A Survey of Parking Functions

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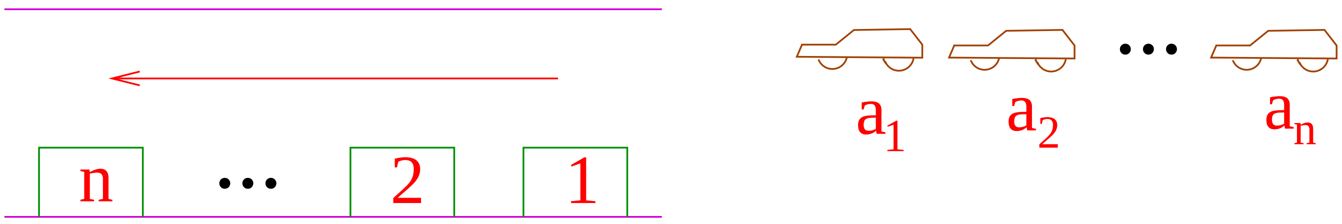
M.I.T.

Parking functions



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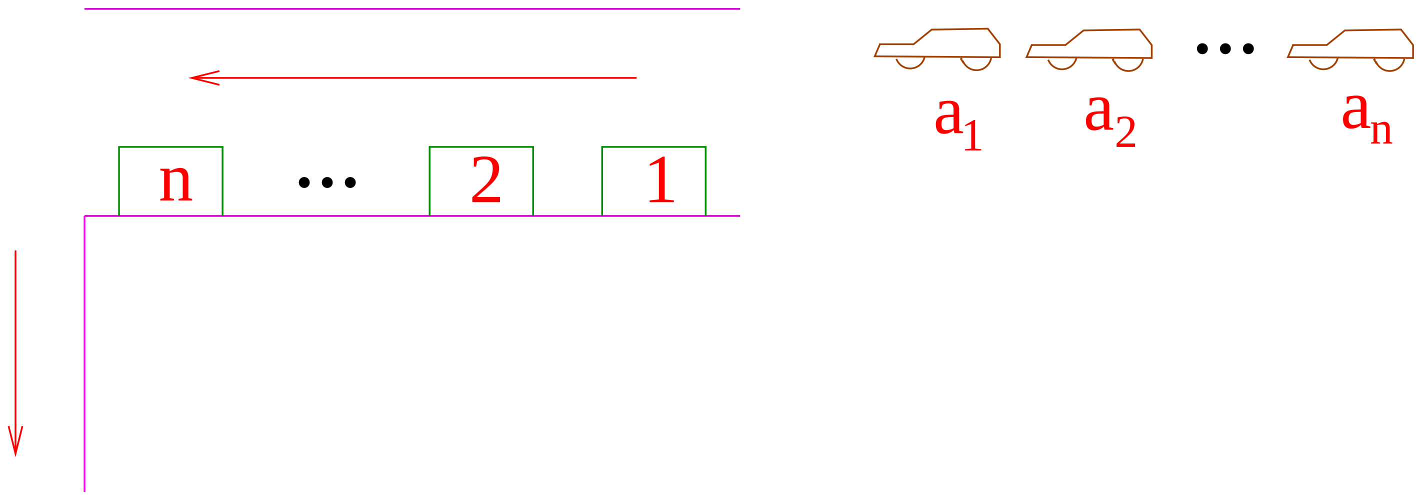

