Smith Normal Form and Combinatorics

Richard P. Stanley

I
Really
Appreciate

Great
Enumerative
Successes
Stated
Exceptionally
Lucidly

A curious connection

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QUASISYMMETRIC FUNCTION

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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NOTE. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of $\det$.

Existence of SNF

If R is a **principal ideal ring** (PIR), such as $\mathbb{Z}$ or $K[x]$ ($K =$ field), then A has a unique SNF up to units.

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Otherwise A “typically” does not have a SNF but may have one in special cases.

Row and column operations

Over a principal ideal ring, can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in R.

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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Algebraic interpretation of SNF

R: a PIR

A: an $n \times n$ matrix over R with rows $v_1, \ldots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \ldots, e_n)$: SNF of A

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Theorem.

$$R^n/(v_1, \ldots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).$$

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$R^n/(v_1, \dots, v_n)$: **(Kastelyn) cokernel** of A

An explicit formula for SNF

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$\mathrm{diag}(e_1, e_2, \ldots, e_n)$: SNF of A

Theorem. $e_1 e_2 \cdots e_i$ *is the gcd of all* $i \times i$ *minors of* A*.*

minor: determinant of a square submatrix.

Special case: e_1 is the gcd of all entries of A.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Laplacian matrices

$\boldsymbol{L_0(G)}$: reduced Laplacian matrix of the graph G

Matrix-tree theorem. $\det L_0(G) = \boldsymbol{\kappa(G)}$, *the number of spanning trees of* G.

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In general, SNF of $L_0(G)$ not understood.

Chip firing

Abelian sandpile: a finite collection σ of indistinguishable chips distributed among the vertices V of a (finite) connected graph. Equivalently,

$$\sigma : V \to \{0, 1, 2, \dots\}.$$

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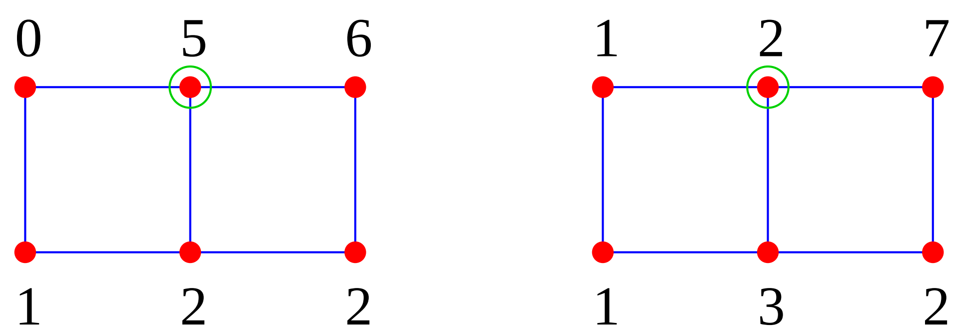
toppling of a vertex v: if $\sigma(v) \geq \deg(v)$, then send a chip to each neighboring vertex.



The sandpile group

Choose a vertex to be a **sink**, and ignore chips falling into the sink.

stable configuration: no vertex can topple

Theorem (easy). *After finitely many topples a stable configuration will be reached, which is independent of the order of topples.*

The monoid of stable configurations

Define a commutative monoid $\boldsymbol{M}$ on the stable configurations by vertex-wise addition followed by stabilization.

ideal of M: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

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Exercise. The (unique) minimal ideal of a finite commutative monoid is a group.

Sandpile group

sandpile group of G: the minimal ideal $\boldsymbol{K(G)}$ of the monoid M

Fact. $K(G)$ is independent of the choice of sink up to isomorphism.

