THE RSK ALGORITHM AND ITS APPLICATIONS

Lecture 1: An overview of RSK

1 2 3 4 5 6	1 2 3 5 4 6	1 2 3 6 4 5
1 2 4 5 3 6	1 2 4 6 3 5	1 2 5 6 3 4
1 3 4 5 2 6	1 3 4 6 2 5	1 3 5 6 2 4

$$f^{4,2} = 9$$

hook lengths :

5 4 2 1
2 1

$$f^{4,2} = \frac{6!}{5 \cdot 4 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 9$$

$w = 4273615$

4	1

2	1
4	2

2 **7**	1 3
4	2

2 **3**	1 3
4 **7**	2 4

2 3 **6**	1 3 5
4 7	2 4

1 3 6	1 3 5
2 7	2 4
4	6

1 3 **5**	1 3 5
2 **6**	2 4
4 **7**	6 7

$\boldsymbol{\chi^\lambda}$: irred. character of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$

$$f^\lambda = \chi^\lambda(1) = \dim \chi^\lambda$$

$$\sum_{\lambda \vdash n} \left(f^\lambda\right)^2 = n!$$

First symmetry property.

$$w = 4273615 \xrightarrow{\text{rsk}} \begin{array}{lll} 1&3&5 \\ 2&6& \\ 4&7& \end{array} \quad \begin{array}{lll} 1&3&5 \\ 2&4& \\ 6&7& \end{array}$$

$$w^{-1} = 6241753 \xrightarrow{\text{rsk}} \begin{array}{lll} 1&3&5 \\ 2&4& \\ 6&7& \end{array} \quad \begin{array}{lll} 1&3&5 \\ 2&6& \\ 4&7& \end{array}$$

Theorem (Schützenberger) *If* $w \stackrel{\text{rsk}}{\to} (P, Q)$, *then*

$$w^{-1} \stackrel{\text{rsk}}{\to} (Q, P).$$

Corollary. *Let* $t(n)$ *denote the number of SYT with* n *squares. Then*

$$t(n) = \#\{w \in \mathfrak{S}_n : w^2 = 1\}$$

$$\sum_{n \geq 0} t(n) \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right).$$

$$t(n) = \sum_{\lambda \vdash n} f^{\lambda} = \sum_{\lambda \vdash n} \chi^{\lambda}(1)$$

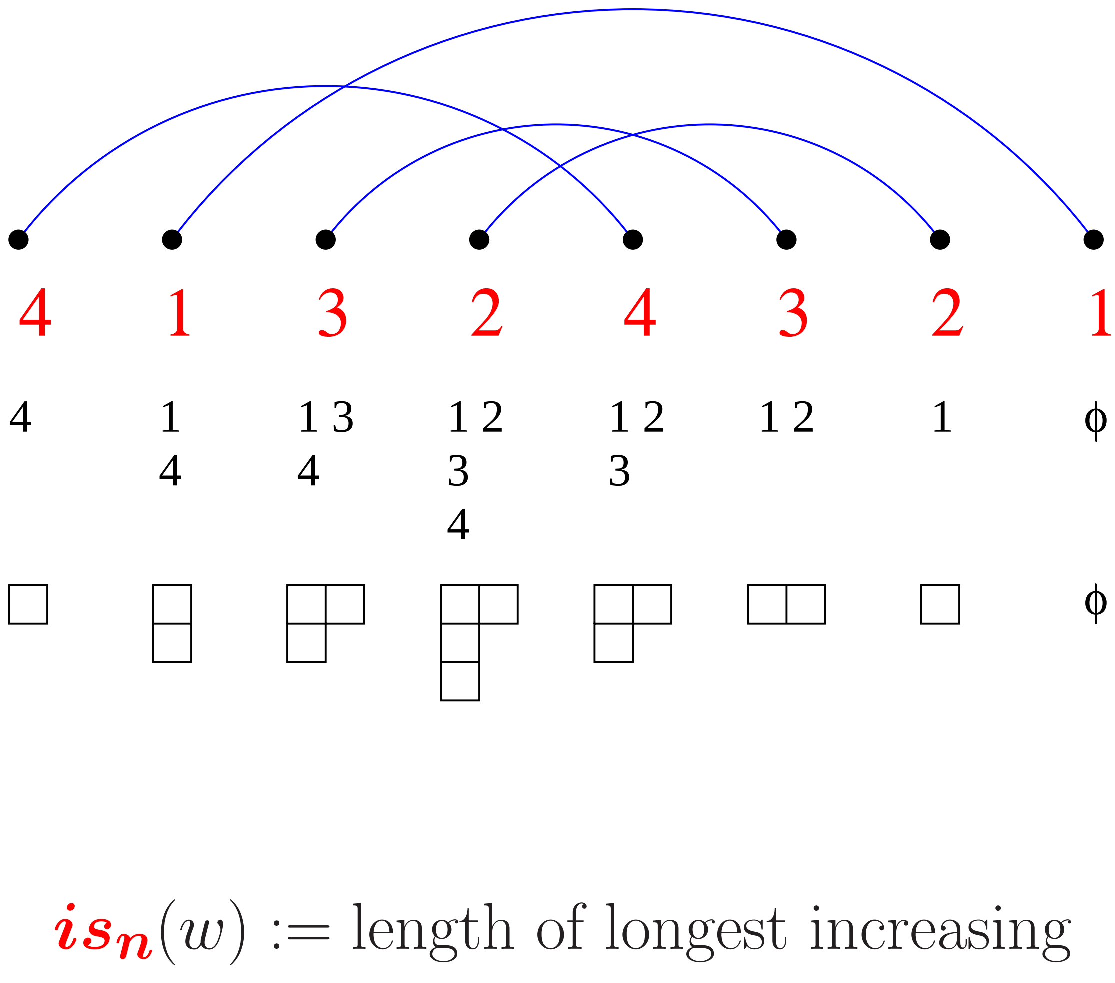
Theorem (Frobenius). *Let G be a finite group and $\hat{G}$ its set of (complex) irreducible characters. Then*

