}$$

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Easy to check (completely analogous the proof that $\mathcal{O}(P)$ is a union of simplices $\mathcal{O}(\sigma)$): Δ_σ's, for $\sigma \in \mathcal{L}(P)$, have disjoint interiors and union $\mathcal{P}$.

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Note. Δ_σ need not be a simplex because σ is a linear extension of P, not $P - t$.

Proof (cont.)

Let $\sigma(t) = i$, and define $\boldsymbol{w} \in \mathfrak{S}_{p-1}$ by

$$\sigma(t_{w(1)}) < \sigma(t_{w(2)}) < \cdots < \sigma(t_{w(p-1)}).$$

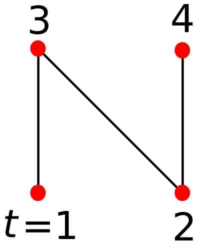
Then Δ_σ consists of all $(x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1}$ satisfying

$$0 \leq x_{w(1)} \leq \cdots \leq x_{w(i-1)} \leq \beta \leq x_{w(i)} \leq \cdots \leq x_{w(p-1)} \leq \alpha + \beta.$$

This is a product of two simplices with volume

$$\mathrm{vol}(\Delta_\sigma) = \frac{\alpha^{p-i}}{(p-i)!} \frac{\beta^{i-1}}{(i-1)!}.$$

An example



1234	$\beta \le x_2 \le x_3 \le x_4 \le \alpha + \beta$
2134	$0 \le x_2 \le \beta \le x_3 \le x_4 \le \alpha + \beta$
1243	$\beta \le x_2 \le x_4 \le x_3 \le \alpha + \beta$
2143	$0 \le x_2 \le \beta \le x_4 \le x_3 \le \alpha + \beta$
2413	$0 \le x_2 \le x_4 \le \beta \le x_3 \le \alpha + \beta$

Conclusion of proof

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$$\begin{aligned}
\Rightarrow \mathrm{vol}(\mathcal{P}) &= \sum_{\sigma\in\mathcal{L}(P)} \mathrm{vol}(\Delta_\sigma) \\
&= \sum_{i=1}^{p} N_i(t)\frac{\alpha^{p-i}}{(p-i)!}\frac{\beta^{i-1}}{(i-1)!} \\
&= \frac{1}{(p-1)!}\sum_{i=0}^{p-1} N_{i+1}(t)\binom{p-1}{i}\alpha^{p-1-i}\beta^{i},
\end{aligned}$$

so $V_i(K,L) = \dfrac{N_{i+1}(t)}{(p-1)!}$. $\square$

The chain polytope

$P = \{t_1, \dots, t_p\}$ as before

Definition. The **chain polytope** $\mathcal{C}(P) \subset \mathbb{R}^p$ is defined by

$$\mathcal{C}(P) = \{(x_1, \dots, x_p) \in \mathbb{R}^p : 0 \le x_i, \ \sum_{t_i \in C} x_i \le 1 \text{ for every chain (or maximal chain) } C \text{ of } P\}.$$

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Example. (a) If P is an antichain, then $\mathcal{C}(P) = \mathcal{O}(P)$, the p-dimensional unit cube $[0,1]^p$.

(b) If P is a chain $t_1 < \cdots < t_p$, then $\mathcal{C}(P)$ is the p-dimensional simplex $x_i \geq 0$, $x_1 + \cdots + x_p \leq 1$.

Two notes

Note. If P is a chain $t_1 < \cdots < t_p$, then define $\phi\colon \mathcal{O}(P) \to \mathcal{C}(P)$ by $\phi(x_1, \ldots, x_p) = (x_1, x_2 - x_1, \ldots, x_p - x_{p-1})$. Then ϕ is a linear isomorphism from $\mathcal{O}(P)$ to $\mathcal{C}(P)$. It preserves volume since the linear transformation has determinant 1 (lower triangular with 1's on the diagonal).

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Note. $\dim \mathcal{C}(P) = p$, since the cube $[0, 1/p]^p$ is contained in $\mathcal{C}(P)$.