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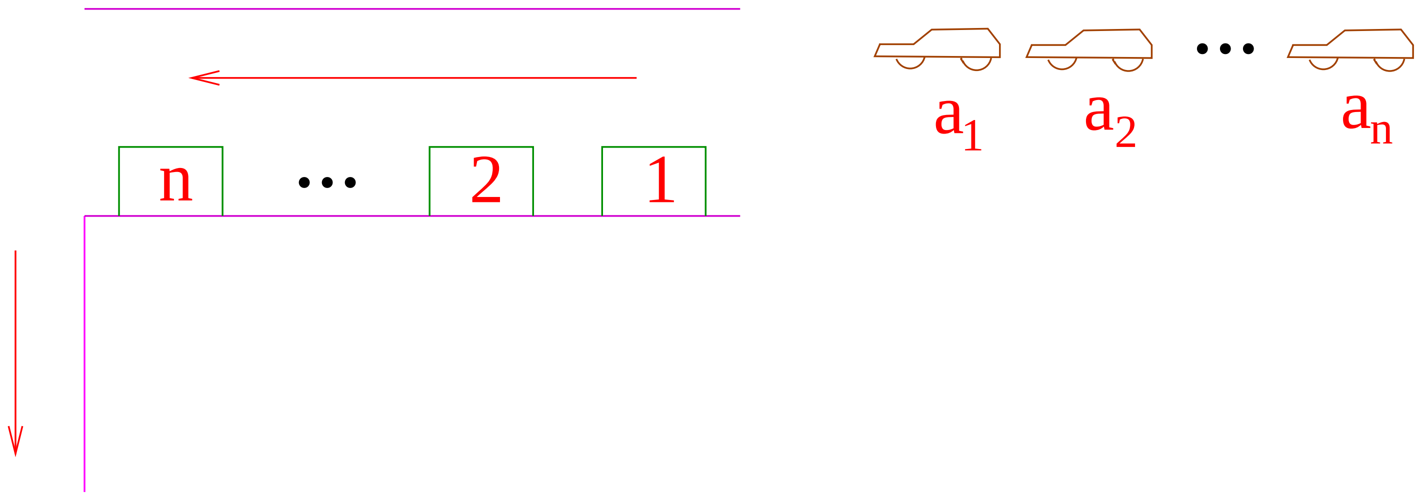


Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

Small examples

$n = \mathbf{2}:$ 11 12 21

$n = \mathbf{3}:$ 111 112 121 211 113 131 311 122

212 221 123 132 213 231 312 321

Parking function characterization

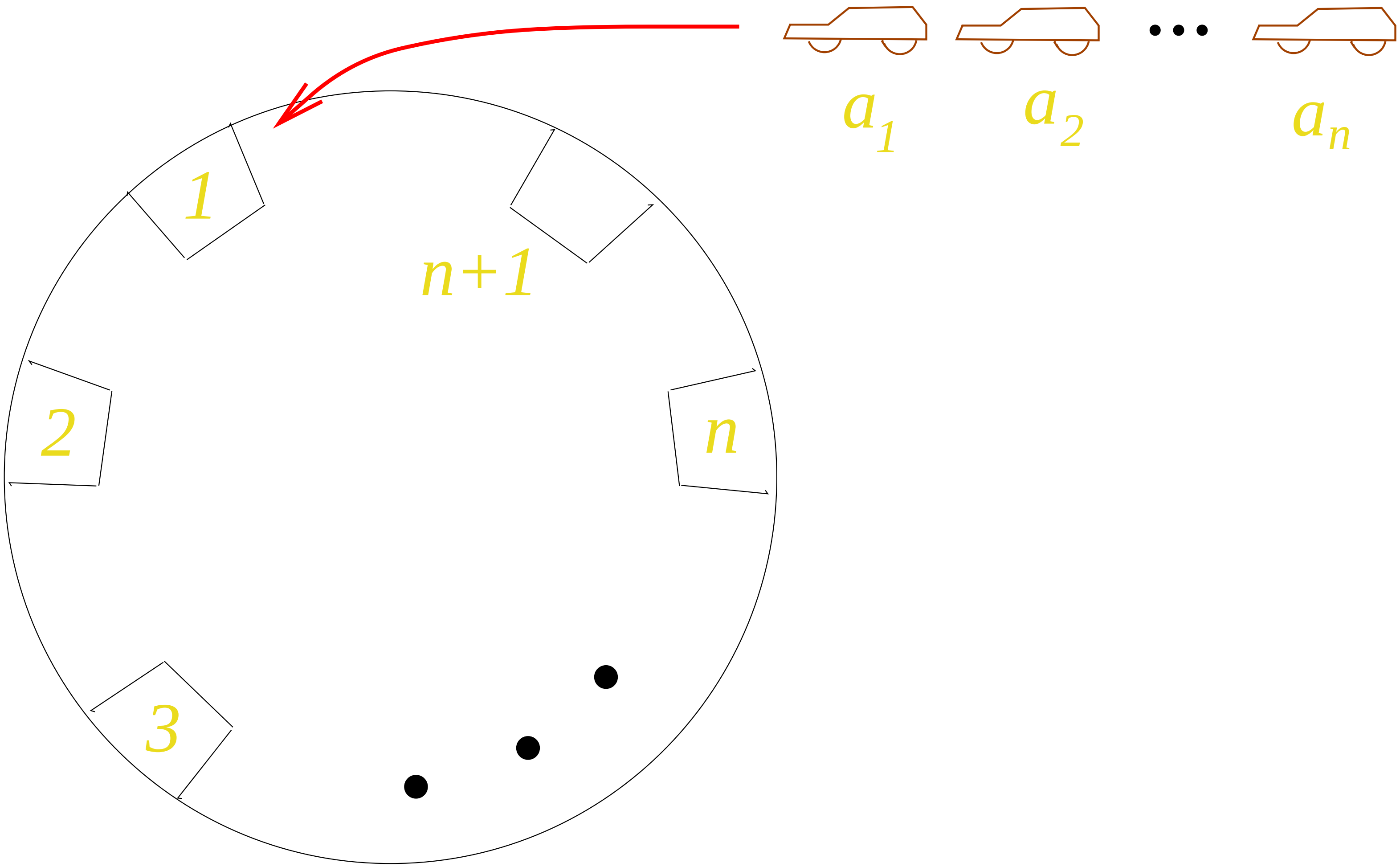
Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Enumeration of parking functions

