Increasing and Decreasing Subsequences

Richard P. Stanley

M.I.T.

Definitions

$$\mathbf{3}\,1\,8\,\mathbf{4}\,9\,\mathbf{6}\,\mathbf{7}\,2\,5 \quad \text{(i.s)}$$

$$3\,1\,\mathbf{8}\,\mathbf{4}\,9\,6\,7\,\mathbf{2}\,5 \quad \text{(d.s)}$$

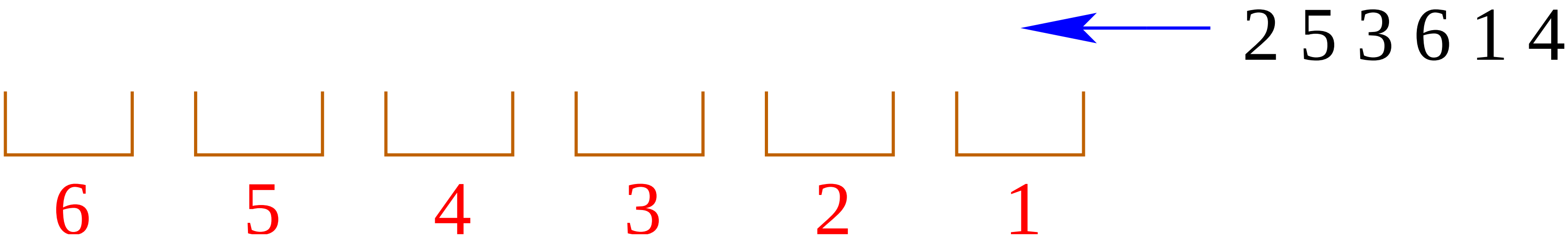
$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$