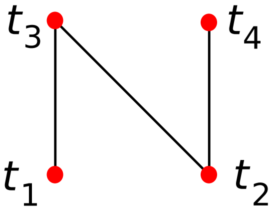
Vertices of $\mathcal{C}(P)$

Recall that if $S \subseteq P$, then

$$\chi_S = \left\{ (x_1, \ldots, x_p) \; : \; x_i = \begin{cases} 0, & t_i \notin S \\ 1, & t_i \in S. \end{cases} \right\}$$

Theorem. *The vertices of $\mathcal{C}(P)$ are the sets χ_A, where A is an antichain of P.*

An example



(0,0,0,0)
(1,0,0,0)
(0,1,0,0)
(0,0,1,0)
(0,0,0,1)
(1,1,0,0)
(1,0,0,1)
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Proof. Clearly $\chi_A \in \mathcal{O}(P)$, and χ_A is a vertex (e.g., since it is a vertex of the binary unit cube containing $\mathcal{C}(P)$). Also, any binary vector in $\mathcal{C}(P)$ has the form χ_A.

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Assume $v \in \mathcal{C}(P)$ is a vertex and $v \neq \chi_A$ for some A.

Idea: show $v = \lambda x + (1 - \lambda)y$ for some $0 < \lambda < 1$ and $x, y \in \mathcal{C}(P)$.

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Let $v = (v_1, \dots, v_p)$, $v \neq \chi_A$. Let $Q = \{t_i \in P : 0 < v_i < 1\}$. Let Q_1 be the set of minimal elements of Q and Q_2 the set of minimal elements of $Q - Q_1$.

Easy: if v is a vertex then $Q_1, Q_2 \neq \varnothing$.

Proof (cont.)

Define
$$\varepsilon = \min\{v_i, 1 - v_i \ : \ t_i \in Q_1 \cup Q_2\}.$$
Define $y = (y_1, \dots, y_p)$, $z = (z_1, \dots, z_p) \in \mathbb{R}^p$ by

$$y_i = \begin{cases} v_i, & t_i \notin Q_1 \cup Q_2 \\ v_i + \varepsilon, & t_i \in Q_1 \\ v_i - \varepsilon, & t_i \in Q_2, \end{cases}$$

$$z_i = \begin{cases} v_i, & t_i \notin Q_1 \cup Q_2 \\ v_i - \varepsilon, & t_i \in Q_1 \\ v_i + \varepsilon, & t_i \in Q_2, \end{cases}$$

Easy to see $y, z \in \mathcal{C}(P)$. Since $y \neq z$ and $v = \frac{1}{2}(y + z)$, it follows that v is not a vertex of $\mathcal{C}(P)$. $\square$

Order ideals and antichains

Recall: vertices of $\mathcal{O}(P)$: $\overline{\chi}_I$, where I is an order ideal

vertices of $\mathcal{C}(P)$: χ_A, where A is an antichain

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$$\begin{aligned} f(I) &= \{\text{maximal elements of } I\} \\ f^{-1}(A) &= \{s \in P : s \leq t \text{ for some } t \in A\} \\ &= \bigcap_{\substack{\text{order ideals } J \\ A \subseteq J}} J. \end{aligned}$$

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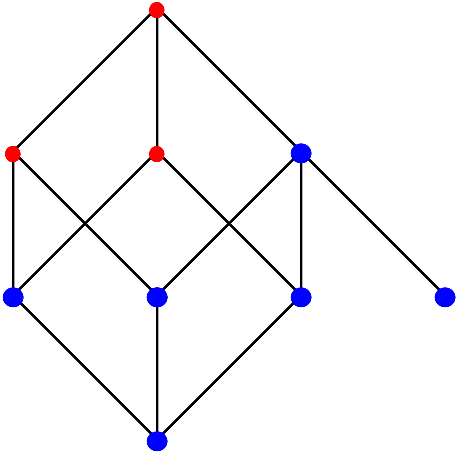
Thus there is a (canonical) bijection $V(\mathcal{O}(P)) \to V(\mathcal{C}(P))$. What about other faces?

Example of order ideal ↔ antichain bijection



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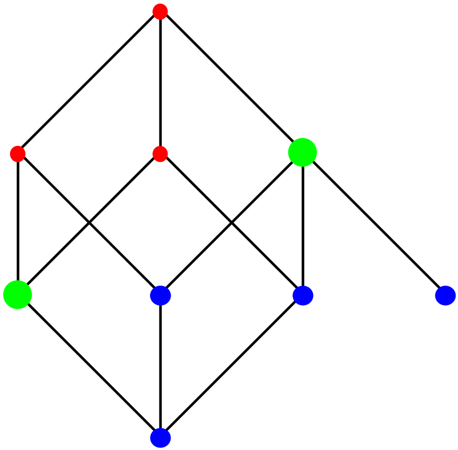
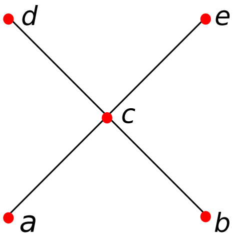