Theorem (**Pyke**, 1959; **Konheim and Weiss**, 1966). *Let* $f(n)$ *be the number of parking functions of length* n. *Then* $f(n) = (n+1)^{n-1}$.

Proof (**Pollak**, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.



a1
a2
an
1
n+1
2
n
3

Conclusion of Pollak's proof

Now all cars can park, and there will be one empty space. α is a parking function $\Leftrightarrow$ if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1 + i, a_2 + i, \ldots, a_n + i) \text{ (modulo } n+1)$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}. \qquad \square$$

The parking function $\mathfrak{S}_n$-action

The symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length n by permuting coordinates.

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Example. $(1,2,3)(4)(5,6) \cdot 314131 = 431113$

Sample properties

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

$$n = 3 : \qquad \mathbf{111 \quad 112 \quad 122 \quad 113 \quad 123}$$

- Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathfrak{S}_n$ (character value at w):

$$\#\mathrm{Fix}(w) = (n+1)^{(\#\,\text{cycles of } w)-1}$$

Symmetric functions

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots) \vdash n$, i.e.,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \ \sum \lambda_i = n.$$

Complete symmetric function:

$$\begin{aligned} \boldsymbol{h_n} &= \sum_{i_1+i_2+\cdots=n} x_1^{i_1} x_2^{i_2} \cdots \quad (h_0 = 1) \\ \boldsymbol{h_\lambda} &= \prod_i h_{\lambda_1} h_{\lambda_2} \cdots \end{aligned}$$

Symmetric function bases

Example: n=2.

$$\begin{aligned} h_2 &= x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + \cdots \\ h_1^2 &= (x_1 + x_2 + x_3 + \cdots)^2 \end{aligned}$$

The h_λ's for $\lambda \vdash n$ are a basis (say over $\mathbb{Q}$) for all homogeneous symmetric formal power series of degree n in $x_1, x_2, \ldots$.

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Other bases: $\boldsymbol{e_\lambda}$ (elementary), $\boldsymbol{m_\lambda}$ (monomial), $\boldsymbol{p_\lambda}$ (power sums), $\boldsymbol{s_\lambda}$ (Schur), $\boldsymbol{f_\lambda}$ (forgotten), ...

Parking function symmetric function

Let $\mathbf{PF}_n = \mathrm{ch}(\mathcal{P}_n)$.

$$\begin{aligned} \mathcal{I}_n &= \{\text{increasing PFs of length } n\} \\ \mathcal{I}_3 &= \{111,\ 112,\ 113,\ 122,\ 123\} \\ \#\mathcal{I}_n &= C_n = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

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If $\boldsymbol{\alpha} = a_1 \cdots a_n \in \mathcal{I}_n$ define

$$\hat{\boldsymbol{\alpha}} = h_{m_1} h_{m_2} \cdots ,$$

where α has $\boldsymbol{m_i}$ i's.

