HYPERPLANE ARRANGEMENTS AND INTERVAL ORDERS

dedicated to the memory of Gian-Carlo Rota

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Transparencies available at:
Let $P$ be a (finite) set of closed intervals $[a, b] \subseteq \mathbb{R}$, with $a < b$. Define a partial ordering of $P$ by

$$[a, b] < [c, d] \text{ if } b < c.$$ 

Any poset (partially ordered set) isomorphic to $P$ is an interval order.
and

unlabelled

labelled

and

give
Theorem. (Fishburn, 1970) A finite poset is an interval order if and only if it contains no induced

\[
\begin{array}{c}
\text{OK} \\
\text{not OK}
\end{array}
\]

Let \( r(\mathcal{A}) \) denote the number of regions of the hyperplane arrangement \( \mathcal{A} \).
**Theorem.** Let $a_1, \ldots, a_n > 0$. Let $I(a_1, \ldots, a_n)$ be the number of (labelled) interval orders such that interval $I_i$ has length $a_i$. Then

$$I(a_1, \ldots, a_n) = r(\mathcal{I}(a_1, \ldots, a_n)),$$

where $\mathcal{I}(a_1, \ldots, a_n)$ is the arrangement (in $\mathbb{R}^n$)

$$x_i - x_j = a_i, \ i \neq j, \ 1 \leq i, j \leq n.$$

**Proof.**

![Diagram showing intervals and their relationships]

\[I_j < I_i \iff x_j < x_i - a_i \]

\[\iff x_i - x_j > a_i.\]
Special case:

\[ \mathcal{I}_n : \quad x_i - x_j = 1, \quad i \neq j, \quad 1 \leq i, j \leq n \]

**Definition.** A poset is a **semiorder** or **unit interval order** if it is an interval order using intervals of length one.

**Theorem.** (Scott & Suppes, 1958) A finite poset is a semiorder if and only if it contains no induced

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad \text{or} \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

**Corollary.** \( r(\mathcal{I}_n) \) is the number of semiorders on \( \{1, 2, \ldots, n\} \).
$\mathcal{I}_n^0: \ x_i - x_j = 0, \pm 1, \ 1 \leq i < j \leq n.$

\[
r(\mathcal{I}_n^0) = n! \cdot \# \text{ nonisomorphic semiorders with } n \text{ elements}
\]

\[
= n! C_n, \ C_n = \frac{1}{n + 1} \binom{2n}{n}
\]

(Catalan number).
Let

\[ F(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}. \]

Then

\[ \sum_{n \geq 0} r(\mathcal{I}_n) \frac{x^n}{n!} = F(1 - e^{-x}). \]

(Chandon, Lemaire, & Pouget, 1978).
Now let $a_1, \ldots, a_n$ be generic, e.g., linearly independent over $\mathbb{Q}$, or

$$a_1 \ll a_2 \ll \cdots \ll a_n.$$  

**Note.** $I(a_1, \ldots, a_n)$ is independent of $a_1, \ldots, a_n$ (though the interval orders themselves depend on $a_1, \ldots, a_n$) (since the intersection posets are the same).

**Example.** $(a_1, a_2, a_3, a_4) = (1, 2, 4, 8, 16)$

Cannot be achieved by $(a_1, a_2, a_3, a_4) = (1, 1.0001, 1.001, 1.01, 1.1)$. 

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Let $h_n = I(a_1, \ldots, a_n)$.

**Theorem. Define**

$$y = 1 + x + 5 \frac{x^2}{2!} + 46 \frac{x^3}{3!} + \cdots$$

by

$$1 = y(2 - e^{xy}).$$

Let

$$z = \sum_{n \geq 0} h_n \frac{x^n}{n!}$$

$$= 1 + x + 3 \frac{x^2}{2!} + 19 \frac{x^3}{3!} + 195 \frac{x^4}{4!} + \cdots.$$ 

Then

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$
**Proof.** Whitney, Zaslavsky $\implies$

$$h_n = \sum_{B} (-1)^{\text{cycle dim}} 2^{\# \text{blocks}},$$

where $B$ ranges over all bipartite graphs with vertices $1, 2, \ldots, n$.

\[= 195\]