$$\sum_{\chi \in \hat{G}} \chi(1) = \#\{w \in G \,:\, w^2 = 1\}$$

if and only if every representation of G is equivalent to a real representation (true for $\mathfrak{S}_n$).

$$w = 318496725$$

$$w \xrightarrow{\text{rsk}} \begin{array}{llll} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & \\ 8 & 9 & & \end{array} \qquad \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & \\ 8 & 9 & & \end{array}$$



$$\boldsymbol{is_n}(w) := \text{length of longest increasing subsequence of } w \in \mathfrak{S}_n$$

Theorem (easy). *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, $\text{shape}(P) = (\lambda_1, \lambda_2, \ldots)$. *Then*

$$\text{is}(w) = \lambda_1.$$

Second symmetry property.

$$w = a_1 a_2 \cdots a_n, \quad w^r := a_n \cdots a_2 a_1$$

Theorem (Schensted) *If* $w \stackrel{\text{rsk}}{\rightarrow} (P, Q)$, *then*

$$w^r \stackrel{\text{rsk}}{\rightarrow} (P^t, \text{evac}(Q)^t).$$

Corollary. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, $\text{shape}(P) = (\lambda_1, \lambda_2, \ldots)$. *Then*

$$\text{ds}(w) = \lambda_1' = \ell(\lambda).$$

Corollary (Erdős-Szekeres). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either*

$$\text{is}(w) > p \ \text{ or } \ \text{ds}(w) > q.$$

Theorem (Greene). *Let* $w \xrightarrow{\text{rsk}} (P, Q)$ *and* $\text{shape}(P) = \lambda = (\lambda_1, \lambda_2, \cdots)$. *Then* $\lambda_1 + \cdots + \lambda_i$ *(resp.,* $\lambda'_1 + \cdots + \lambda'_i$*) is equal to the maximum cardinality of the union of* i *increasing (resp., decreasing) subsequences of* w.

Example. $w = 247951368$

$$\lambda_1 = 5, \quad \lambda_1 + \lambda_2 = 8 \ (= 4 + 4)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 9 \Rightarrow \lambda = (5, 3, 1)$$

$$\lambda'_1 = 3, \quad \lambda'_1 + \lambda'_2 = 5$$

$$\lambda'_1 + \lambda'_2 + \lambda'_3 = 3, \quad \lambda'_1 + \cdots + \lambda'_4 = 8$$

$$\lambda'_1 + \cdots + \lambda'_5 = 9 \Rightarrow \lambda' = (3, 2, 2, 1, 1)$$

$$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is}_n(w) \le k\}.$$

Gessel:

$$\sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{i-j}(2x) \right]_{i,j=1}^{k},$$

where

$$I_m(x) = \sum_{j \ge 0} \frac{x^{m+2j}}{j!(m+j)!}.$$

$$\sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{i-j}(2x) \right]_{i,j=1}^{k},$$

where

$$I_m(x) = \sum_{j \ge 0} \frac{x^{m+2j}}{j!\,(m+j)!}.$$

Baik-Deift-Johansson: Let $\mathbf{Ai}(\boldsymbol{x})$ denote the **Airy function**:

$$\frac{d^2}{dx^2}\mathrm{Ai}(x) = x\,\mathrm{Ai}(x)$$

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad (x \to \infty)$$

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*)$$

$$u(x) \sim -\mathrm{Ai}(x)\ (x \to \infty).$$

(∗) is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

Tracy-Widom distribution:

$$\boldsymbol{F(t)}$$
$$= \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

Let $\boldsymbol{\chi}$ be a random variable with distribution F, and let $\boldsymbol{\chi_n}$ be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

Theorem. *As* $n \to \infty$,

$$\chi_n \to \chi \quad \text{in distribution},$$

i.e.,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \leq t) = F(t).$$

Theorem. *For any* $m = 0, 1, 2, \ldots,$

$$\lim_{n\to\infty} E(\chi_n^m) = E(\chi^m).$$

Corollary. $\displaystyle\lim_{n\to\infty} \frac{\mathrm{Var}(\mathrm{is}_n)}{n^{1/3}}$

$$= \int t^2\, dF(t) - \left(\int t\, dF(t)\right)^2$$
$$= 0.8132\cdots$$

$$\lim_{n\to\infty} \frac{E(\mathrm{is}_n) - 2\sqrt{n}}{n^{1/6}}$$

$$= \int t\, dF(t)$$
$$= -1.7711\cdots$$

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution $F(t)$ come from?