Formula for PF_n

Example. $\alpha = 11344446 \Rightarrow \hat{\alpha} = h_1^2 h_2 h_4$

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$$\mathrm{PF}_n = \sum_{\alpha \in \mathcal{I}_n} \hat{\alpha}$$

An example: $n = 3$

$$\begin{array}{ll} 111 & h_3 \\ 112 & h_2h_1 \\ 113 & h_2h_1 \\ 122 & h_2h_1 \\ 123 & h_1^3 \end{array}$$

$$\Rightarrow \mathrm{PF}_3 = h_3 + 3h_2h_1 + h_1^3$$

Some properties

$$
\begin{aligned}
\mathrm{PF}_n &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} z_\lambda^{-1} \boldsymbol{p_\lambda} \\
&= \sum_{\lambda \vdash n} \frac{1}{n+1} s_\lambda(1^{n+1}) \boldsymbol{s_\lambda} \\
&= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{\lambda_i + n}{n} \right] \boldsymbol{m_\lambda}
\end{aligned}
$$

More properties

$$
\begin{aligned}
\mathrm{PF}_n &= \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1(\lambda)!\cdots m_n(\lambda)!} \boldsymbol{h_\lambda} \\
&= \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{(n+2)(n+3)\cdots(n+\ell(\lambda))}{m_1(\lambda)!\cdots m_n(\lambda)!} \boldsymbol{e_\lambda} \\
&= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{n+1}{\lambda_i} \right] \boldsymbol{f_\lambda}
\end{aligned}
$$

r, k-parking functions

There are numerous generalizations of parking functions.

(r, k)-parking functions $(r, k \geq 1)$:

$(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies

$$b_i \leq (i-1)r + k.$$

Ordinary parking function: $r = k = 1$.

$(2,1)$-parking functions

Example. $n=3, r=2, k=1$, so $(b_1,b_2,b_3) \leq (1,3,5)$. Increasing $(2,1)$-parking functions of length 3 (with size of $\mathfrak{S}_3$-orbit):

$$\begin{array}{llll} 111\,(1) & 114\,(3) & 123\,(6) & 133\,(3) \\ 112\,(3) & 115\,(3) & 124\,(6) & 134\,(6) \\ 113\,(3) & 122\,(3) & 125\,(6) & 135\,(6). \end{array}$$

Thus total number is **49**, number of $\mathfrak{S}_3$-orbits is **12**.

Parking algorithm

rn cars and $rn + k - 1$ spaces

$\alpha = (a_1, \ldots, a_n)$: cars $C_{r(i-1)+1}, \ldots, C_{ri}$ all prefer a_i.

Same parking algorithm.

Pollak's proof generalized

Arrange $rn+k$ spaces on a circle and park as in Pollak's proof.

α is an (r,k)-parking function $\Leftrightarrow$ space $rn+k$ is empty.

Theorem (**Pyke**, essentially).

$$\#\mathcal{P}_n^{(r,k)} = k(rn+k)^{n-1}$$

Further properties

with **Yinghui Wang**

Further properties

with **Yinghui Wang** (王颖慧)

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Many further properties of (r, k)-parking functions.

A generating function

For simplicity, assume $r = 1$.

Define

$$\begin{aligned} F(t) &:= \sum_{n\geq 0} \mathrm{PF}_n t^n \\ &= 1 + h_1 t + (h_2 + h_1^2)t^2 + \cdots . \end{aligned}$$

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Many interesting properties of $F(t)^k$, $k \in \mathbb{Z}$. Here we consider $k = -1$.

Motivation

Let

$$\begin{aligned} \boldsymbol{A(t)} &= \sum_{n\geq 0} a_n t^n \\ \boldsymbol{B(t)} &= \sum_{n\geq 0} b_n t^n \\ &= \frac{1}{1-A(t)} = \sum_{k\geq 0} A(t)^k. \end{aligned}$$

Thus a_n counts **“prime”** objects and b_n all objects.

$B(t) = F(t)$

Note. $B(t) = \frac{1}{1-A(t)} \Leftrightarrow A(t) = 1 - \frac{1}{B(t)}$.

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Suggests: $1 - \frac{1}{F(t)}$ might be connected with **"prime"** parking functions.

Prime parking functions

Definition (**I. Gessel**). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

Note. A sequence $b_1 \leq b_2 \leq \cdots \leq b_n$ is an increasing parking function if and only if $1 \leq b_1 \leq \cdots \leq b_n$ is an increasing prime parking function.

The prime parking function sym. fn.