The X-poset



$\mathcal{O}(P)$: $0 \le a, b$ (2 equations), $d, e \le 1$ (2 equations), $a \le c$, $b \le c$, $c \le d$, $c \le e$, so **8** facets (maximal faces)

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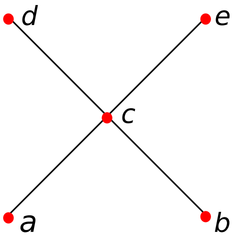

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Hence $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are not **combinatorially equivalent** (i.e., they don't have isomorphic face posets)

Length two posets

Exercise. If P has no 3-element chain, then there is a bijective linear transformation $\tau: \mathcal{O}(P) \to \mathcal{C}(P)$ of determinant 1. Thus $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are combinatorial equivalent and have the same Ehrhart polynomial (and hence the same volume).

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Note. For the characterization of P for which there is a bijective linear transformation $\tau\colon \mathcal{O}(P) \to \mathcal{C}(P)$ of determinant 1, see Hibi and Li, `arXiv:1208.4029`.

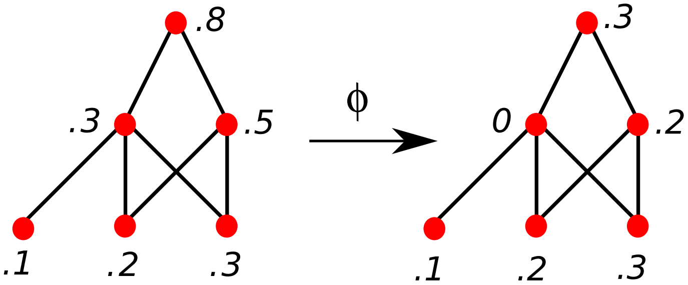
The transfer map

Definition. Define the **transfer map** $\phi : \mathcal{O}(P) \to \mathcal{C}(P)$ as follows: if $x = (x_1, \ldots, x_p) \in \mathcal{O}(P)$ and $t_i \in P$, then $\phi(x) = (y_1, \ldots, y_p)$, where

$$\boldsymbol{y_i} = \min\{x_i - x_j \ : \ t_i \text{ covers } t_j \text{ in } P\}.$$

(If t_i is a minimal element of P, then $y_i = x_i$.)

An example



Transfer theorem

Theorem. (a) *The transfer map* ϕ *is a continuous, piecewise-linear bijection from* $\mathcal{O}(P)$ *onto* $\mathcal{C}(P)$.

Note. **Piecewise-linear** means that we can express $\mathcal{O}(P)$ as a finite union $\mathcal{O}(P) = X_1 \cup \cdots \cup X_k$ such that ϕ restricted to each X_i is linear.

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(b) *Let $n \in \mathbb{P} = \{1, 2, \ldots\}$ and $x \in \mathcal{O}(P)$. Then $nx \in \mathbb{Z}^p$ if and only if $n\phi(x) \in \mathbb{Z}^p$.*

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Corollary. $\mathrm{vol}(\mathcal{C}(P)) = \mathrm{vol}(\mathcal{O}(P)) = e(P)$ *and* $i(\mathcal{C}(P), n) = i(\mathcal{O}(P), n) = \Omega_P(n+1)$.

Transfer theorem proof.

(a) *The transfer map ϕ is a continuous, piecewise-linear bijection from $\mathcal{O}(P)$ onto $\mathcal{C}(P)$.*

Proof. Continuity is immediate from the definition. Recall the decomposition $\mathcal{O}(P) = \bigcup_{\sigma \in \mathcal{L}(P)} \mathcal{O}(\sigma)$, where $\sigma = (t_{i_1}, \ldots, t_{i_p})$ and

$$\mathcal{O}(\sigma) = \{(x_1, \ldots, x_n) \in \mathbb{R}^p : 0 \le x_{i_1} \le x_{i_2} \le \cdots \le x_{i_p} \le 1\}.$$