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Theorem. *Let*

$$L_0(G) \xrightarrow{\mathrm{SNF}} \mathrm{diag}(e_1, \ldots, e_{n-1}).$$

Then

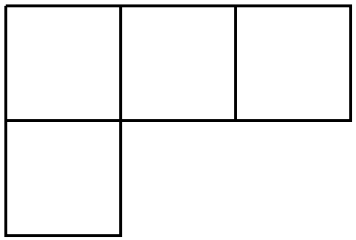
$$K(G) \cong \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_{n-1}\mathbb{Z}. \qquad \blacksquare$$

Second example

Some matrices connected with Young diagrams

Extended Young diagrams

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram



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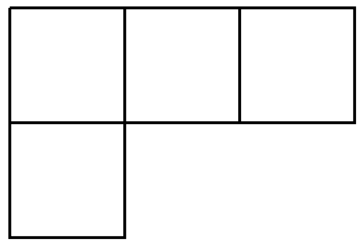

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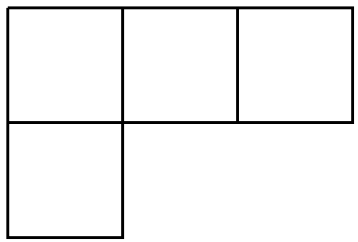

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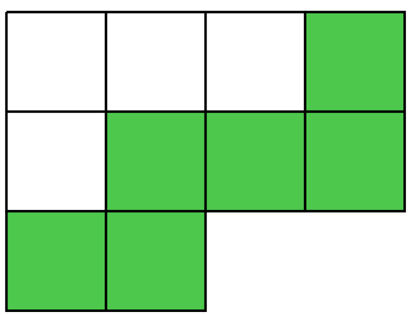
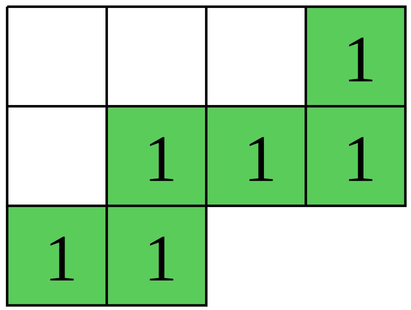


(3,1)* = (4,4,2)

Initialization

Insert 1 into each square of λ^*/λ.



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M_t

Let $t \in \lambda$. Let M_t be the largest square of λ^* with t as the upper left-hand corner.

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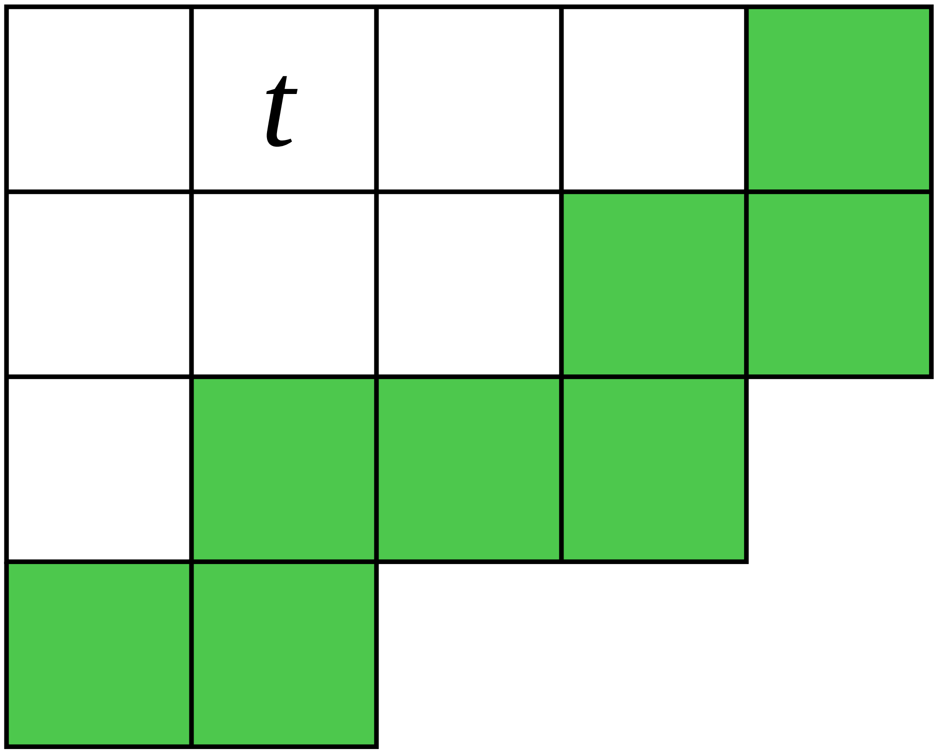