E.g., $n = 4$: increasing prime parking functions are

$$1111,\ 1112,\ 1113,\ 1122,\ 1123.$$

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$$\Rightarrow \mathbf{PPF_4} = h_4 + 2h_3h_1 + h_2^2 + h_2h_1^2.$$

Factorization of increasing PF's

1	2	3	4	5	6	7	8	9	10	11
1	1	3	3	4	4	7	8	8	9	10

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$$\to (1,1),\ (1,1,2,2),\ (1),\ (1,1,2,3)$$

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Theorem. $F(t)^{-1} = 1 - \sum_{n\geq 1} \mathrm{PPF}_n t^n$

Parking functions & invariant theory

Background: invariants of $\mathfrak{S}_n$

The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$\boldsymbol{R^{\mathfrak{S}_n}} = \{f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}.$$

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Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n],$$

where $$\boldsymbol{e_k} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The coinvariant algebra

Let

$$\begin{aligned} \boldsymbol{R_+} &= \{f \in R : f(0,\ldots,0) = 0\} \\ \boldsymbol{D} &:= R/\left(R_+^{\mathfrak{S}_n}\right) \\ &= R/(e_1,\ldots,e_n). \end{aligned}$$

The coinvariant algebra

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Then $\dim_{\mathbb{C}} D = n!$, and $\mathfrak{S}_n$ acts on D according to the **regular representation**.

Diagonal action of $\mathfrak{S}_n$

Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

As before, let

$$\begin{aligned} \boldsymbol{R^{\mathfrak{S}_n}} &= \{f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\} \\ D &= R/\left(R_+^{\mathfrak{S}_n}\right). \end{aligned}$$

Haiman's theorem

Theorem (**Haiman**, 1994, 2001).

$$\dim D = (n+1)^{n-1},$$

and the action of $\mathfrak{S}_n$ on D is isomorphic to the action on $\mathcal{P}_n$, tensored with the sign representation.

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Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.

The Shi arrangement: background

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$.

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$$\begin{aligned} \boldsymbol{\mathcal{R}} &= \text{set of regions of } \mathcal{B}_n \\ \#\mathcal{R} &= ?? \end{aligned}$$

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Let $\boldsymbol{R_0}$ be the **base region**

$$R_0: \; x_1 > x_2 > \cdots > x_n.$$

Labeling the regions

Label R_0 with

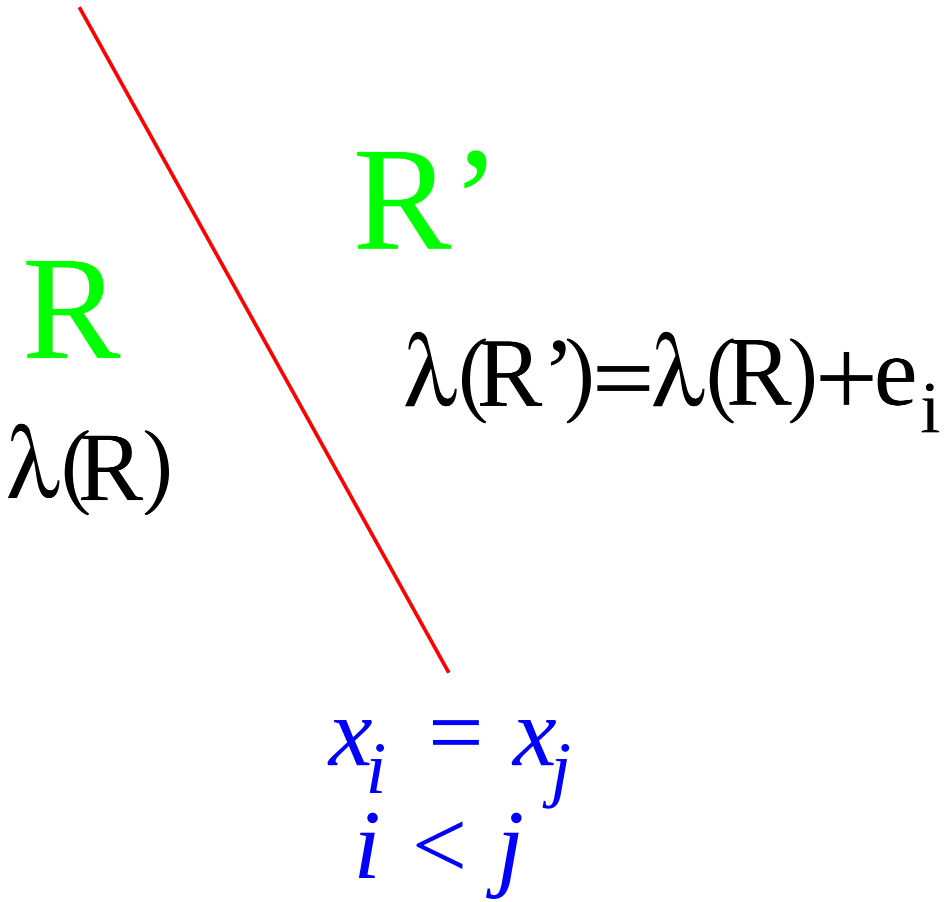
$$\boldsymbol{\lambda(R_0)} = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