$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

Application: airplane boarding

Naive model: passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

Boarding process



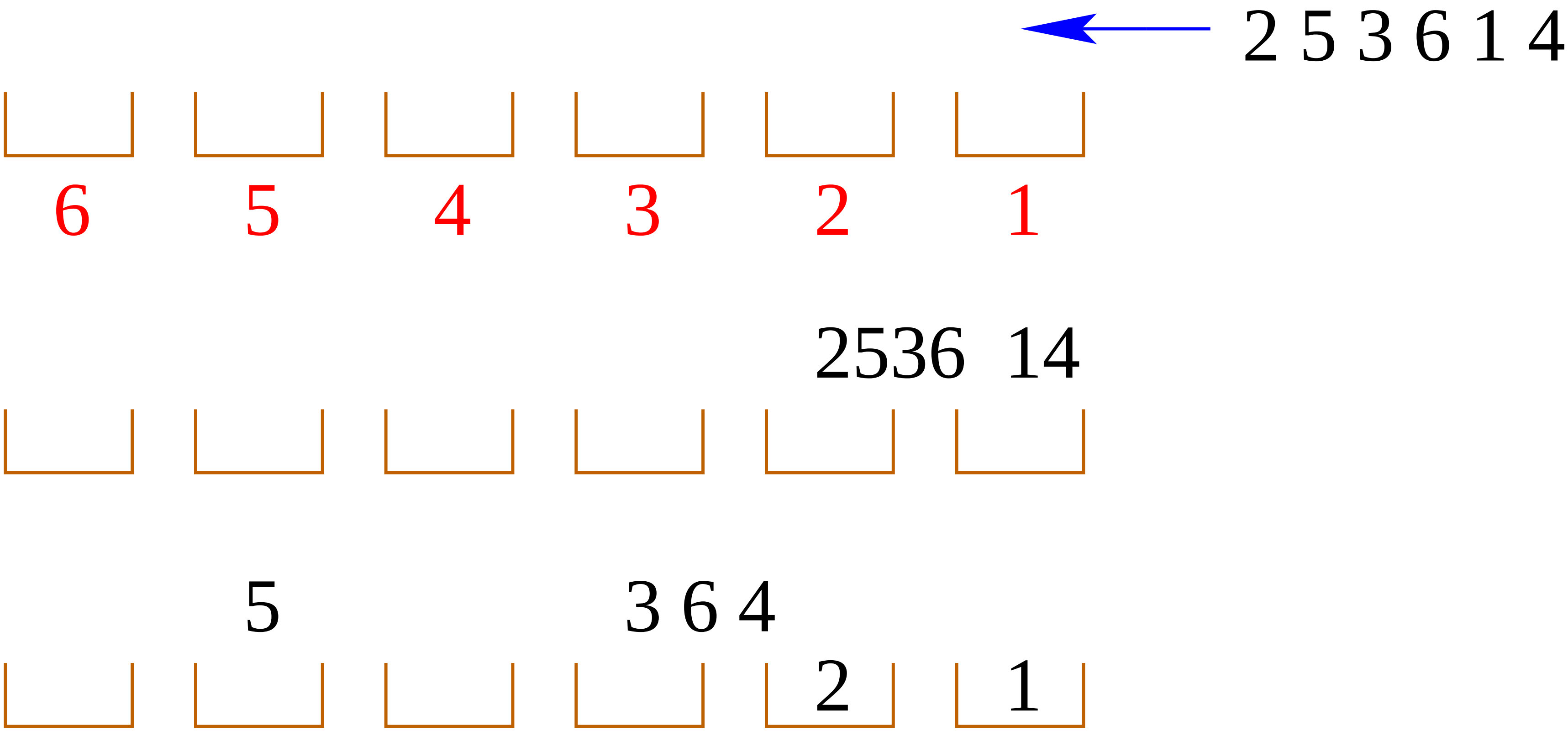
Boarding process

2 5 3 6 1 4

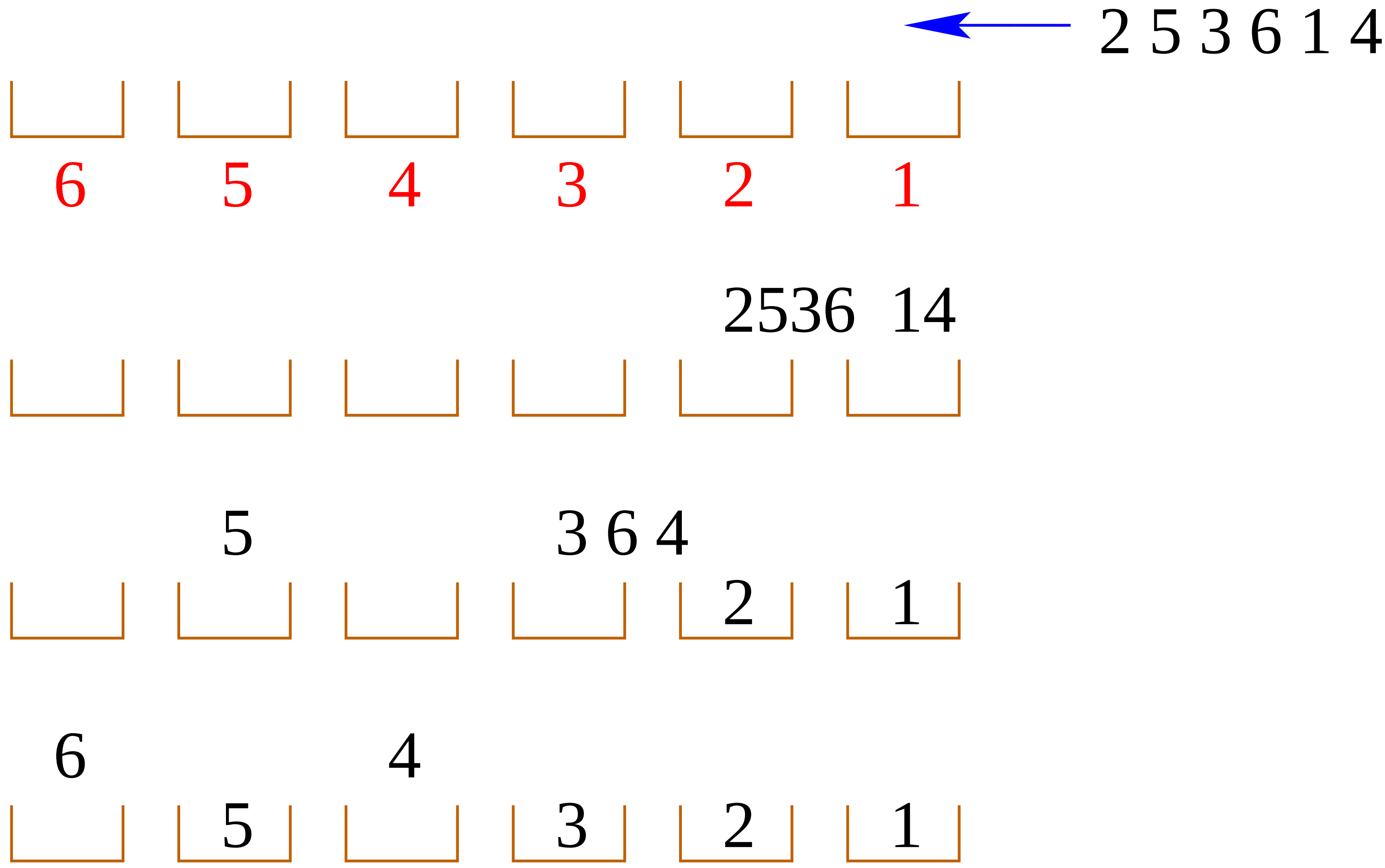
6 5 4 3 2 1

2536 14

Boarding process



Boarding process



Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

Two conclusions:

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

Partitions

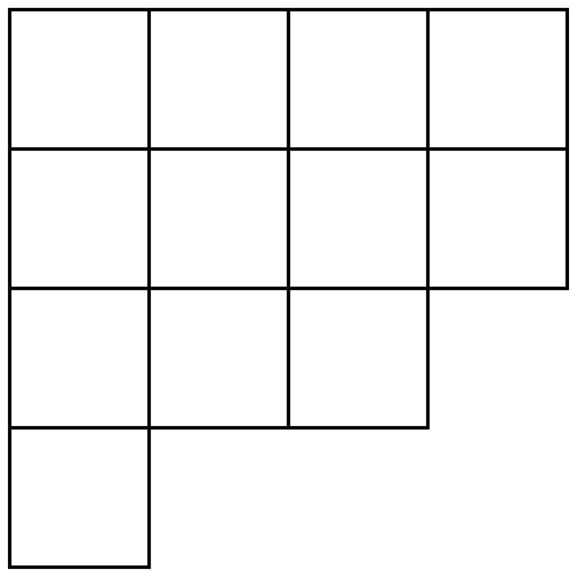
partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