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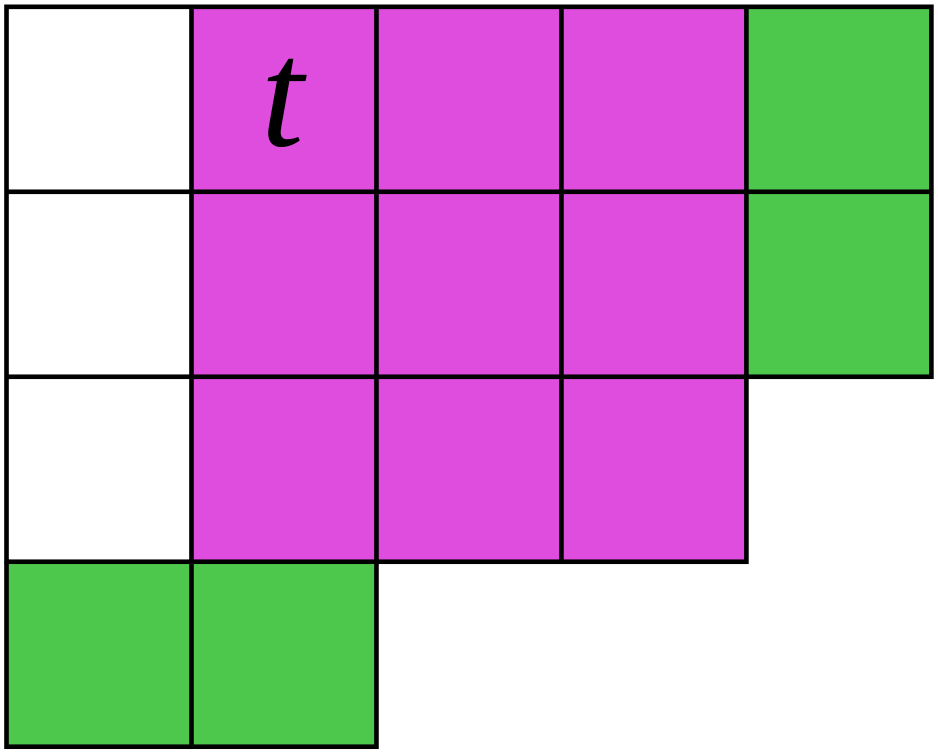

Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.

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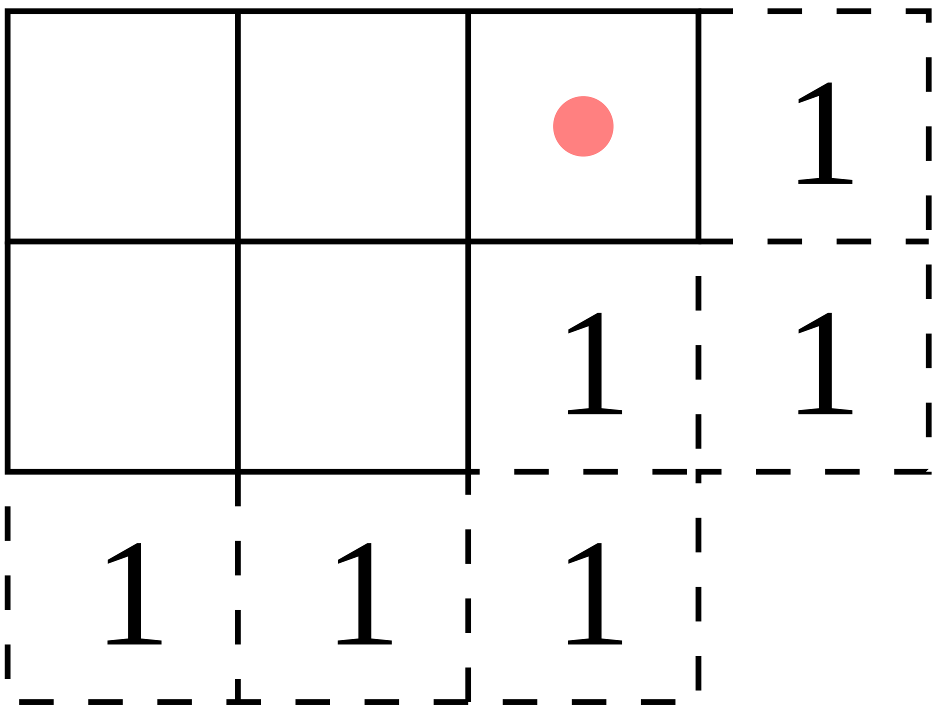