$$\boldsymbol{F(t)}$$

$$= \exp\left(- \int_t^\infty (x-t)u(x)^2\, dx \right)$$

Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $\boldsymbol{M} = (\boldsymbol{M_{ij}})$ with probability distribution

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii}$$

$$\cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

Let $\boldsymbol{\alpha_1}$ denote the largest eigenvalue of $\boldsymbol{M}$. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

More generally, let

$$\boldsymbol{\lambda_1 + \cdots + \lambda_k}$$

be the largest number of elements in the union of k increasing subsequences of $w \in \mathfrak{S}_n$, and let $\boldsymbol{\alpha_k}$ be the kth largest eigenvalue of the GUE matrix M. Then as $n \to \infty$, λ_k and α_k are equidistributed, up to scaling (Okounkov, Borodin-Okounkov-Olshanski, Johansson).

Is the connection between is(w) and GUE a coincidence?

The proof of Okounkov provides a connection, *via* the theory of **random topologies on surfaces**.

THE RSK ALGORITHM AND ITS APPLICATIONS

Lecture 2: Some algebraic connections

A **semistandard tableau** T of shape $(5, 4, 3)$:

$$\begin{matrix} 6 & 6 & 4 & 2 & 2 \\ 4 & 4 & 3 & 1 & \\ 2 & 1 & 1 & & \end{matrix}$$

$$x^T = x_1^3 x_2^3 x_3 x_4^3 x_6^2$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 3 & 3 & 2 & 3 \end{pmatrix}$$

$$\mathbf{1} \qquad\qquad 1$$

$$1\,\mathbf{2} \qquad\qquad 1\,1$$

$$\begin{array}{l} 1\,\mathbf{1} \\ \mathbf{2} \end{array} \qquad\qquad \begin{array}{l} 1\,1 \\ 2 \end{array}$$

$$\begin{array}{l} 1\,1\,\mathbf{3} \\ 2 \end{array} \qquad\qquad \begin{array}{l} 1\,1\,2 \\ 2 \end{array}$$

$$\begin{array}{l} 1\,1\,3\,\mathbf{3} \\ 2 \end{array} \qquad\qquad \begin{array}{l} 1\,1\,2\,2 \\ 2 \end{array}$$

$$\begin{array}{l} 1\,1\,\mathbf{2}\,3 \\ 2\,\mathbf{3} \end{array} \qquad\qquad \begin{array}{l} 1\,1\,2\,2 \\ 2\,3 \end{array}$$

$$\begin{array}{l} 1\,1\,2\,3\,\mathbf{3} \\ 2\,3 \end{array} \qquad\qquad \begin{array}{l} 1\,1\,2\,2\,3 \\ 2\,3 \end{array}$$

1 1	1 1	1 2	1 3
2	3	2	3

2 3	2 2	1 2	1 3
3	3	3	2

Schur function:

$$s_{2,1}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2$$
$$+ x_2 x_3^2 + x_2^2 x_3 + 2 x_1 x_2 x_3$$

Cauchy identity:

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$$

Define the group homomorphism (**polynomial representation**)

$$\varphi : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(3,\mathbb{C})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$$

If $A \in \mathrm{GL}(2,\mathbb{C})$ has eigenvalues x, y, then $\varphi(A)$ has eigenvalues x^2, xy, y^2.

$$\begin{aligned} \textbf{char}(\varphi) &= \mathrm{tr}\,\varphi(A) \\ &= x^2 + xy + y^2 \\ &= s_2(x,y). \end{aligned}$$

The irreducible polynomial representations φ_λ of $\mathrm{GL}(n, \mathbb{C})$ are indexed by **partitions**

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n,$$

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

$$\begin{aligned} \textbf{char}(\varphi_\lambda) &:= \operatorname{tr} \varphi_\lambda(A) \\ &= s_\lambda(x_1, \dots, x_n), \end{aligned}$$

the Schur function indexed by λ, where A has eigenvalues $x_1, \dots, x_n$.

Cauchy identity:

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x)s_\lambda(y)$$

$$S(V\otimes W)=\bigoplus_\lambda (V_\lambda\otimes W_\lambda)$$

$$(x_1+x_2+\cdots)^n=\sum_\lambda f^\lambda s_\lambda(x)$$

$$V^{\otimes n}=\bigoplus_\lambda \left(M^\lambda\otimes V_\lambda\right)$$

$$n!=\sum_\lambda (f^\lambda)^2$$

$$\mathbb{Q}\mathfrak{S}_n=\bigoplus_{\lambda\vdash n}(M^\lambda\otimes M^\lambda)$$

Plane partitions.