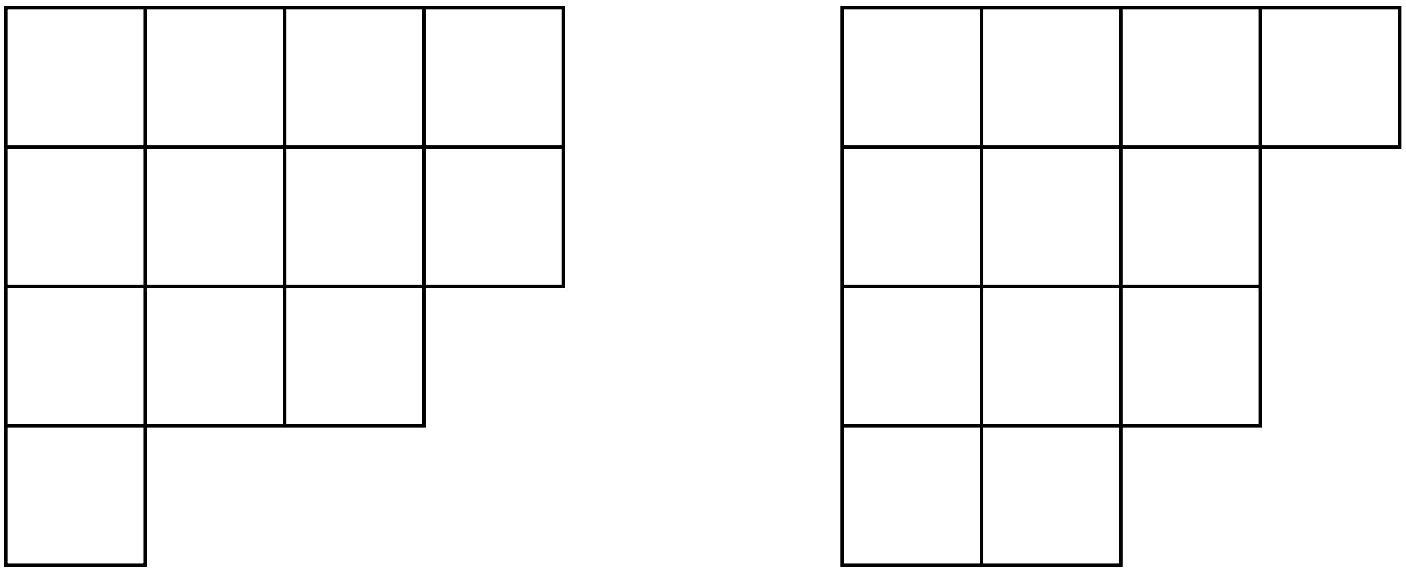
Young diagrams

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Conjugate partitions

$\lambda' = (4, 3, 3, 2)$, the **conjugate** partition to
$\lambda = (4, 4, 3, 2)$



λ λ'

Standard Young tableau

standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$$
\begin{array}{cc}
 & < \\
\wedge &
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 7 & 10 \\
\hline
3 & 5 & 8 & 12 \\
\hline
4 & 6 & 11 & \multicolumn{1}{|c}{} \\
\cline{1-3}
9 & \multicolumn{3}{|c}{} \\
\cline{1-1}
\end{array}
\end{array}
$$

f^{λ}

$f^{\lambda} = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

$1\,2\,3$	$1\,2\,4$	$1\,2\,5$	$1\,3\,4$	$1\,3\,5$
$4\,5$	$3\,5$	$3\,4$	$2\,5$	$2\,4$

f^λ

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

$1\,2\,3$	$1\,2\,4$	$1\,2\,5$	$1\,3\,4$	$1\,3\,5$
$4\,5$	$3\,5$	$3\,4$	$2\,5$	$2\,4$

$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

f^λ

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

$$\begin{array}{ccccc} 1\,2\,3 & 1\,2\,4 & 1\,2\,5 & 1\,3\,4 & 1\,3\,5 \\ 4\,5 & 3\,5 & 3\,4 & 2\,5 & 2\,4 \end{array}$$

$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the **symmetric group** of all permutations of $1, 2 \ldots, n$.

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

R = Gilbert de Beauregard Robinson
S = Craige Schensted (= Ea Ea)
K = Donald Ervin Knuth

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

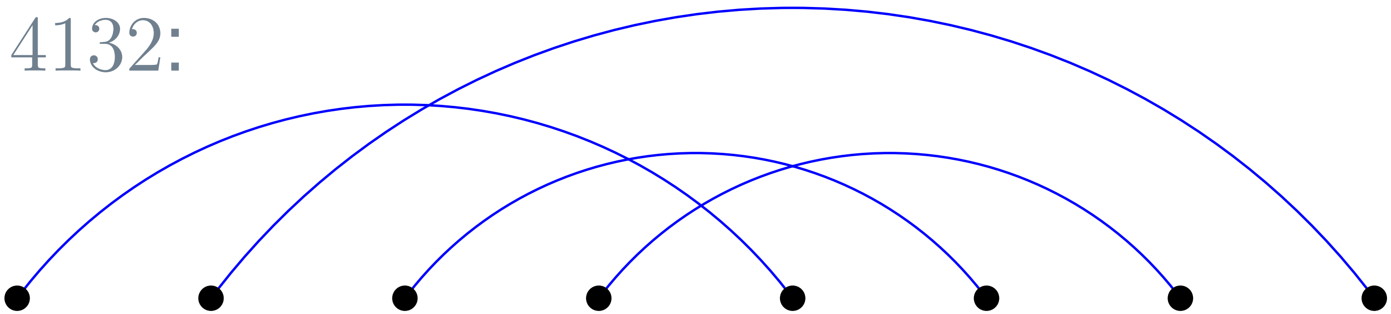
Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

R = Gilbert de Beauregard Robinson
S = Craige Schensted (= Ea Ea)
K = Donald Ervin Knuth

ea.ea.home.mindspring.com

Example of RSK

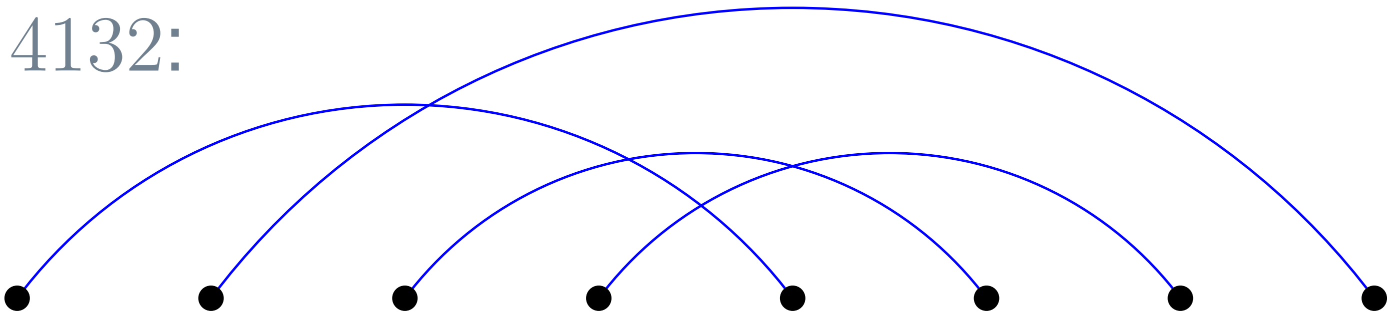
$w = 4132$:



4	1	3	2	4	3	2	1
4	1	13	12	12	12	1	ϕ
	4	4	3	3			
			4				

Example of RSK

$w = 4132$:



4	1	3	2	4	3	2	1
4	1 4	13 4	12 3 4	12 3	12	1	ϕ

$$(P, Q) = \begin{pmatrix} \begin{matrix} 1\,2 \\ 3 \\ 4 \end{matrix} & , & \begin{matrix} 1\,3 \\ 2 \\ 4 \end{matrix} \end{pmatrix}$$

Schensted's theorem

Theorem. *Let* $w \xrightarrow{\mathrm{rsk}} (P,Q)$, *where* *sh*$(P) = \mathrm{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \mathrm{is}(w) &= \text{longest row length} = \lambda_1 \\ \mathrm{ds}(w) &= \text{longest column length} = \lambda_1'. \end{aligned}$$

Schensted's theorem

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* *sh*$(P) = \text{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda'_1. \end{aligned}$$

Example. $4132 \xrightarrow{\text{rsk}} \left(\begin{array}{ll} 1\,2 & \\ 3 & \\ 4 & \end{array}, \begin{array}{ll} 1\,3 & \\ 2 & \\ 4 & \end{array} \right)$

$$\text{is}(w) = 2, \quad \text{ds}(w) = 3.$$

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$*. Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$*.*

Proof. Let $\lambda = \mathrm{sh}(w)$. If $\mathrm{is}(w) \leq p$ and $\mathrm{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda_1' \leq q$, so $\sum \lambda_i \leq pq$. $\square$

An extremal case

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \ : \ \mathrm{is}(w) = p, \ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

An extremal case

Corollary. *Say* $p \le q$. *Then*

$$\#\{w \in \mathfrak{S}_{pq} \ : \ \mathrm{is}(w) = p, \ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

Romik's theorem

Romik: let

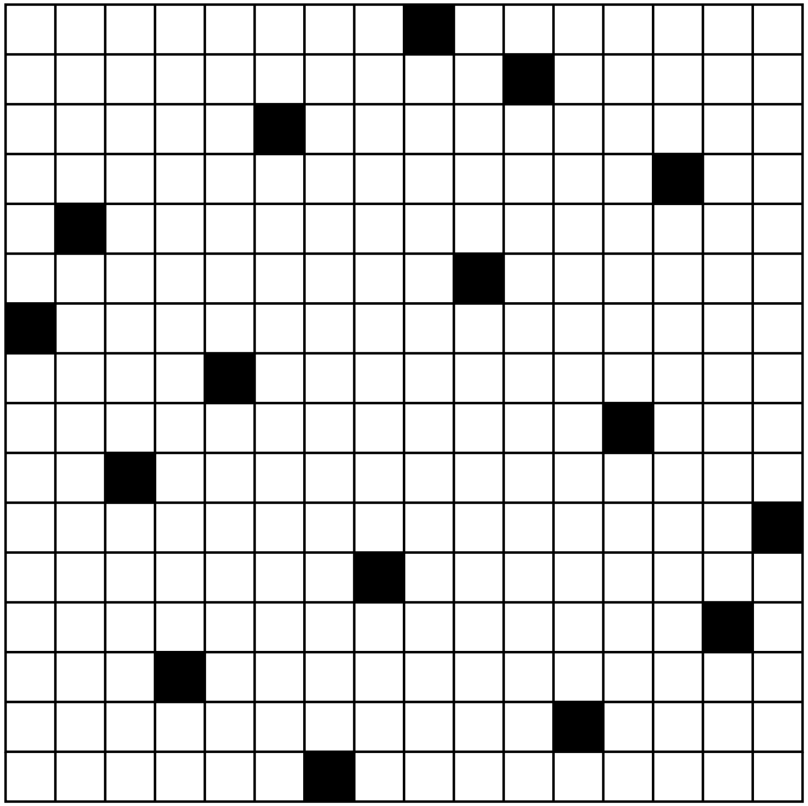
$$w \in \mathfrak{S}_{n^2}, \ \mathrm{is}(w) = \mathrm{ds}(w) = n.$$

Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $n \to \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

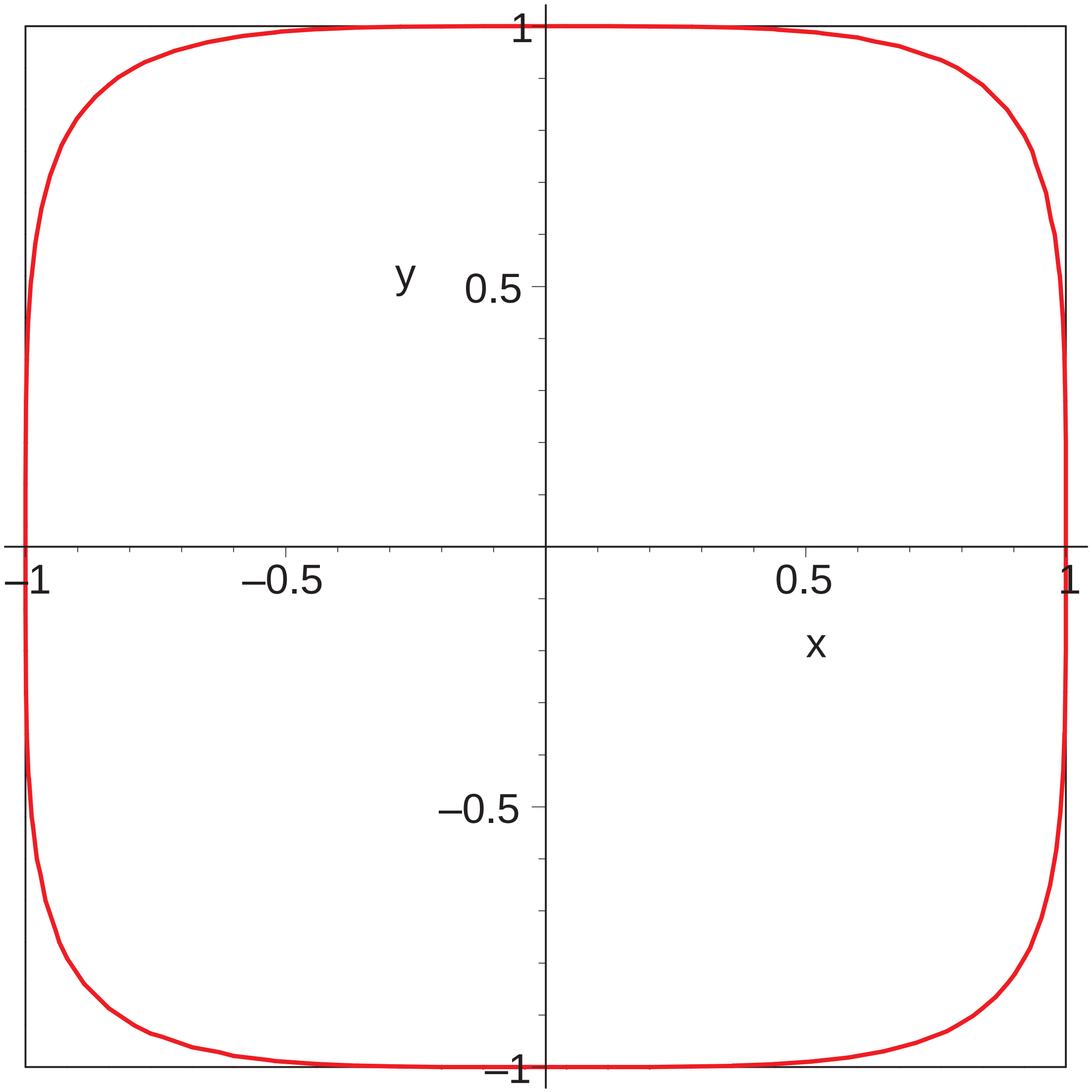
and will remain isolated outside this region.

An example



$$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 17$$

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$$



Area enclosed by curve

$$\begin{aligned}\alpha &= 8\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-(t/3)^2)}}dt \\ &\quad -6\int_0^1 \sqrt{\frac{1-(t/3)^2}{1-t^2}}dt \\ &= 4(0.94545962\cdots)\end{aligned}$$

Expectation of is(w)

$$
\begin{aligned}
\boldsymbol{E(n)} &= \text{expectation of } \mathrm{is}(w),\ w \in \mathfrak{S}_n \\
&= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \\
&= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^{\lambda}\right)^2
\end{aligned}
$$

Expectation of $\mathrm{is}(w)$

$$\begin{aligned} \boldsymbol{E(n)} &= \text{expectation of } \mathrm{is}(w),\ w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \end{aligned}$$

Ulam: what is distribution of $\mathrm{is}(w)$? rate of growth of $E(n)$?

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \le c \le e.$$

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

Conjectured $c = 2$.

$c = 2$

Logan-Shepp, Vershik-Kerov (1977): $c = 2$

$c = 2$

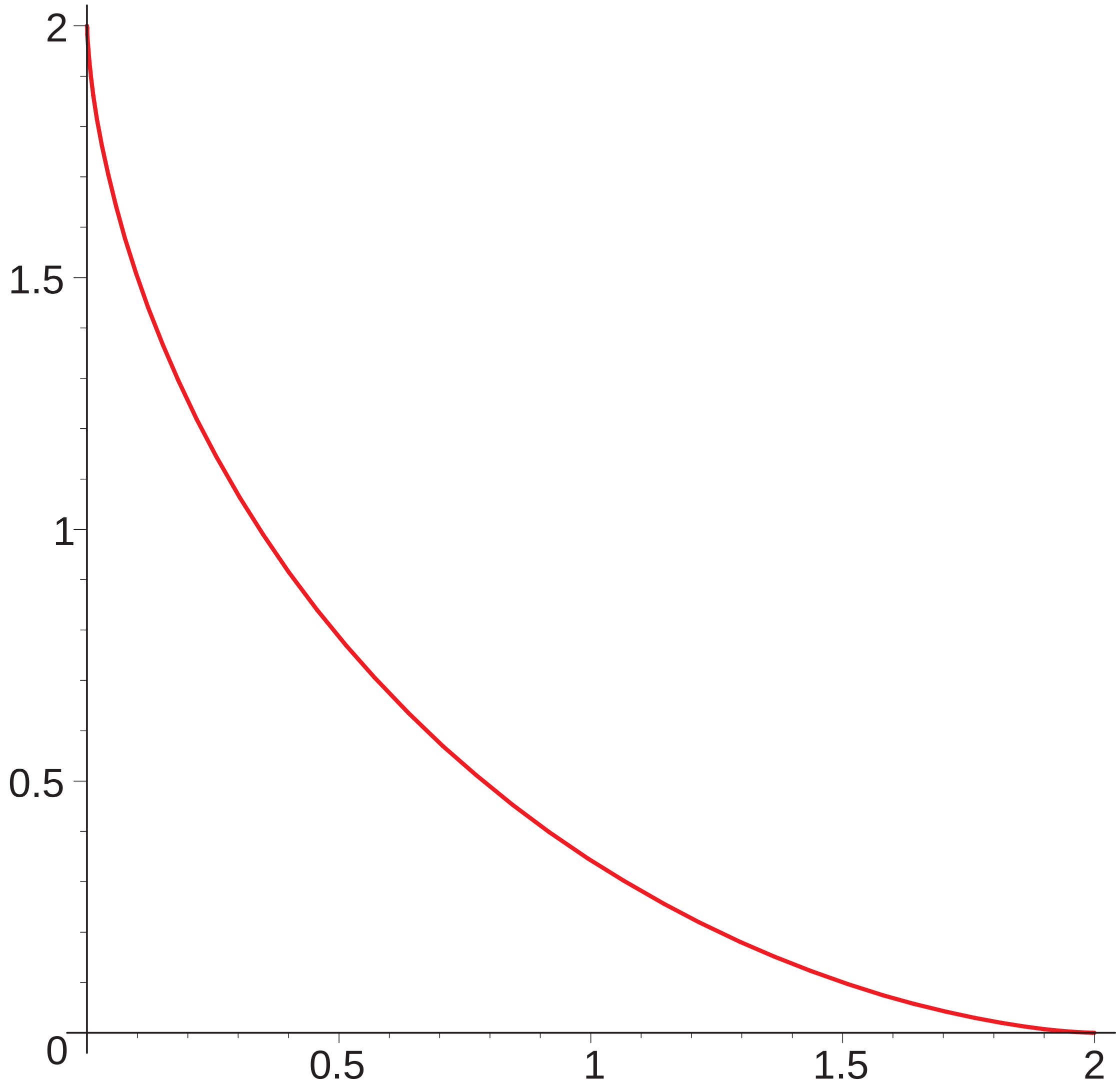
Logan-Shepp, Vershik-Kerov (1977): $c = 2$

Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \\ &\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 . \end{aligned}$$

Find “limiting shape” of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

The limiting curve



Equation of limiting curve

$$
\begin{aligned}
x &= y + 2\cos\theta \\
y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta)
\end{aligned}
$$

$$0 \le \theta \le \pi$$

$\mathbf{is}(\boldsymbol{w}) \leq \mathbf{2}$

$$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \;:\; \mathrm{is}_n(w) \leq k\}.$$

$\mathbf{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \,:\, \mathrm{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

$\mathbf{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

For $\geq$160 combinatorial interpretations of C_n, see

www-math.mit.edu/∼rstan/ec

Gessel's theorem

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0}\frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m.

The case $k = 2$

Example. $$\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2}$$

$$= I_0(2x)^2 - I_1(2x)^2$$

$$= \sum_{n\geq 0} C_n \frac{x^{2n}}{n!^2}.$$

Painlevé II equation

Baik-Deift-Johansson:

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

Painlevé II equation

Baik-Deift-Johansson:

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

The Tracy-Widom distribution

$$\boldsymbol{F(t)} = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

where $u(x)$ is the Painlevé II function.

The Baik-Deift-Johansson theorem

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

The Baik-Deift-Johansson theorem

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

Theorem. *As* $n \to \infty$,

$$\chi_n \to \chi \quad \text{in distribution,}$$

i.e.,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \leq t) = F(t).$$

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

Corollary to BDJ theorem.

$$
\begin{aligned}
E(n) &= 2\sqrt{n} + \left(\int t\, dF(t)\right) n^{1/6} + o(n^{1/6}) \\
&= 2\sqrt{n} - (1.7711\cdots)n^{1/6} + o(n^{1/6})
\end{aligned}
$$

Proof of BDJ theorem

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Origin of Tracy-Widom distribution

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

Tracy-Widom theorem

Tracy-Widom (1994): let α_1 denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right)$$

$$= F(t).$$

Random topologies

Is the connection between is(w) and GUE a coincidence?

Random topologies

Is the connection between is(w) and GUE a coincidence?

Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Two variations

1. Matchings

2. Alternating subsequences

Collaborators

Joint with:

Bill Chen 陈永川

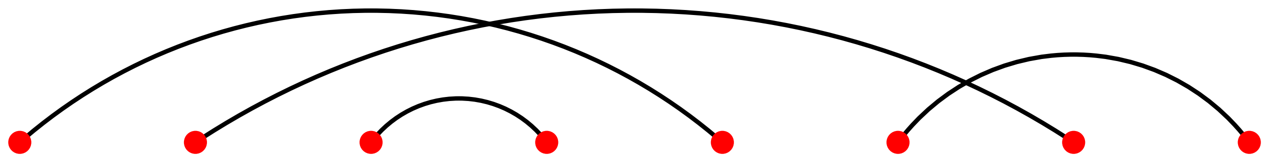
Eva Deng 邓玉平

Rosena Du 杜若霞

Catherine Yan 颜华菲

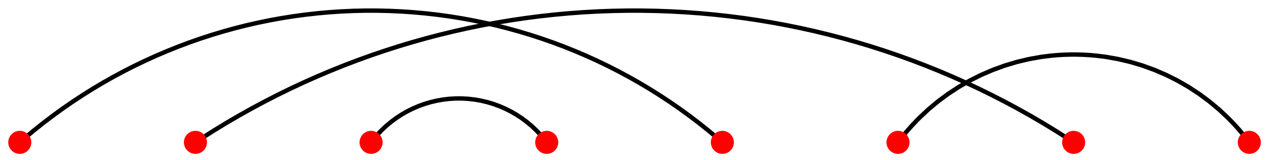
Complete matchings

(complete) matching:



Complete matchings

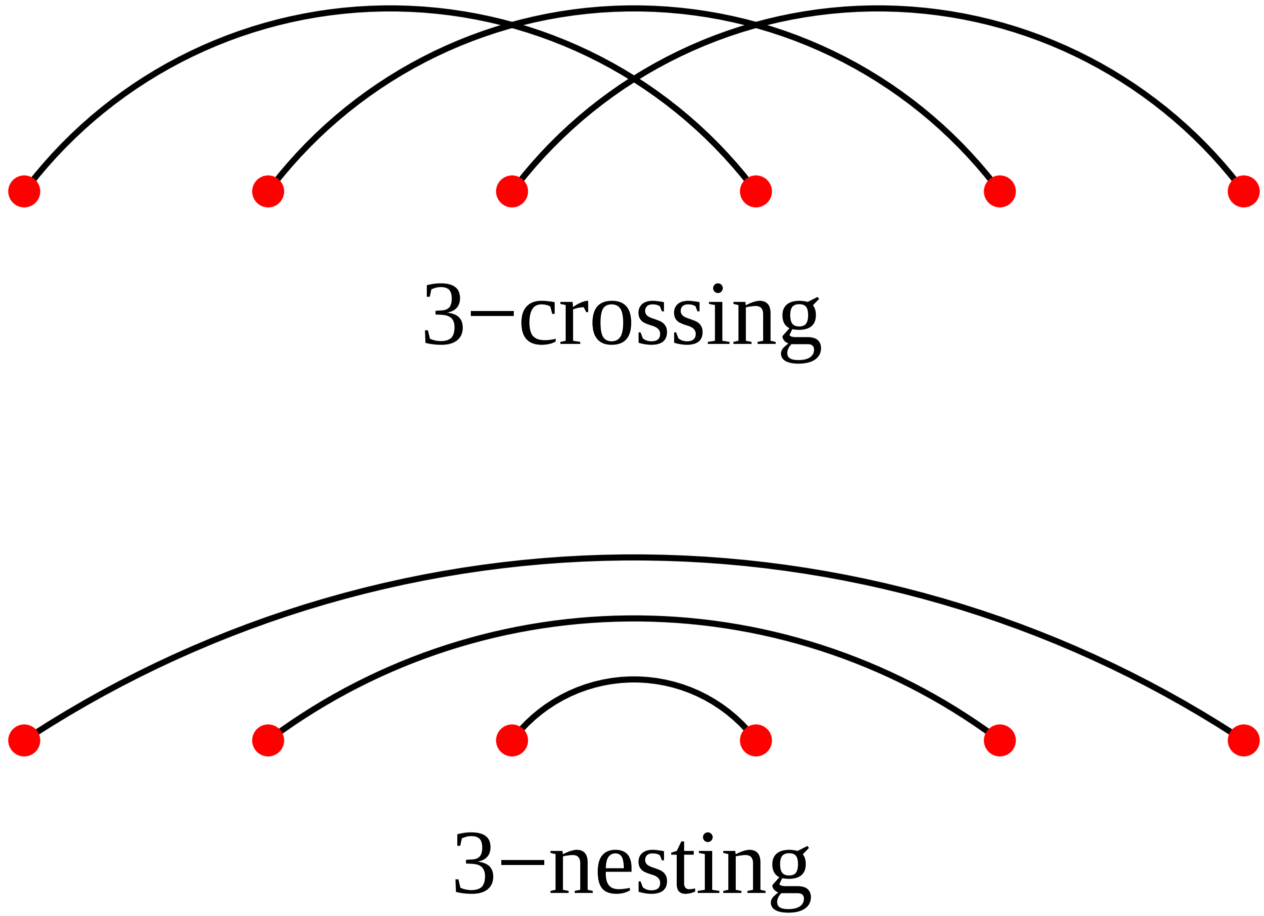
(complete) matching:



total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$$\mathbf{(2n-1)!!} := 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Crossings and nestings



Crossing and nesting number

$$\begin{aligned} \boldsymbol{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k \,:\, \exists\, k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k \,:\, \exists\, k\text{-nesting}\} \end{aligned}$$

Crossing and nesting number

$$\begin{aligned} \boldsymbol{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\} \end{aligned}$$

Theorem. *The number of matchings on* $[2n]$ *with no crossings (or with no nestings) is*

$$\boldsymbol{C_n} := \frac{1}{n+1}\binom{2n}{n}.$$

Main result on matchings

Theorem. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then*

$$\boldsymbol{f_n(i,j) = f_n(j,i)}.$$

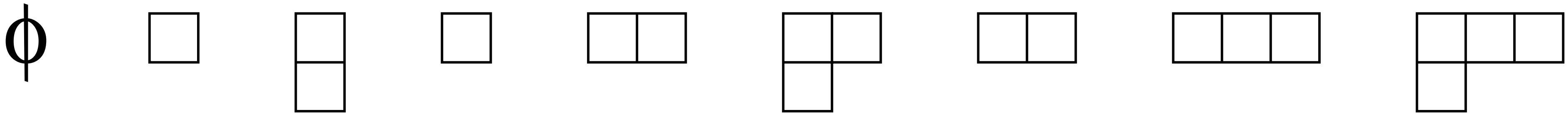
Main result on matchings

Theorem. *Let* $f_n(i,j) =$ *# matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$*. Then*

$$\boldsymbol{f_n(i,j) = f_n(j,i)}.$$

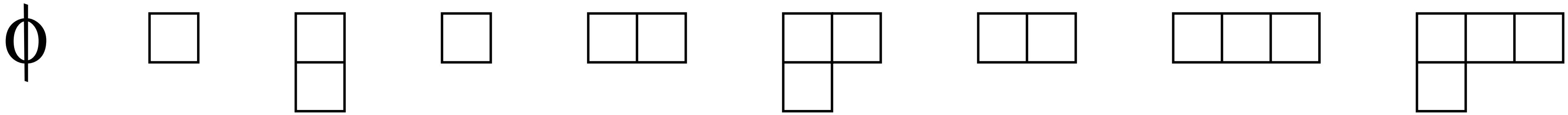
Corollary. *# matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = k$ *equals # matchings* M *on* $[2n]$ *with* $\mathrm{ne}(M) = k$*.*