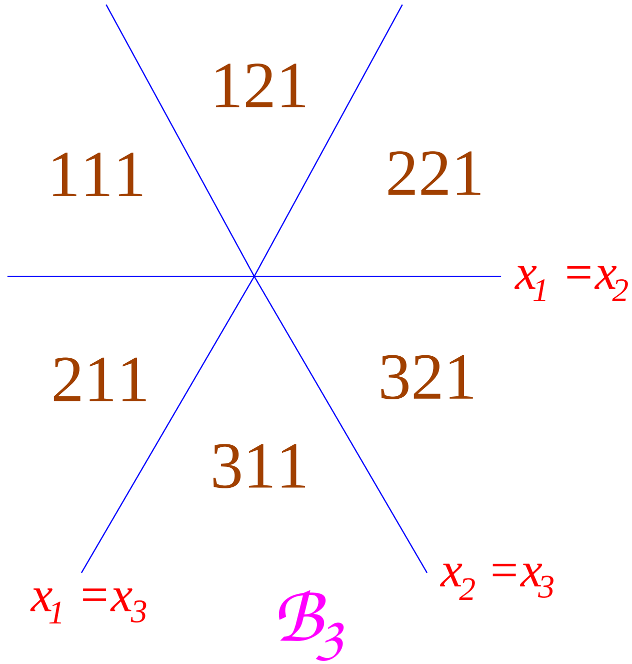
$$\boldsymbol{\lambda(R')} = \lambda(R) + e_i,$$

where $\boldsymbol{e_i} = i$th unit coordinate vector.

The labeling rule



Description of labels



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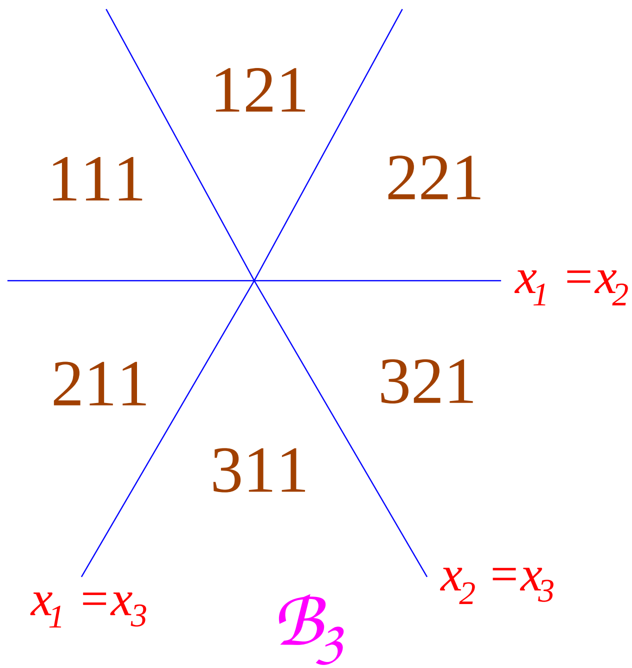

Theorem (easy). *The labels of $\mathcal{B}_n$ are the sequences $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $1 \le b_i \le n - i + 1$.*

The Shi arrangement

Shi Jianyi

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Shi Jianyi (时俭益)

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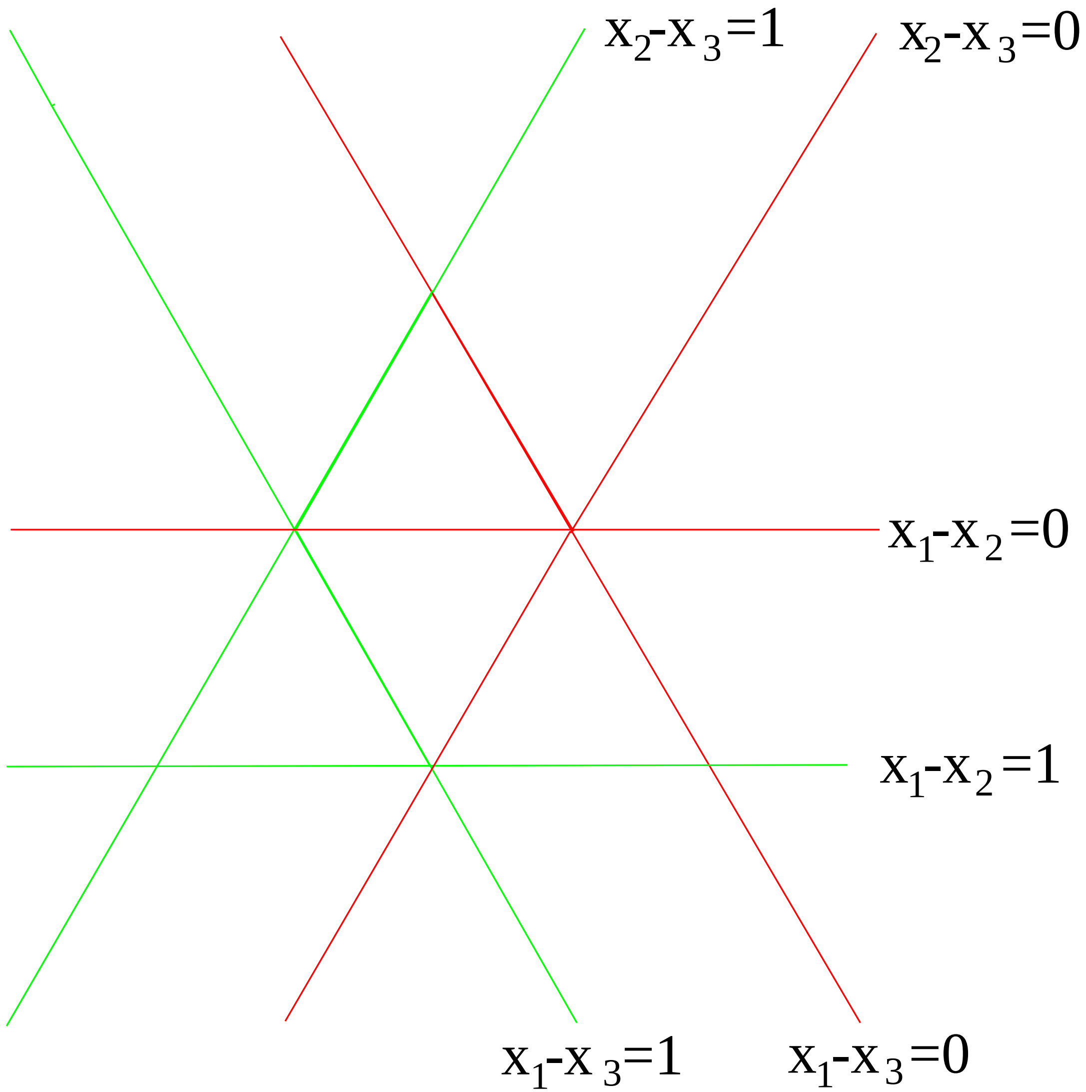
Shi arrangement $\boldsymbol{S_n}$: the set of hyperplanes

$$x_i - x_j = 0, 1,$$

$1 \le i < j \le n$, in $\mathbb{R}^n$.

The case $n = 3$



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base region:

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The labeling rule

R'

R

$\lambda(R') = \lambda(R) + e_i$

$\lambda(R)$

$x_i = x_j$

$i < j$

R'

R

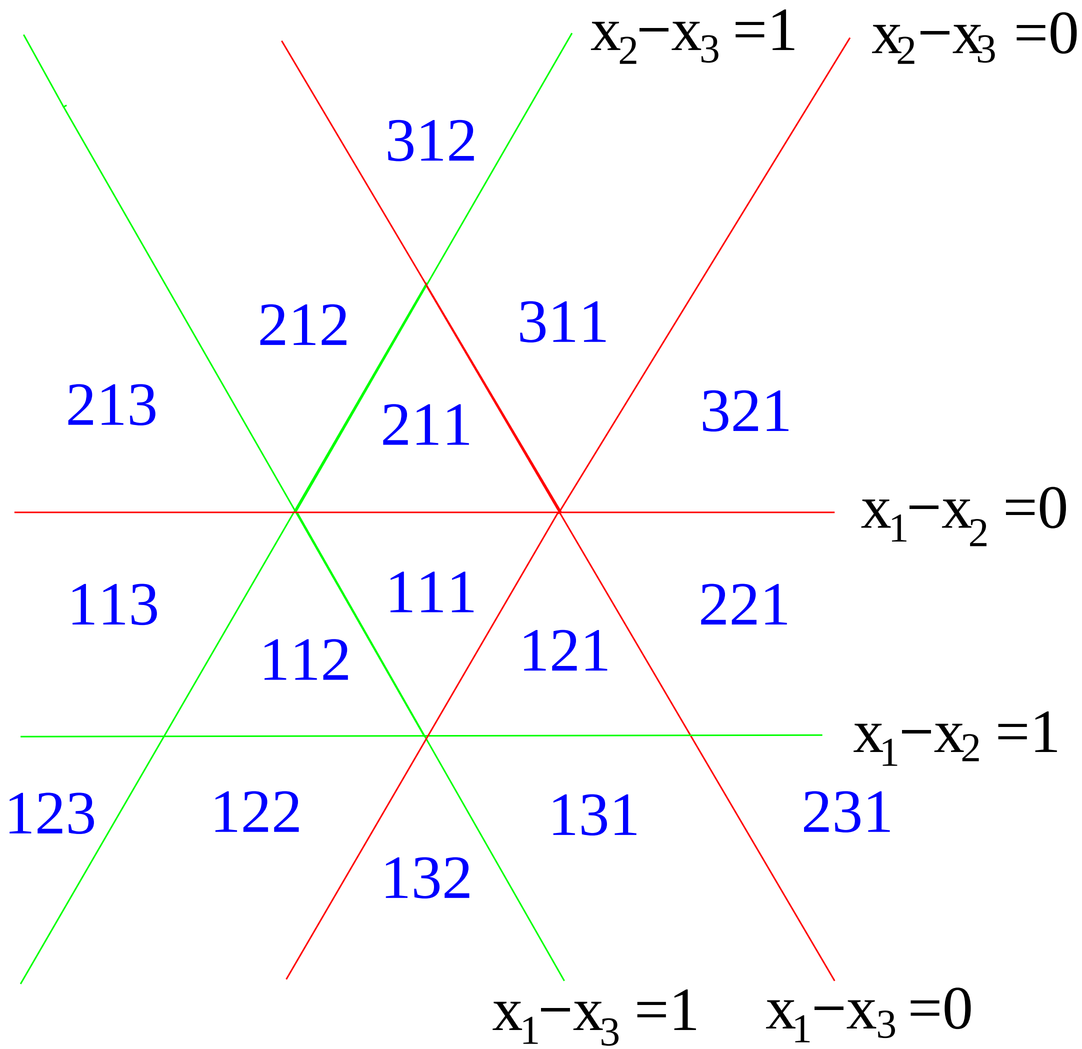
$\lambda(R') = \lambda(R) + e_j$

$\lambda(R)$

$x_i = x_j + 1$

$i < j$

The labeling for $n = 2$



Description of the labels

Theorem (**Pak, S.**). *The labels of $\mathcal{S}_n$ are the parking functions of length n (each occurring once).*

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Corollary (**Shi**, 1986).

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

The parking function polytope

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\boldsymbol{P_n = P(x_1, \ldots, x_n)} \subset \mathbb{R}^n$ by:

$(y_1, \ldots, y_n) \in P_n$ if

$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

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for $1 \leq i \leq n$.

(also called **Pitman-Stanley polytope**)

Volume of P

Theorem. *Let* $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$*. Then*

$$n!\, V(P_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}.$$

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Note. If each $x_i > 0$, then P_n has the combinatorial type of an n-cube.


THE END