$$\begin{array}{llllll} 5 & 5 & 3 & 2 & 2 & 1 \\ 5 & 3 & 3 & 1 & 1 & \\ 5 & 2 & 2 & 1 & 1 & \\ 2 & 2 & 1 & & & \end{array}$$

$$\begin{array}{cccccc} 3 & 21 & 111 & 2 & 1 & 11 \\ & & & 1 & 1 & 1 \\ & & & & 1 & \end{array}$$

$$a(3) = 6$$

MacMahon:

$$\sum_{n \geq 0} a(n) x^n = \prod_{i \geq 1} (1 - x^i)^{-i}.$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{rsk}} \quad \begin{matrix} 3\,3\,3\,3\,2 \\ 2\,2\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 4\,4\,3\,2\,1 \\ 3\,2\,2 \\ 1 \end{matrix}$$

$$\xrightarrow{\text{merge}} \quad \begin{matrix} 3\,3\,3\,3\,2 \\ 3\,3\,3\,1 \\ 3\,3\,2 \\ 2\,1 \end{matrix}$$

$$\xrightarrow{\text{rowconj}} \quad \begin{matrix} 5\,5\,4 \\ 4\,3\,3 \\ 3\,3\,2 \\ 2\,1 \end{matrix} = \pi_A$$

$$|\pi_A| = \sum_{i,j} (i + j - 1) a_{ij}$$

$$\#(\text{rows of } \pi_A) = \#(\text{rows of } A)$$

$$\#(\text{columns of } \pi_A) = \#(\text{columns of } A)$$

$$\begin{aligned}
\Rightarrow \sum_{\pi} x^{|\pi|} &= \sum_{A} x^{(i+j-1)a_{ij}} \\
&= \prod_{i,j \geq 1} \left(\sum_{a_{ij} \geq 0} x^{(i+j-1)a_{ij}} \right) \\
&= \prod_{i,j \geq 1} (1 - x^{i+j-1})^{-1} \\
&= \prod_{i \geq 1} (1 - x^{i})^{-i}
\end{aligned}$$

RSK $\Rightarrow$

$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols}}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} (1 - x^{i+j-1})^{-1}$$

More difficult (MacMahon):

$$\sum_{\substack{\pi \\ \leq r \text{ rows} \\ \leq s \text{ cols} \\ \max \leq t}} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1 - x^{i+j+k-1}}{1 - x^{i+j+k-2}}$$

Symplectic tableau:

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}$$

$$\begin{array}{llllllll} 1 & 1 & \bar{1} & 2 & 4 & \bar{4} & \bar{4} & \bar{5} \\ 2 & 2 & \bar{2} & \bar{2} & \bar{4} & 5 & \bar{6} & \\ \bar{3} & 4 & 4 & \bar{5} & \bar{5} & & & \\ 5 & \bar{5} & 6 & 6 & & & & \end{array}$$

$$\begin{aligned} x^T &= x_1^{2-1}\, x_2^{3-2}\, x_3^{0-1}\, x_4^{3-3}\, x_5^{2-4}\, x_6^{2-1} \\ &= x_1\, x_2\, x_3^{-1}\, x_5^{-2}\, x_6. \end{aligned}$$

$$\mathrm{sp}_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_T x^T,$$

an irreducible character of $\mathrm{Sp}(2n)$.

Symplectic Cauchy identity (D. E. Littlewood):

$$\prod_{1\le i<j\le n}(1-t_it_j)\prod_{i,j=1}^{n}(1-t_ix_j)^{-1}\left(1-t_ix_j^{-1}\right)^{-1}$$
$$=\sum_{\substack{\lambda\\ \ell(\lambda)\le n}}\mathrm{sp}_\lambda(x_1^{\pm1},\dots,x_n^{\pm1})s_\lambda(t_1,\dots,t_n).$$

Bijective proof by S. Sundaram (1990) based on insertion algorithm of A. Berele (1986).

Brauer algebra $\mathfrak{B}_n(k)$: centralizer algebra of action of $\mathrm{Sp}(V)$ on $V^{\otimes n}$, where $\dim V = 2k$. For $k \geq n$,

$\dim \mathfrak{B}_n(k) = 1{\cdot}3{\cdot}5 \cdots (2n-1) = (2n-1)!!.$

$$V^{\otimes n} = \bigoplus_{\lambda} (N^\lambda \otimes W_\lambda)$$

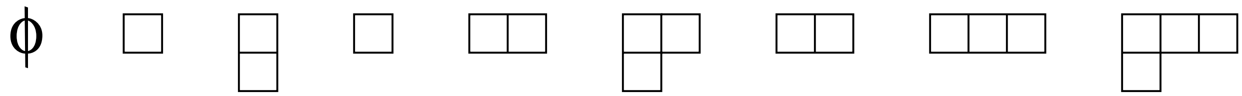
$$\Rightarrow (x_1 + x_2 + \cdots)^n = \sum_{\lambda} \tilde{f}^\lambda \, \mathrm{sp}_\lambda,$$

where $\tilde{f}^\lambda = \dim N^\lambda$. Thus

$$\begin{aligned} \sum_{\lambda} \left(\tilde{f}^\lambda\right)^2 &= \dim \mathfrak{B}_n(k) \\ &= (2n-1)!! \\ &= \#\{w \in \mathfrak{S}_n \,:\, w^2 = 1,\ w(i) \neq i\} \end{aligned}$$

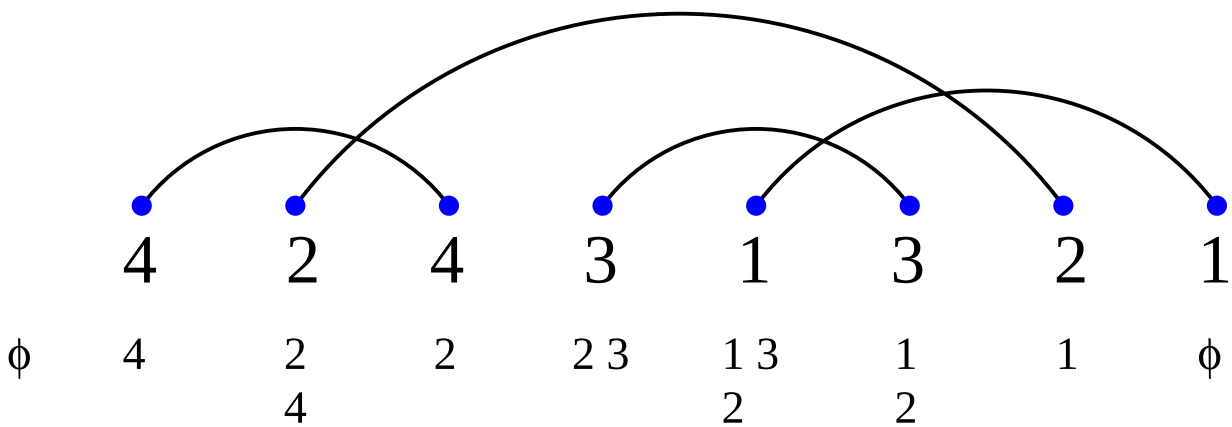
(matching)

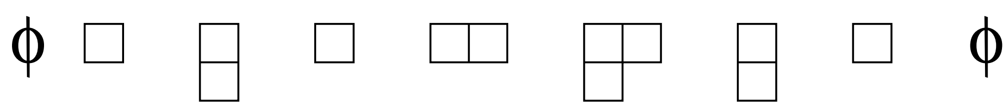
Oscillating tableaux.



shape $(3,1)$, length 8

$$\tilde{f}^{\lambda} = \#\{\text{osc. tab. of shape } \lambda, \text{ length } n\}$$



$\Phi(M) = ($  $)$

$$\Rightarrow \sum_{\lambda} \left(\tilde{f}^{\lambda}\right)^2 = (2n-1)!!$$

The nilpotent flag variety

$$\dim_{\mathbb{C}} V = n, \quad A : V \to V \text{ nilpotent}$$

$$\boldsymbol{\mathcal{N}(A)} =$$

$$\{\emptyset = V_0 \subset V_1 \subset \cdots \subset V_n = V \ : \ AV_i \subseteq V_i\},$$

subvariety of complete flag variety $\mathrm{GL}(V)/B$.

$$\boldsymbol{\lambda(A)} = (\lambda_1, \lambda_2, \ldots) \vdash n,$$

where $\lambda_1, \lambda_2, \ldots$ are the Jordan block sizes of λ.

Lemma. *Let*

$$\mathcal{F} : \emptyset = V_0 \subset V_1 \subset \cdots \subset V_n$$

be a flag in $\mathcal{N}(A)$. *Let* $\lambda^i = \lambda(A|_{V_i})$. *Then*

$$\emptyset = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n = \lambda,$$

an SYT $T = T(\mathcal{F})$.

Theorem. *The irreducible components* $\mathcal{N}_T(A)$ *of* $\mathcal{N}(A)$ *are given by*

$$\mathcal{N}_T(A) = \{\mathcal{F} \in \mathcal{N}(A) \, : \, T = T(\mathcal{F})\}.$$

Hence $\mathcal{N}(A)$ *has* f^λ *irreducible components.*

Relative position of flags. Let

$$\mathcal{F} : \emptyset = V_0 \subset V_1 \subset \cdots \subset V_n$$

$$\mathcal{G} : \emptyset = W_0 \subset W_1 \subset \cdots \subset W_n$$

$(v_1, \ldots, v_n)$: ordered basis such that

$$V_i = \text{span}\{v_1, \ldots, v_i\}$$

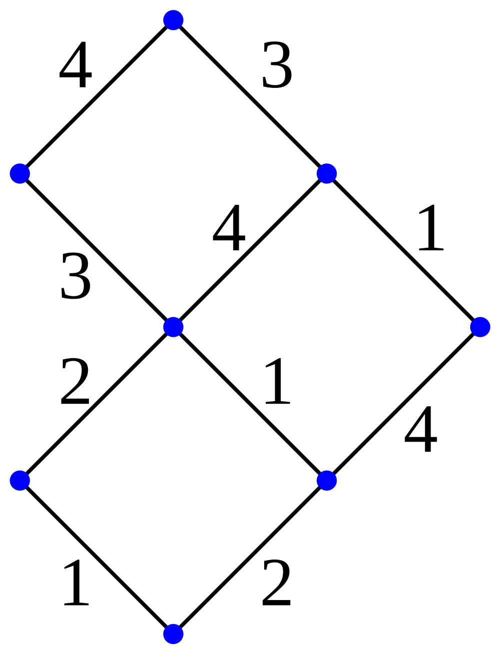
There is a unique ordered basis $(w_1, \ldots, w_n)$ such that

$$W_i = \text{span}\{w_1, \ldots, w_i\}$$

and w.r.t. $(v_1, \ldots, v_n)$ we have, e.g.,

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

$$d(\mathcal{F}, \mathcal{G}) = 2413 \in \mathfrak{S}_4$$



Theorem (Steinberg) *Let* $\mathcal{F} \in \mathcal{N}_P(A)$ *and* $\mathcal{G} \in \mathcal{N}_Q(A)$. *Let* $w = d(\mathcal{F}, \mathcal{G})$. *Then generically (Zariski-dense)*

$$w \xrightarrow{\text{rsk}} (P, Q).$$

THE RSK ALGORITHM AND ITS APPLICATIONS

Lecture 3: Some combinatorial connections

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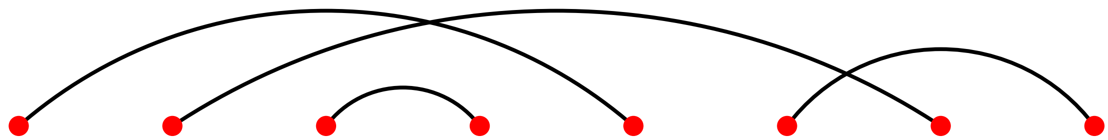
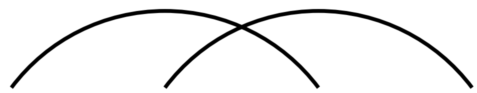
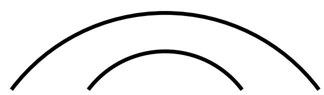
(complete) matching:



crossing: 

nesting: 

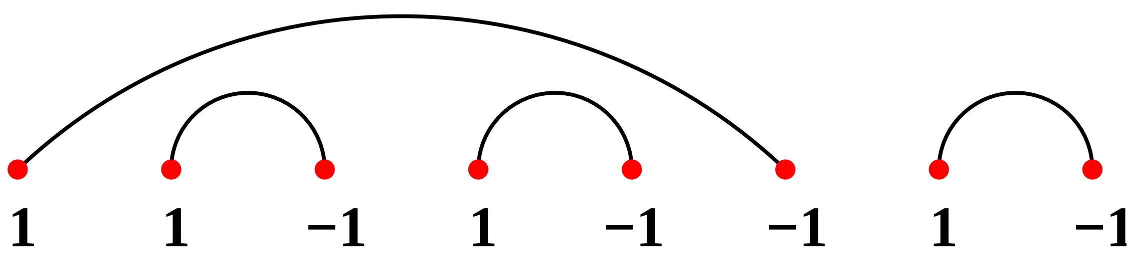
Theorem. *The number of matchings on* $[2n]$ *with no crossings (or with no nestings) is*

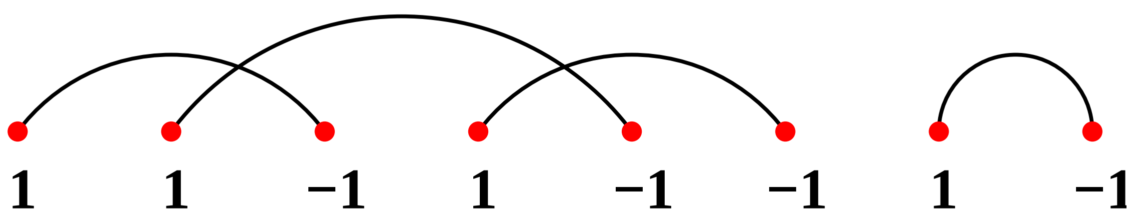
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

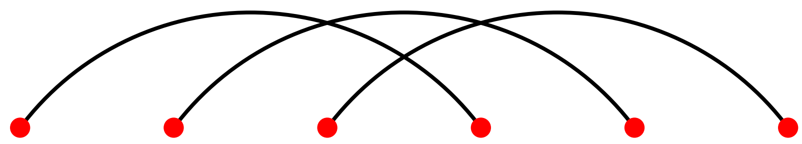
Recall:

$$C_n = \#\{a_1 \cdots a_{2n} : a_i = \pm 1,$$
$$a_1 + \cdots + a_i \geq 0, \sum a_i = 0\}$$

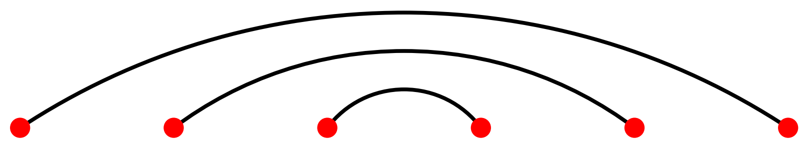
(**ballot sequence**).







3−crossing



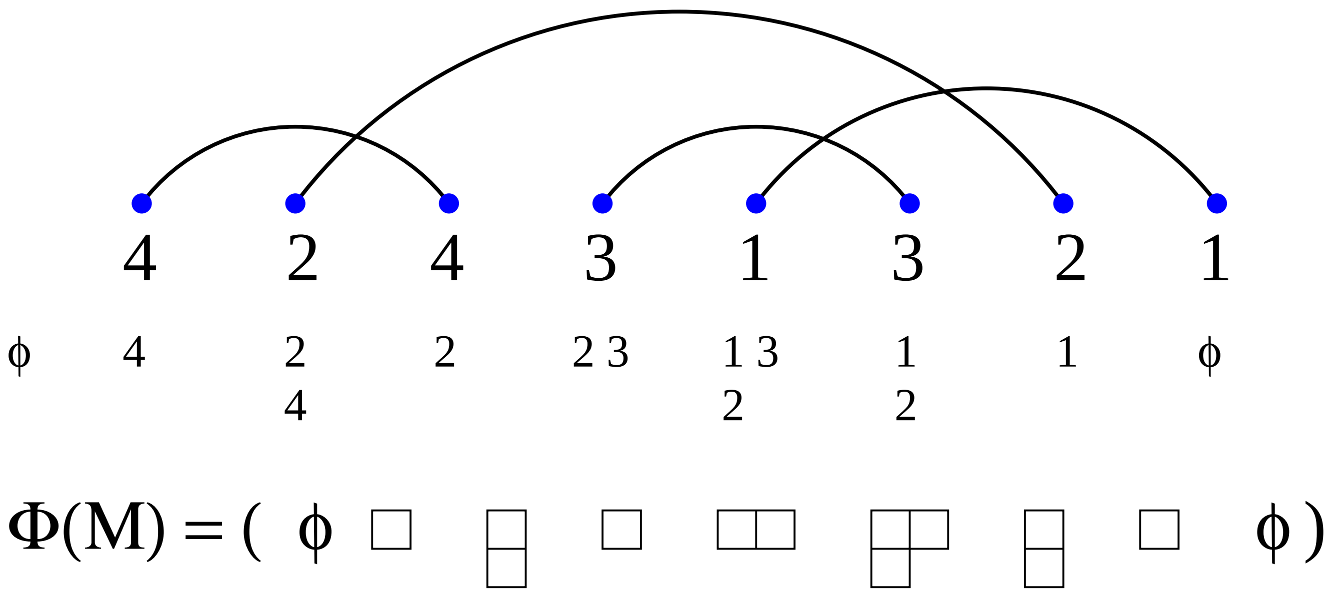
3−nesting

$$\begin{aligned} \boldsymbol{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\}. \end{aligned}$$

Theorem. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Corollary. *#* *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = k$ *equals* *#* *matchings* M *on* $[2n]$ *with* $\mathrm{ne}(M) = k$.

Recall “oscillating RSK”:



Theorem. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \mathrm{cr}(M) &= \text{max column length of any } \lambda^i \\ \mathrm{ne}(M) &= \text{max row length of any } \lambda^i. \end{aligned}$$

Corollary. $f_n(i,j) = f_n(j,i)$.

Proof. $\Phi(M') =$

$$(\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).$$

Enumeration of k-noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\mathrm{cr}(M) = 1$, (or with no nestings) is $\boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}$.

What about the number with $\mathrm{cr}(M) \leq k$?

Let $\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$. If $\mathrm{cr}(M) \leq k$, then

$$\lambda^i = (\lambda^i_1, \ldots, \lambda^i_k) \in \mathbb{N}^k.$$

Thus $\Phi(M)$ is a lattice path in a certain region of $\mathbb{R}^k$ (a fundamental chamber for the Weyl group of type B_n).

Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

$$\boldsymbol{f_k(n)} = \#\{M \text{ on } [2n] : \operatorname{cr}(M) \le k\}$$

$$\boldsymbol{F_k(x)} = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}$$

Theorem.

$$F_k(x) = \det \left[I_{i-j}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \ge 0} \frac{x^{m+2j}}{j!\,(m+j)!}.$$

Example. $k = 1$ (noncrossing matchings):

$$F_1(x) = I_0(2x) - I_2(2x) = \sum_{j\geq 0} C_j \frac{x^{2j}}{(2j)!}.$$

Compare:

$$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is_n}(w) \leq k\}.$$

Gessel:

$$\sum_{n\geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{i-j}(2x) \right]_{i,j=1}^{k}.$$

$$s_i = (i, i+1) \in \mathfrak{S}_n$$

reduced decomposition $(a_1, \ldots, a_p)$ of $w \in \mathfrak{S}_n$:

