ACYCLIC FLOW POLYTOPES AND KOSTANT'S PARTITION FUNCTION

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Transparencies available at:

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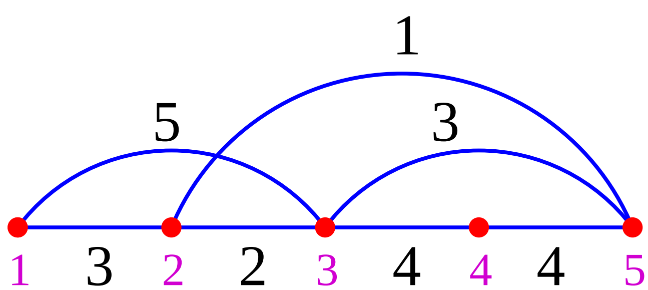
(with A. Postnikov)



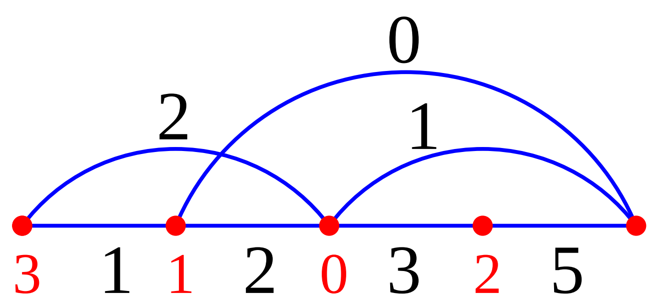
N- flow
size = 8
V(G) = {*1*, ..., *m*+*1*}

flow polytope $\mathcal{F}_G \subset \mathbb{R}^{E(G)}$:

flows $f : E(G) \to \mathbb{R}_{\geq 0}$ of size 1

excess flow vector

$$\gamma = (a_1, \ldots, a_m) \in \mathbb{N}^m$$



γ = (3, 1, 0, 2)

(restricted) Kostant's partition function:

$$\boldsymbol{e_i} = (0 \cdots 0 \overset{i}{1} 0 \cdots 0) \in \mathbb{R}^{m+1}$$

$$\boldsymbol{e_{ij}} = e_i - e_j$$

$$\boldsymbol{\nu} \in \mathbb{Z}^{m+1}, \quad \sum \nu_i = 0$$

$$\boldsymbol{A_m^+} = \{e_{ij} : 1 \le i < j \le m+1\} \subset \mathbb{Z}^{m+1}$$

$$\boldsymbol{S} \subseteq A_m^+$$

$$\boldsymbol{K_S(\nu)} = \#\left\{ \left(b_{ij}\right)_{e_{ij} \in S} : \nu = \sum b_{ij} e_{ij} \right\}$$

Proposition. *Let*

$$S = S(G) = \{e_{ij} : (i,j) \in E(G)\}.$$

The number of $\mathbb{N}$*-flows with excess flow* $(a_1, \ldots, a_m)$ *is equal to*

$$K_S(a_1, \ldots, a_m, -\sum a_i).$$

Main Theorem (D. Peterson, unpublished, for $S = A_m^+$). *Let*

$$\boldsymbol{d_i} = \mathrm{outdeg}(i) - 1.$$

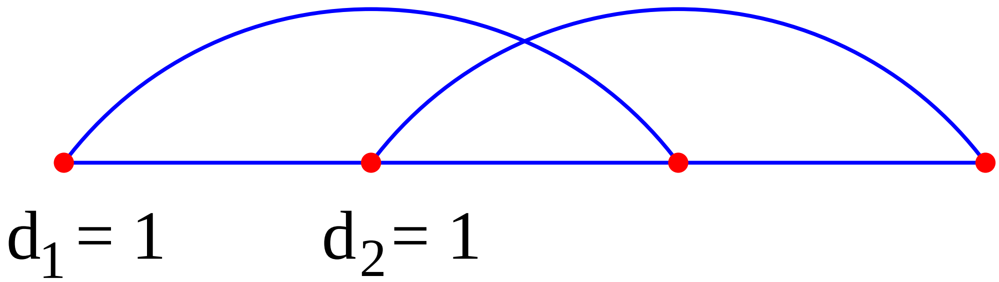
Then

$$K_S(a_1, \ldots, a_m, -\sum a_i) =$$

$$\sum K_S(\nu_1 - d_1, \ldots, \nu_{m-1} - d_{m-1})$$
$$\cdot \binom{a_1 + d_1}{\nu_1} \cdots \binom{a_{m-1} + d_{m-1}}{\nu_{m-1}},$$

summed over all $\nu_1, \ldots, \nu_{m-1} \in \mathbb{N}$ *satisfying*

$$\begin{aligned} \nu_1 + \cdots + \nu_i &\geq d_1 + \cdots + d_i \\ \sum \nu_i &= d_1 + \cdots + d_{m-1}. \end{aligned}$$



$$(\nu_1, \nu_2) = (2, 0), (1, 1)$$

$$S = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{34}\}$$

$$K_S(a, b, c, -a - b - c) =$$

$$K_S(1, -1)\binom{a+1}{2} + K_S(0, 0)\binom{a+1}{1}\binom{b+1}{1}$$

$$= \binom{a+1}{2} + (a+1)(b+1).$$

Idea of proof. Consider

$$\prod_{(i,j)\in E(G)} \frac{1}{1 - x_i x_j^{-1}} =$$

$$\sum_{\beta} K_S(\beta) x^{\beta}.$$

Systematically apply the identity

$$\frac{1}{1 - x_i x_j^{-1}} \cdot \frac{1}{1 - x_j x_k^{-1}} =$$

$$\frac{1}{1 - x_i x_k^{-1}} \left(\frac{1}{1 - x_i x_j^{-1}} + \frac{x_j x_k^{-1}}{1 - x_j x_k^{-1}} \right)$$

(Elliott-MacMahon algorithm).