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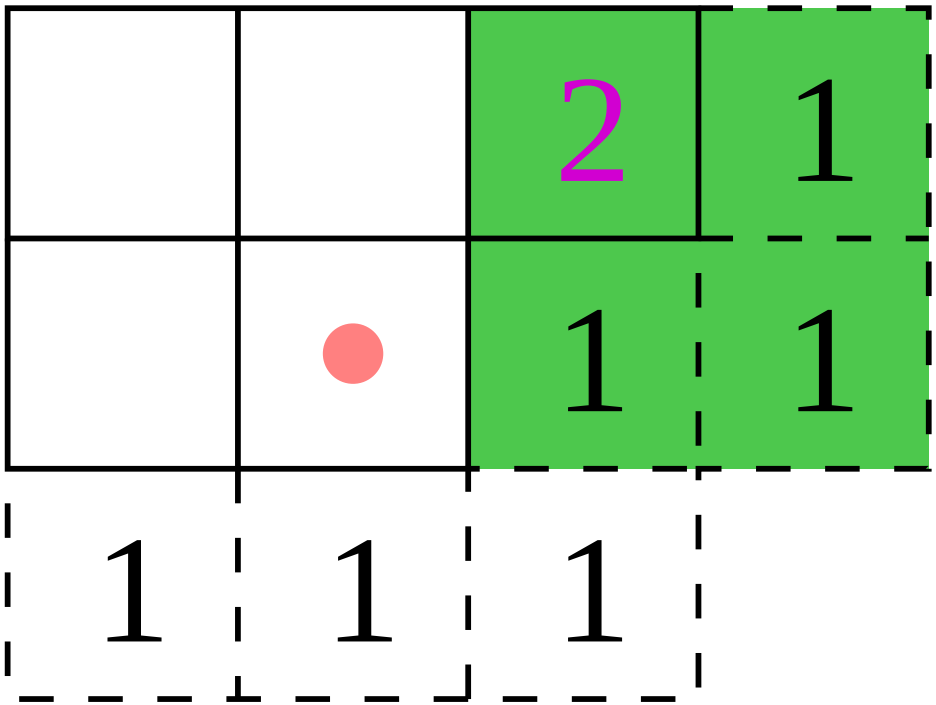

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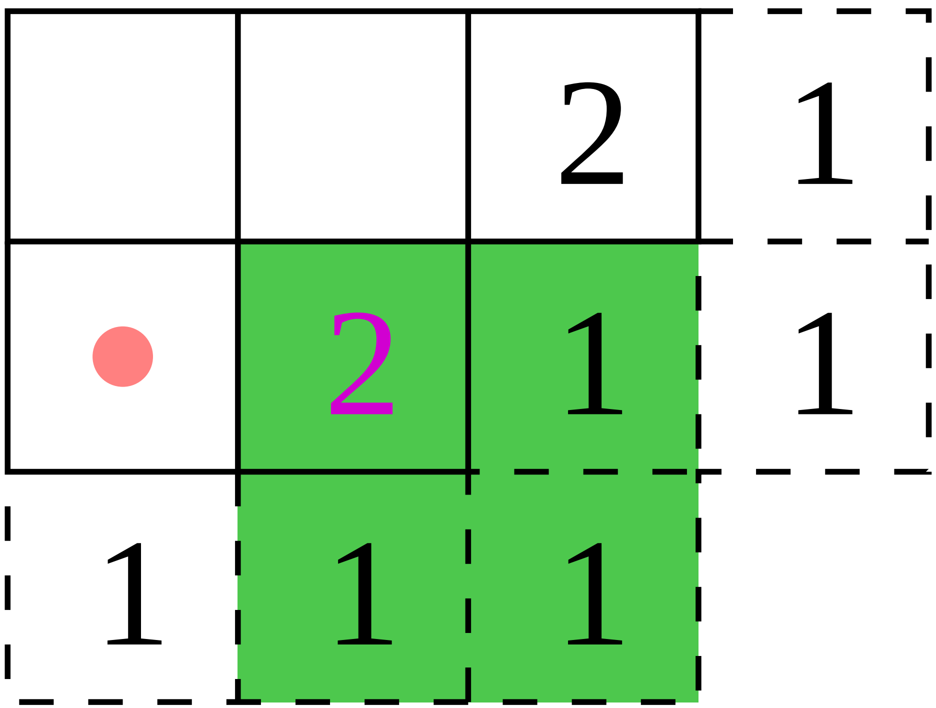

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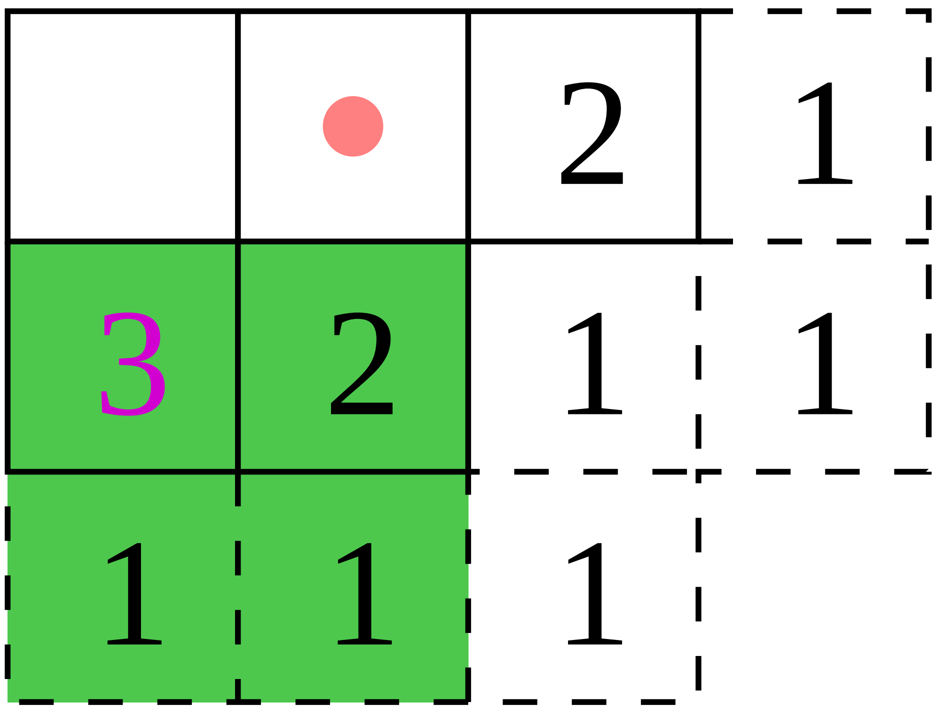

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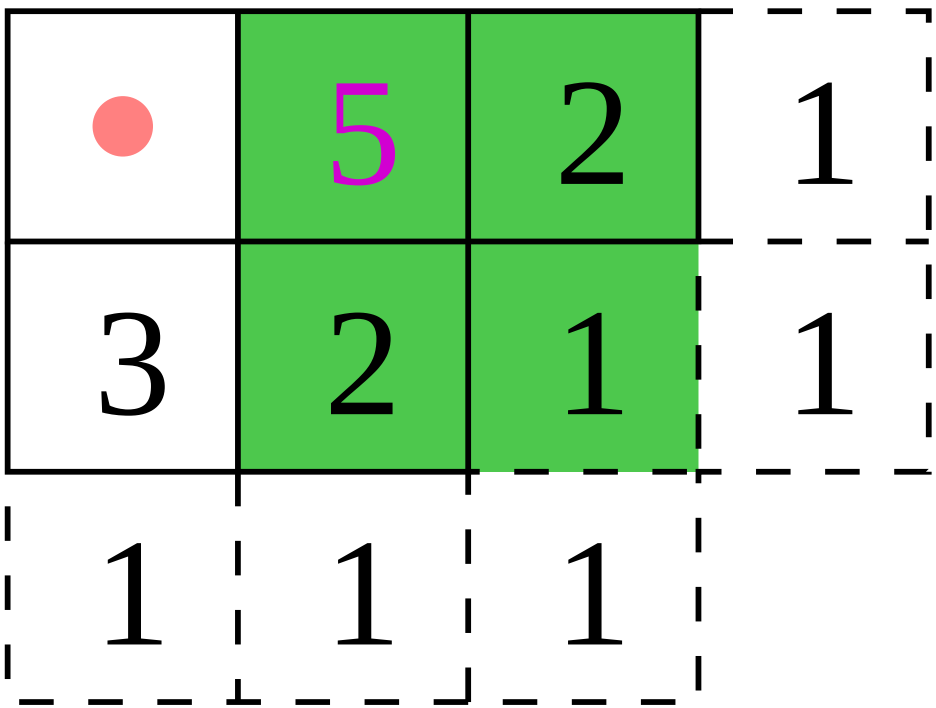

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9	5	2	1
3	2	1	1
1	1	1	

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

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Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

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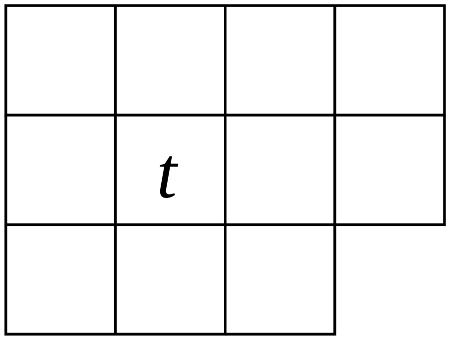

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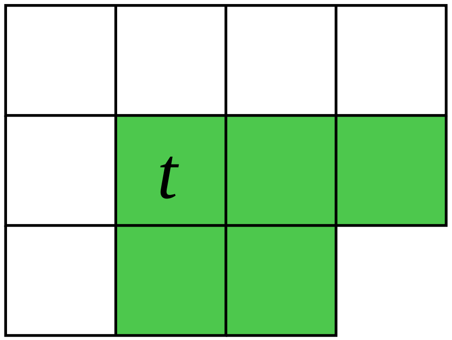

$$\lambda = (4,4,3)$$

$$\lambda(t) = (3,2)$$

u_λ

$$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$$

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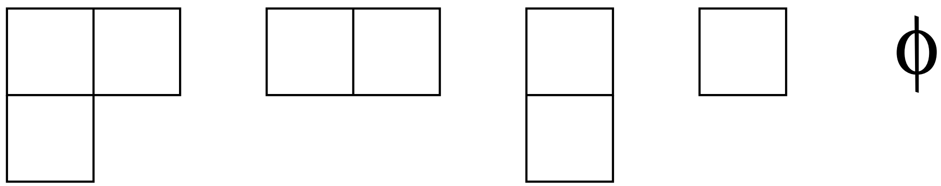
Example. $u_{(2,1)} = 5$:



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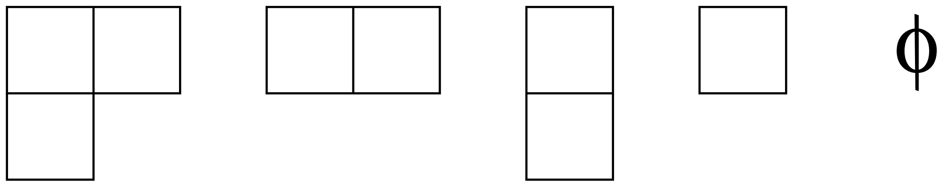

There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

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Theorem. $n_t = u_{\lambda(t)}$

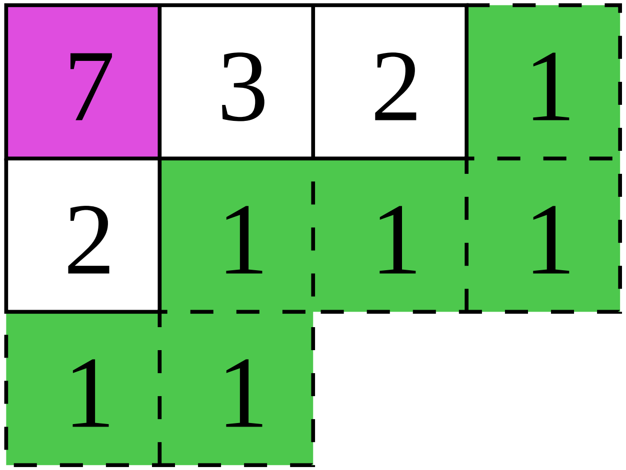
Proofs. 1. Induction (row and column operations).

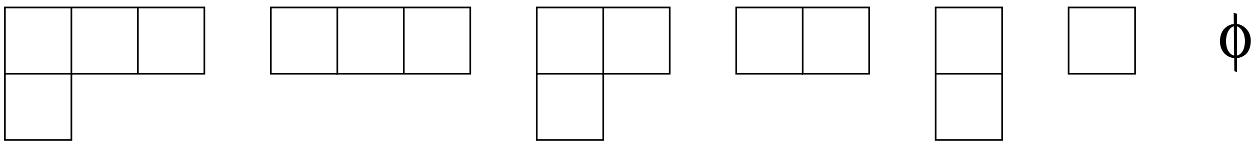
2. Nonintersecting lattice paths.

An example

7	3	2	1
2	1	1	1
1	1		

An example





Many indeterminates

For each square $(i, j) \in \lambda$, associate an indeterminate $\boldsymbol{x_{ij}}$ (matrix coordinates).

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x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	

A refinement of u_λ

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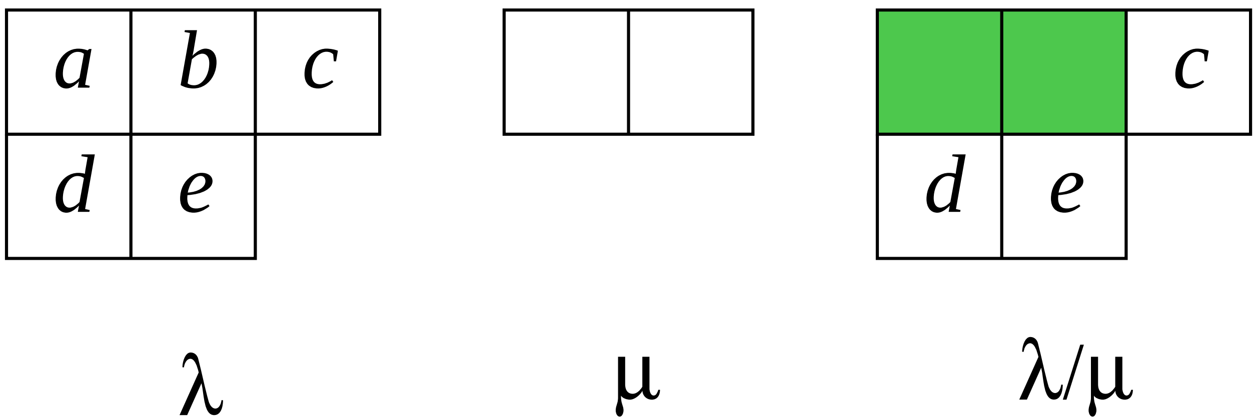

$$\prod_{(i,j) \in \lambda/\mu} x_{ij} = cde$$

An example

a	b	c
d	e	

$abcde+bcde+bce+cde+ce+de+c+e+1$	$bce+ce+c+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
1	1	1	