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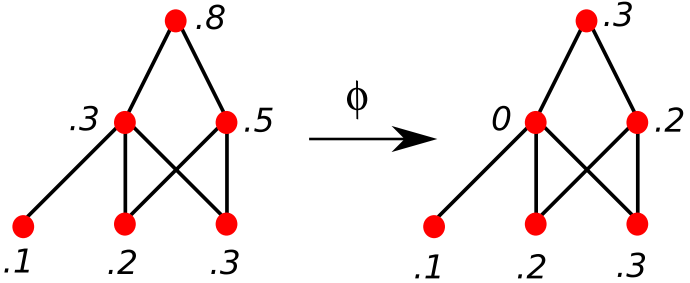
Define $\psi : \mathcal{C}(P) \to \mathcal{O}(P)$ by $\psi(y_1, \ldots, y_p) = (x_1, \ldots, x_p)$, where

$$x_j = \max\{y_{i_1} + y_{i_2} + \cdots + y_{i_k} : t_{i_1} < t_{i_2} < \cdots < t_{i_k} = t_j\}.$$

Easy to check: $\psi\phi(x) = x$ and $\phi\psi(y) = y$ for all $x \in \mathcal{O}(P)$ and $y \in \mathcal{C}(P)$. Hence ϕ is a bijection with inverse ψ.

An example (redux)



Conclusion of proof

Remains to show that $nx \in \mathbb{Z}^p$ if and only if $n\phi(x) \in \mathbb{Z}^p$. This is clear from the formulas for $\phi(x)$ and $\psi(y)$.

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Alternatively, the restriction of ϕ to each $\mathcal{O}(\sigma)$ belongs to $\mathrm{SL}(p, \mathbb{Z})$, i.e., the matrix of of ϕ with respect to the standard basis of $\mathbb{R}^p$ is integral of determinant 1. (In fact, it's lower triangular with 1's on the diagonal.)

Interesting corollary

$\Delta(P)$: set of chains of P (**order complex**)

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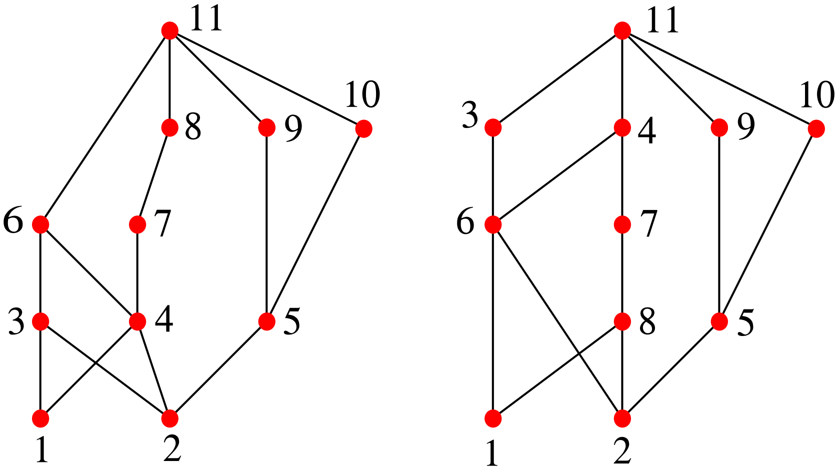
Proof. The set $\mathcal{A}$ of antichains of P depends only on $\Delta(P)$. Recall

$$\mathcal{C}(P) = \mathrm{conv}\{\chi_A : A \in \mathcal{A}\}.$$

Thus $\mathcal{C}(P)$ depends only on $\Delta(P)$. Since $\Omega_P(n+1) = i(\mathcal{C}(P), n)$, the proof follows. $\square$

An example where $\Delta(P) = \Delta(Q)$



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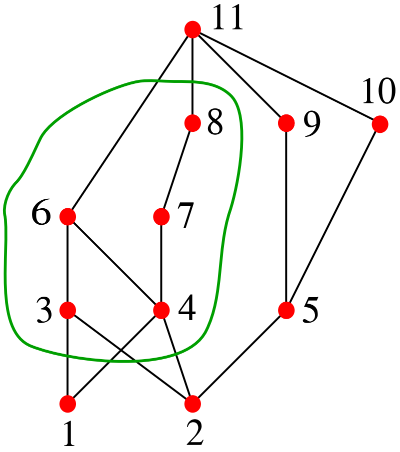

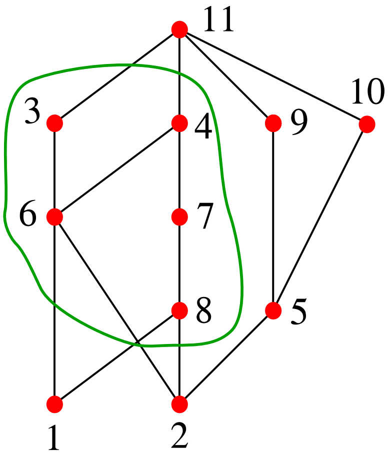

Aside: when does $\Delta(P) = \Delta(Q)$?