Oscillating tableaux

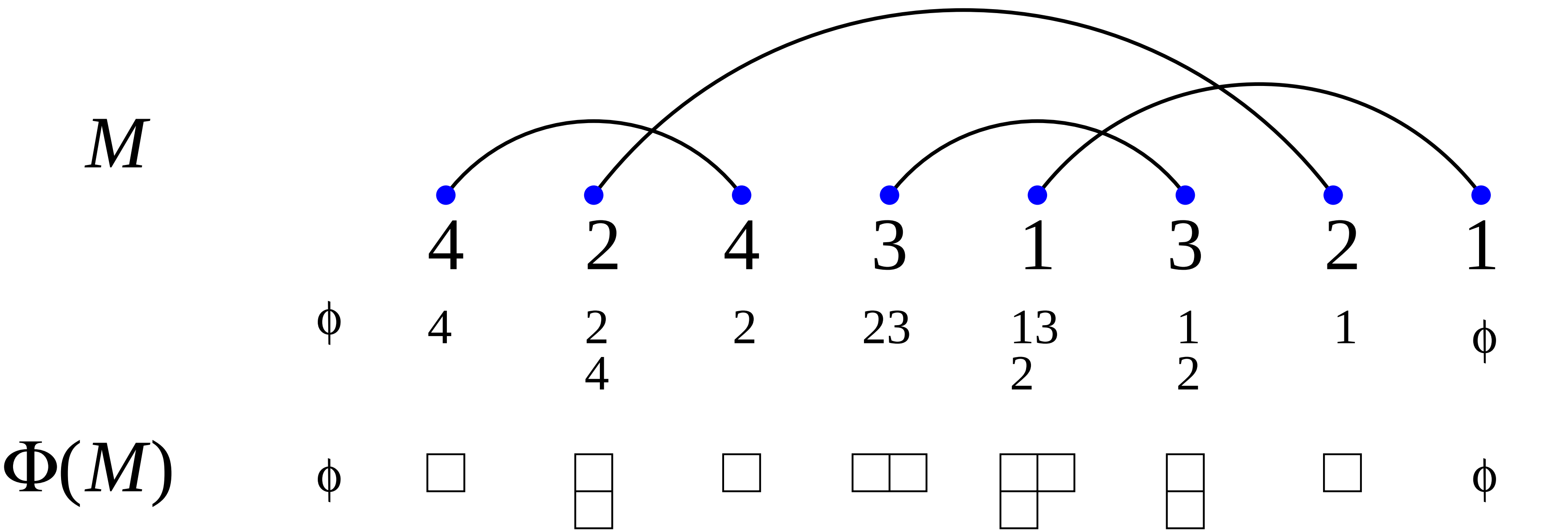


shape $(3,1)$, length 8

Oscillating tableaux



shape $(3, 1)$, length 8



Proof sketch

Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary. *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$.*

Proof sketch

Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary. *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$.*

(related to **Brauer algebra** of dimension $(2n-1)!!$).

Schensted for matchings

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \le i \le n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \le i \le n\}. \end{aligned}$$

Schensted for matchings

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \le i \le n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \le i \le n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

M'

Now let $\mathrm{cr}(M) = i$, $\mathrm{ne}(M) = j$, and

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Define M' by

$$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).$$

By Schensted's theorem for matchings,

$$\mathrm{cr}(M') = j, \quad \mathrm{ne}(M') = i.$$

Conclusion of proof

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne.

$\Rightarrow$ **Theorem.** *Let* $f_n(i,j) = $ *# matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Simple description?

Open: simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds.

$g_k(n)$

$$g_k(n) = \text{number of matching } M \text{ on } [2n] \text{ with } \mathrm{cro}(M) \le k$$

(matching analogue of $u_k(n)$)

Grabiner-Magyar theorem

Theorem. *Define*

$$\boldsymbol{H_k(x)} = \sum_n g_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Noncrossing example

Example. $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

Baik-Rains theorem

Baik-Rains (implicitly):

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{cr}_n(M)-\sqrt{2n}}{(2n)^{1/6}} \le \frac{t}{2}\right) = F_1(t),$$

where

$$\boldsymbol{F_1(t)} = \sqrt{F(t)}\exp\left(\frac{1}{2}\int_t^\infty u(s)ds\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(t)$ the Painlevé II function.

Alternating permutations

Alternating sequence of length k:

$$b_1 > b_2 < b_3 > b_4 < \cdots\ b_k$$

$\boldsymbol{E_n}$: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

Alternating permutations

Alternating sequence of length k:

$$b_1 > b_2 < b_3 > b_4 < \cdots\ b_k$$

$\boldsymbol{E_n}$: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

Desirée André (1879):

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Alternating subsequences?

$\mathbf{as}(\boldsymbol{w})$ = length of longest alternating subseq. of w

Alternating subsequences?

$\mathbf{as}(\boldsymbol{w})$ = length of longest alternating subseq. of w

$$w = \mathbf{5}6\mathbf{2}1\mathbf{834}7 \Rightarrow \mathrm{as}(w) = 5$$

The main lemma

MAIN LEMMA. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains* n.

The main lemma

MAIN LEMMA. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains* n.

$$\begin{aligned} \boldsymbol{a_k(n)} &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\} \\ \boldsymbol{b_k(n)} &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) \le k\}. \end{aligned}$$

Recurrence for $a_k(n)$

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

The main generating function

$$\text{Define } \boldsymbol{B(x,t)} = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\begin{aligned} \Rightarrow \quad b_1(n) &= 1 \\ b_2(n) &= n \\ b_3(n) &= \tfrac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \tfrac{1}{8}(4^n - (2n-4)2^n) \\ &\vdots \end{aligned}$$

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow \quad \begin{aligned} b_1(n) &= 1 \\ b_2(n) &= n \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \frac{1}{8}(4^n - (2n-4)2^n) \\ &\vdots \end{aligned}$$

no such formulas for longest increasing subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{A(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \mathrm{as}(w),$$

the **expectation** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{A(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w),$$

the **expectation** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Recall $E(n) \sim 2\sqrt{n}$.

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{A(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w),$$

the **expectation** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Recall $E(n) \sim 2\sqrt{n}$.

Corollary.

$$A(n) = \frac{4n+1}{6},\ n \geq 2$$

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\left(\mathrm{as}(w) - \frac{4n+1}{6}\right)^2,\ n\geq 2$$

the **variance** of $\mathrm{as}(n)$ for $w\in\mathfrak{S}_n$

Variance of $\mathrm{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \; n \geq 2$$

the **variance** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Corollary.

$$V(n) = \frac{8}{45} n - \frac{13}{180}, \; n \geq 4$$

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\left(\mathrm{as}(w) - \frac{4n+1}{6}\right)^2, \ n \geq 2$$

the **variance** of $\mathrm{as}(n)$ for $w \in \mathfrak{S}_n$

Corollary.

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \ n \geq 4$$

similar results for higher moments

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \leq t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\,ds$$

(Gaussian distribution)

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \le t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\,ds$$

(Gaussian distribution)


THE END