Etc.
The Linial Arrangement

\[ \mathcal{L}_n : \quad x_i - x_j = 1, \quad 1 \leq i < j \leq n \]
\[ r(\mathcal{L}_1) = 1 \]
\[ r(\mathcal{L}_2) = 2 \]
\[ r(\mathcal{L}_3) = 7 \]
\[ r(\mathcal{L}_4) = 36 \]
\[ r(\mathcal{L}_5) = 246 \]
\[ r(\mathcal{L}_6) = 2104 \]
\[ r(\mathcal{L}_7) = 21652 \]
\[ r(\mathcal{L}_8) = 260720 \]
\[ r(\mathcal{L}_9) = 3598120 \]
\[ r(\mathcal{L}_{10}) = 56010096 \]
\[ r(\mathcal{L}_{11}) = 971055240 \]
\[ r(\mathcal{L}_{12}) = 18558391936 \]
**Theorem.** $r(\mathcal{L}_n)$ is the number of posets on \{1, 2, \ldots, n\} obtained by intersecting a semiorder (unit interval order) with the chain

$$1 < 2 < \cdots < n.$$
**Theorem** (A. Postnikov). Let
\[ a < b < c < d. \]

The obstructions to being an intersection of a semiorder with the chain
\[ 1 < 2 < \cdots < n \]
are the induced posets

\[
\begin{array}{ccc}
  \bullet & \bullet & \bullet \\
  a & b & c \\
  \bullet & \bullet & \bullet \\
  d & \bullet & \bullet \\
  \bullet & \bullet & \bullet \\
  a & \bullet & \bullet \\
  \bullet & \bullet & \bullet \\
  c & \bullet & \bullet \\
  \bullet & \bullet & \bullet \\
  d & \bullet & \bullet \\
  \bullet & \bullet & \bullet \\
  b & \bullet & \bullet \\
  \bullet & \bullet & \bullet \\
  a & \bullet & \bullet \\
  \end{array}
\]
Theorem (Athanasiadis, Postnikov)

Let

\[ y = \sum_{n \geq 0} r(\mathcal{L}_n) \frac{x^n}{n!}. \]

Then

\[ y = \exp \frac{x}{2}(1 + y) \]

\[ r(\mathcal{L}_n) = \frac{1}{2n} \sum_{k=0}^{n} \binom{n}{k}(k+1)^{n-1}. \]
Define

\[ [a, a + 2] < [b, b + 2] \text{ if } a + 1 < b \]
\[ [a, a + 2] < [b, b + 2] \text{ if } a + 2 < b. \]

A **double semiorder** (easily generalized to \( k \)-interval orders and \( k \)-semiorders) is the “double poset” obtained from a (finite) set of intervals of length two and the two relations \(<\) and \(<\).
Theorem. (a) The double semiorders on $1, 2 \ldots, n$ are in one-to-one correspondence with the regions of the arrangement

$I_{n,2} : x_i - x_j = \pm 1, \pm 2, \ 1 \leq i < j \leq n.$

(b) Let

$I_{n,2}^0 : x_i - x_j = 0, \pm 1, \pm 2, \ 1 \leq i < j \leq n.$

Then

\[ r(I_{n,2}^0) = n! \cdot \# \text{ nonisomorphic double semiorders with } n \text{ elements} \]

\[ = n! \frac{1}{2n + 1} \binom{3n}{n} \]
(c) Let
\[
G(x) = \sum_{n \geq 0} r(\mathcal{I}_{n,2}^0) \frac{x^n}{n!} = \sum_{n \geq 0} \frac{1}{2n + 1} \binom{3n}{n} x^n.
\]
Then
\[
\sum_{n \geq 0} r(\mathcal{I}_{n,2}) \frac{x^n}{n!} = G(1 - e^{-x}).
\]

Compare:
\[
F(x) = \sum_{n \geq 0} r(\mathcal{I}_n^0) \frac{x^n}{n!} = \sum_{n \geq 0} \frac{1}{n + 1} \binom{2n}{n} x^n
\]
\[
\sum_{n \geq 0} r(\mathcal{I}_n) \frac{x^n}{n!} = F(1 - e^{-x}).
\]
Reference: