PARKING FUNCTIONS

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ENNUMERATION OF PARKING FUNCTIONS

Car $C_i$ prefers space $a_i$. If $a_i$ is occupied, then $C_i$ takes the next available space. We call $(a_1, \ldots, a_n)$ a parking function (of length $n$) if all cars can park.
\[ n = 2 : 11 \ 12 \ 21 \]
\[ n = 3 : 111 \ 112 \ 121 \ 211 \\
\quad 113 \ 131 \ 311 \ 122 \\
\quad 212 \ 221 \ 123 \ 132 \\
\quad 213 \ 231 \ 312 \ 321 \]

**Easy:** Let \( \alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n \). Let \( b_1 \leq b_2 \leq \cdots \leq b_n \) be the increasing rearrangement of \( \alpha \). Then \( \alpha \) is a parking function if and only if \( b_i \leq i \).

**Corollary.** Every permutation of the entries of a parking function is also a parking function.
**Theorem** (Konheim and Weiss, 1966). Let $f(n)$ be the number of parking functions of length $n$. Then

$$f(n) = (n + 1)^{n-1}.$$ 

**Proof** (Pollak, c. 1974). Add an additional space $n + 1$, and arrange the spaces in a circle. Allow $n + 1$ also as a preferred space.

![Diagram of parking functions]

The circle represents the arrangement of spaces, with $a_1$, $a_2$, ..., $a_n$ indicating the positions where cars (or parking functions) can be placed.
Now all cars can park, and there will be one empty space. $\alpha$ is a parking function if and only if the empty space is $n + 1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car $C_i$ parking at space $p_i$, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n + 1$) will lead to car $C_i$ parking at space $p_i + j$. Hence exactly one of the vectors 

$$(a_1+i, a_2+i, \ldots, a_n+i) \pmod{n+1}$$

is a parking function, so 

$$f(n) = \frac{(n + 1)^n}{n + 1} = (n + 1)^{n-1}.$$
FORESTS

Let $F$ be a rooted forest on the vertex set $\{1, \ldots, n\}$.

THEOREM (Sylvester-Borchardt-Cayley). The number of such forests is $(n + 1)^{n-1}$.
**Theorem.** The number of parking functions of length $n$ with $k$ 1’s is the number of rooted forests on the vertex set $\{1, 2, \ldots, n\}$ with exactly $k$ components (trees).

**Note:** This number is equal to $\binom{n-1}{k-1} n^{n-k}$.

**Exercise.** Find a combinatorial proof analogous to Pollak’s proof.
NONCROSSING PARTITIONS

A partition of a finite set $S$ is a collection $\{B_1, \ldots, B_k\}$ of subsets $\emptyset \neq B_i \subseteq S$ satisfying:

- $B_1 \cup B_2 \cup \cdots \cup B_k = S$
- $B_i \cap B_j = \emptyset$ if $i \neq j$

$n = 3$: 1-2-3, 12-3, 13-2, 1-23, 123 (five in all)

$n = 4$: 1-2-3-4 [1], ab-c-d [6], ab-cd [3], abc-d [4], 1234 [1] (15 in all)
A **noncrossing partition** of \(\{1, 2, \ldots, n\}\) is a partition \(\{B_1, \ldots, B_k\}\) of \(\{1, \ldots, n\}\) such that

\[
a < b < c < d, \ a, c \in B_i, \ b, d \in B_j \implies i = j.
\]
Theorem (H. W. Becker, 1948–49)

The number of noncrossing partitions of \{1, \ldots, n\} is the Catalan number

\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]

Example. Of the 15 partitions of \{1, 2, 3, 4\}, only 13-24 is not noncrossing. Hence \( C_4 = 15 - 1 = 14 \).

For over 100 combinatorial interpretations of \( C_n \), see

www-math.mit.edu/~rstan/ec.html
A maximal chain \( m \) of noncrossing partitions of \( \{1, \ldots, n+1\} \) is a sequence

\[ \pi_0, \pi_1, \pi_2, \ldots, \pi_n \]

of noncrossing partitions of the set \( \{1, \ldots, n + 1\} \) such that \( \pi_i \) is obtained from \( \pi_{i-1} \) by merging two blocks into one. (Hence \( \pi_i \) has exactly \( n + 1 - i \) blocks.)

\[
\begin{align*}
1 - 2 - 3 - 4 - 5 \\
1 - 25 - 3 - 4 \\
1 - 25 - 34 \\
125 - 34 \\
12345
\end{align*}
\]
Define:
\[
\min B = \text{least element of } B \\
\forall k \in B.
\]

Suppose \(\pi_i\) is obtained from \(\pi_{i-1}\) by merging together blocks \(B\) and \(B'\), with \(\min B < \min B'\). Define
\[
\Lambda_i(m) = \max\{j \in B : j < B'\} \\
\Lambda(m) = (\Lambda_1(m), \ldots, \Lambda_n(m)).
\]

For above example:

\[
1 - 2 - 3 - 4 - 5, \ 1 - 25 - 3 - 4, \ 1 - 25 - 34, \ 125 - 34, \ 12345
\]

we have
\[
\Lambda(m) = (2, 3, 1, 2).
\]
Theorem. \( \Lambda \) is a bijection between the maximal chains of noncrossing partitions of \( \{1, \ldots, n + 1\} \) and parking functions of length \( n \).

Corollary (Kreweras, 1972) The number of maximal chains of noncrossing partitions of \( \{1, \ldots, n + 1\} \) is
\[
(n + 1)^{n-1}.
\]

\[
\begin{align*}
1 &- 2 - 3, \quad 12 - 3, \quad 123 : \quad (1, 2) \\
1 &- 2 - 3, \quad 13 - 2, \quad 123 : \quad (1, 1) \\
1 &- 2 - 3, \quad 23 - 1, \quad 123 : \quad (2, 1)
\end{align*}
\]
THE SHI ARRANGEMENT

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$.

$\mathcal{R}$ = set of regions of $\mathcal{B}_n$

$\#\mathcal{R} = n!$

Let $R_0$ be the “base region”

$$R_0: \quad x_1 > x_2 > \cdots > x_n.$$
Label $R_0$ with
\[ \lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n. \]

If $R$ is labelled, $R'$ is separated from $R$ only by $x_i - x_j = 0 \ (i < j)$, and $R'$ is unlabelled, then set
\[ \lambda(R') = \lambda(R) + e_i, \]
where $e_i = i$th unit coordinate vector.
**Theorem** (easy). The labels of $\mathcal{B}_n$ are the sequences $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $1 \leq a_i \leq n - i + 1$. 
Shi arrangement $S_n$: the set of hyperplanes

$x_i - x_j = 0, 1; 1 \leq i < j \leq n$, in $\mathbb{R}^n$. 
base region

\[ R_0 : \quad x_n + 1 > x_1 > \cdots > x_n \]

- \( \lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n \)
- If \( R \) is labelled, \( R' \) is separated from \( R \) only by \( x_i - x_j = 0 \) (\( i < j \)), and \( R' \) is unlabelled, then set
  \[ \lambda(R') = \lambda(R) + e_i. \]
- If \( R \) is labelled, \( R' \) is separated from \( R \) only by \( x_i - x_j = 1 \) (\( i < j \)), and \( R' \) is unlabelled, then set
  \[ \lambda(R') = \lambda(R) + e_j. \]
$\lambda(R') = \lambda(R) + e_i$

$x_i = x_j$

$i < j$

$\lambda(R') = \lambda(R) + e_j$

$x_i = x_j + 1$

$i < j$
**Theorem** (Pak, S.). The labels of $S_n$ are the parking functions of length $n$ (each occurring once).
Corollary (Shi, 1986)

\[ r(S_n) = (n + 1)^{n-1} \]
A GENERALIZATION

Let
\[ \lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n > 0. \]

A \textbf{λ-parking function} is a sequence \((a_1, \ldots, a_n) \in \mathbb{P}^n\) whose increasing rearrangement \(b_1 \leq \cdots \leq b_n\) satisfies \(b_i \leq \lambda_{n-i+1}\).

Ordinary parking functions:
\[ \lambda = (n, n - 1, \ldots, 1) \]

\textbf{Number} (Steck 1968, Gessel 1996):

\[ \mathbf{N}(\lambda) = n! \det \left[ \frac{\lambda_{j-i+1}}{n-i+1} \frac{j-i+1}{(j-i+1)!} \right]_{i,j=1}^{n} \]
THE PARKING FUNCTION POLYTOPE

Given $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$, define $\mathcal{P} = \mathcal{P}(a_1, \ldots, a_n) \subset \mathbb{R}^n$ by: $(x_1, \ldots, x_n) \in \mathcal{P}_n$ if

$$x_i \geq 0$$

$$x_1 + \cdots + x_i \leq a_1 + \cdots + a_i$$

for $1 \leq i \leq n$. 
\( n = 2 : \quad x, y \geq 0 \)
\[
\begin{align*}
  x &\leq a \\
  x + y &\leq a + b
\end{align*}
\]

\[(0,0) \quad (a,0) \quad (0,b) \quad (a,a+b)\]

\[
\text{area} = \frac{1}{2}(a^2 + 2ab)
\]
**Theorem.** (a) Let \( a_1, \ldots, a_n \in \mathbb{N} \). Then
\[
n! \, V(\mathcal{P}_n) = N(\lambda),
\]
where \( \lambda_{n-i+1} = a_1 + \cdots + a_i \).

(b) \( n! \, V(\mathcal{P}_n) = \sum_{\text{parking functions}} a_{i_1} \cdots a_{i_n}. \)

**Example.** \( n = 2 \):
\[
\begin{align*}
11 & \quad a^2 \\
12 & \quad ab \\
21 & \quad ba
\end{align*}
\]
\[\Rightarrow \text{area} = \frac{1}{2}(a^2 + 2ab)\]

**Note:** If each \( a_i > 0 \), then \( \mathcal{P}_n \) has the combinatorial type of an \( n \)-cube.
ALGEBRAIC ASPECTS OF PARKING FUNCTIONS

The symmetric group acts on the set $\mathcal{P}_n$ of all parking functions of length $n$ by permuting coordinates.

Sample properties:

• Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1} \binom{2n}{n}$
  
  $n = 3 : \quad 111 \ 211 \ 221 \ 311 \ 321$

• Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathcal{S}_n$ (character value at $w$):
  
  $\#\text{Fix}(w) = (n + 1)^{(#\text{cycles of } w)} - 1$

• Multiplicity of any irreducible representation: simple product formula
REFERENCES


12. J. Lewis, Parking functions and regions of the Shi arrangement, preprint dated 1 August 1996.