$$w = s_{a_1} \cdots s_{a_p},$$

where p is minimal, i.e.,

$$p = \ell(w) = \#\{(i,j) \,:\, i < j,\ w(i) > w(j)\}.$$

$\boldsymbol{R(w)}$: set of reduced decomps. of w

$$\boldsymbol{r(w)} := \#R(w)$$

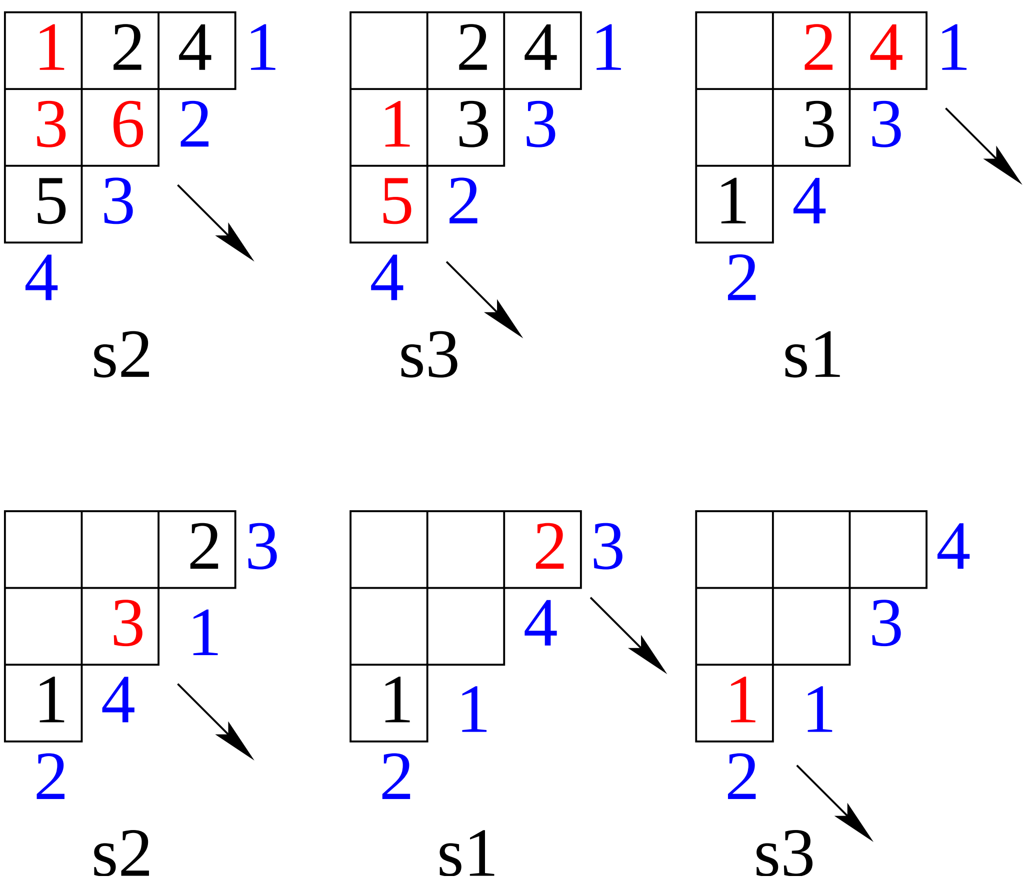
E.g., $w = 321$, $R(w) = \{(1,2,1), (2,1,2)\}$, $r(w) = 2$.

$$w_0 = n, n-1, \ldots, 1 \in \mathfrak{S}_n$$

Theorem.

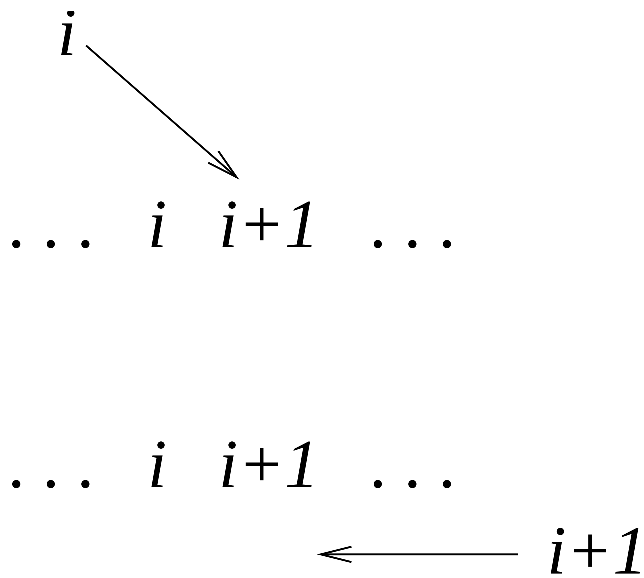
$$r(w_0) = f^{(n-1,n-2,\ldots,1)} = \frac{\binom{n}{2}!}{1^{n-1}\,3^{n-2}\,5^{n-3}\cdots(2n-3)}$$

Proof (Edelman-Greene).



$$(2,3,1,2,1,3) \in R(4321)$$

Inverse:



Example. $312132 \in R(4321)$

$$\begin{array}{l} \mathbf{3} \end{array} \qquad \begin{array}{l} 1 \end{array}$$

$$\begin{array}{l} \mathbf{1} \\ \mathbf{3} \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array}$$

$$\begin{array}{ll} 1 & \mathbf{2} \\ 3 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array}$$

$$\begin{array}{ll} 1 & \mathbf{2} \\ \mathbf{2} & \\ \mathbf{3} & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \end{array}$$

$$\begin{array}{lll} 1 & 2 & \mathbf{3} \\ 2 & & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{array}$$

$$\begin{array}{lll} 1 & 2 & \mathbf{3} \\ 2 & \mathbf{3} & \\ 3 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array}$$

Variant:

$$f(w) = \sum_{(a_1,\ldots,a_p)\in R(w)} a_1 a_2 \cdots a_p$$

E.g., $w = 321$, $R(321) = \{121, 212\}$,

$$f(321) = 1\cdot 2\cdot 1 + 2\cdot 1\cdot 2 = 6 = 3!$$

Theorem (Macdonald, Fomin-S).

$$f(w_0) = \binom{n}{2}!$$

More generally, $f(w) = \mathfrak{S}_w(1, 1, \ldots, 1)\ell(w)!$, where $\mathfrak{S}_w$ is a **Schubert polynomial**.

Corollary. $f(w) = \ell(w)!$ if and only if there never holds

$$i < j < k \Rightarrow w(i) < w(k) < w(j).$$

Number of such $w \in \mathfrak{S}_n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.