Corollary. *Let* $d = \dim \mathcal{F}_G$. *Then*

$$d! \cdot \mathrm{vol}(\mathcal{F}_G) := \tilde{V}(\mathcal{F}_G)$$

$$= K_S(d_{m-1}, d_{m-2}, \ldots, d_1, -\sum d_i).$$

For $G = K_{m+1}$, we have

$$\tilde{V}(\mathcal{F}_{K_{m+1}}) = K\left(1, 2, \ldots, m-2, -\binom{m-1}{2}\right).$$

Chan-Robbins conjecture:

Theorem (Zeilberger). *We have*

$$\tilde{V}\left(\mathcal{F}_{K_{m+1}}\right) = C_1 \cdots C_{m-2},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ (**Catalan number**).

Can also do

$$K\left(a, a+1, \ldots, b, -\binom{b+1}{2} + \binom{a}{2}\right).$$

Theorem (easy). $K(a_1, \ldots, a_m, -\sum a_i)$ *is divisible by*

$$(a_1 + 1)(a_1 + 2) \cdots (a_1 + m - 1).$$

Theorem (J. R. Schmidt and A. M. Bincer, 1984; A. N. Kirillov, 1999) *Also divisible by*

$$a_1 + a_2 + \cdots + a_{m-2} + 3a_{m-1} + 3.$$

In fact,

$$\begin{aligned} &3K(a_1, \ldots, a_m, -\sum a_i) = \\ &\quad (a_1 + \cdots + a_{m-2} + 3a_{m-1} + 3) \\ &\quad \cdot K_{\text{no } e_{m-1,m}}(a_1, \ldots, a_m, -\sum a_i). \end{aligned}$$

Bijective proof?

MIXED LATTICE POINT ENUMERATORS

$\boldsymbol{\mathcal{P}_1, \ldots, \mathcal{P}_m}$ integer polytopes in $\mathbb{R}^d$

$$a_1, \ldots, a_m \in \mathbb{N}$$

$$\begin{aligned} \boldsymbol{\mathcal{P}} &= a_1\mathcal{P}_1 + \cdots + a_m\mathcal{P}_m \\ &= \{a_1 v_1 + \cdots + a_m v_m \,:\, v_i \in \mathcal{P}_i\}. \end{aligned}$$

(Minkowski sum)

McMullen: Let

$$\boldsymbol{N(\mathcal{P})} = \#\left(\mathbb{Z}^d \cap \mathcal{P}\right),$$

the **mixed lattice point enumerator** of $\mathcal{P}_1, \ldots, \mathcal{P}_m$ ($m = 1$: **Ehrhart polynomial**). Then

$$N(\mathcal{P}) \in \mathbb{Q}[a_1, \ldots, a_m].$$

Let $\boldsymbol{N(\mathcal{P}_d)}$ = terms of total degree d. Then

$$\begin{aligned} N(\mathcal{P})_d &= \mathrm{vol}(\mathcal{P}) \\ &= \mathrm{vol}(a_1\mathcal{P}_1 + \cdots + a_m\mathcal{P}_m) \\ &= \sum_{i_1+\cdots+i_m=d} \binom{d}{i_1, \ldots, i_m} \\ &\qquad \cdot \underbrace{V(\mathcal{P}_1^{i_1}, \ldots, \mathcal{P}_m^{i_m})}_{\textbf{mixed volume}} . \end{aligned}$$

Now let $V(G) = \{1, 2, \ldots, m+1\}$ and $\gamma = (a_1, \ldots, a_m)$ as before. Let

$$\mathbf{G_i} = G|_{i,i+1,\ldots,m+1}$$

$$\boldsymbol{\mathcal{F}_i} = \mathcal{F}_{G_i} \subset \mathbb{R}^{E(G)}.$$

Then

$$N(a_1\mathcal{F}_1 + \cdots + a_m\mathcal{F}_m) =$$

$$\#(\mathbb{N}, \gamma)\text{-flows in } G$$

$$= K_S(a_1, \ldots, a_m, -\sum a_i).$$

In particular,

$$V(a_1\mathcal{F}_1 + \cdots + a_m\mathcal{F}_m) =$$

$$\sum_{\text{certain } \nu_i \in \mathbb{N}} \underbrace{K_S(\nu_1 - d_1, \ldots, \nu_{m-1} - d_{m-1})}_{\text{mixed volumes}}$$

$$\cdot \frac{a_1^{\nu_1}}{\nu_1!} \cdots \frac{a_{m-1}^{\nu_{m-1}}}{\nu_{m-1}!}.$$