P: finite poset

autonomous subset $Q \subseteq P$: if $t \in P - Q$, then either:

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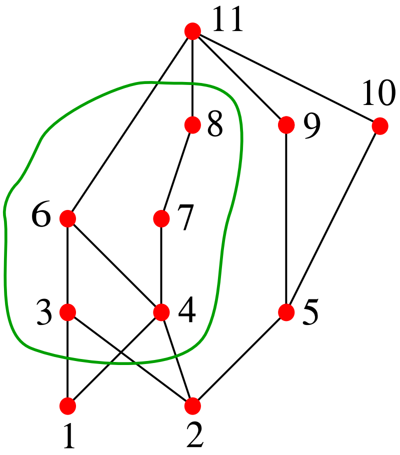
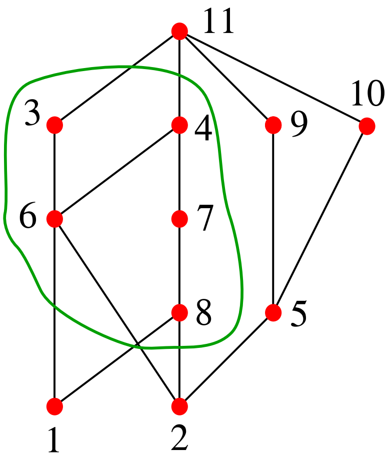
Special case: if $Q = P$, then the flip of Q gives P^*, the **dual** of P

Flip example





Flip theorem

Theorem (**Dreesen**, **Poguntke**, **Winkler**, 1985; implicit in earlier work of **Gallai** and others). *Let P and Q be finite posets. Then $\Delta(P) = \Delta(Q)$ if and only if Q can be obtained from P by a sequence of flips.*

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Easy to see that flips preserve the order polynomial $\Omega_P(n)$. This gives another proof that if $\Delta(P) = \Delta(Q)$ then $\Omega_P(n) = \Omega_Q(n)$, so also $e(P) = e(Q)$ (**Golumbic**, 1980).

Chain polytope analogue of $N_1(t), \ldots, N_p(t)$.

Recall: for $t \in P$,

$$\mathbf{N_i(t)} = \#\{\sigma \in \mathcal{L}(P) : \sigma(t) = i\}.$$

Then $N_i(t)^2 \geq N_{i-1}(t) N_{i+1}(t)$, and no internal zeros.

Proof based on Aleksandrov-Fenchel inequalities for polytopes related to $\mathcal{O}(P)$.

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Would like to "transfer" this result to $\mathcal{C}(P)$. What is the analogue of $N_i(t)$?

$M_i(t)$

Given $t \in P$ and $\sigma \in \mathcal{L}(P)$ with $\sigma(t) = j$, define

$$\textbf{spread}_{\boldsymbol{\sigma}}\boldsymbol{(t)} = \max\{i : \sigma^{-1}(j-1), \sigma^{-1}(j-2), \ldots, \sigma^{-1}(j-i) \text{ are all incomparable with } t\}.$$

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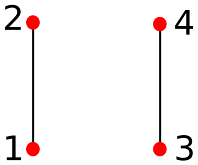
Proof. "Transfer" the proof that $N_i(t)^2 \geq N_{i-1}(t)N_{i+1}(t)$ (messy details omitted).

An example



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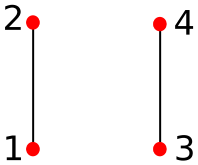

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3 1 4 2	1
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$$\Rightarrow (M_0, M_1, M_2, M_3) = (3, 2, 1, 0)$$

Decreasing property

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Suffices to give a injection (1-1 correspondence) $\mathcal{M}_i(t) \to \mathcal{M}_{i-1}(t), \quad 1 \leq i \leq p-1$.

Given a linear extension

$$\sigma = (t_{k_1}, \dots, t_{k_{j-1}}, t_{k_j} = t, t_{k_{j+1}}, \dots, t_{k_p})$$

of spread at least 1, map it to

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END OF TOPIC 2