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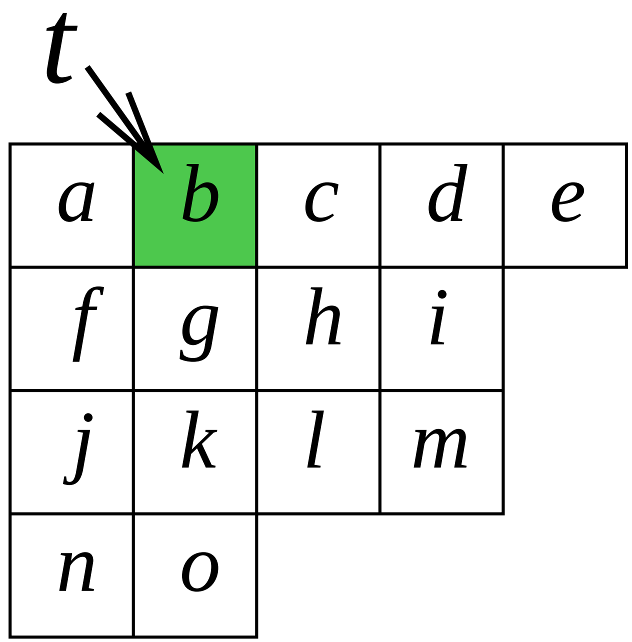

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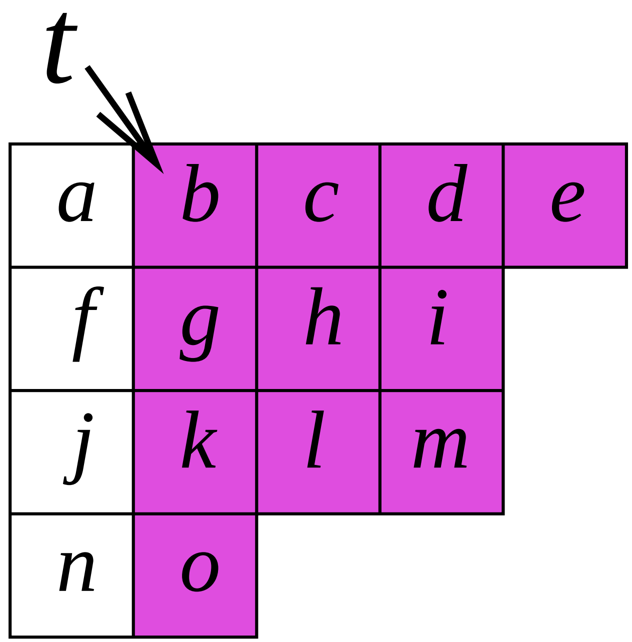

$$A_t = bcdeghiklmo$$

The main theorem

Theorem. *Let* $t = (i, j)$*. Then* M_t *has SNF*

$$\mathrm{diag}(1, \ldots, A_{i-2,j-2}, A_{i-1,j-1}, A_{ij}).$$

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Theorem. *Let $t = (i, j)$. Then M_t has SNF*

$$\mathrm{diag}(1, \ldots, A_{i-2,j-2}, A_{i-1,j-1}, A_{ij}).$$

Proof. 1. Explicit row and column operations putting M_t into SNF.

2. (**C. Bessenrodt**) Induction.

An example

a	b	c
d	e	

$abcde+bcde+bce+cde$ $+ce+de+c+e+1$	$bce+ce+c$ $+e+1$	$c+1$	1
$de+e+1$	$e+1$	1	1
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An example

a	b	c
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1	1	1	

SNF $= \mathrm{diag}(1, e, abcde)$

A special case

Let λ be the **staircase** $\boldsymbol{\delta_n} = (n-1, n-2, \ldots, 1)$.
Set each $x_{ij} = q$.

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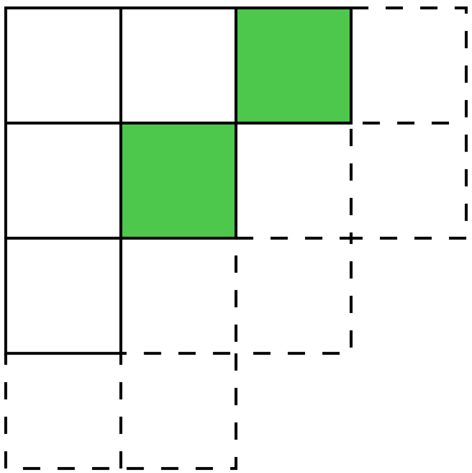

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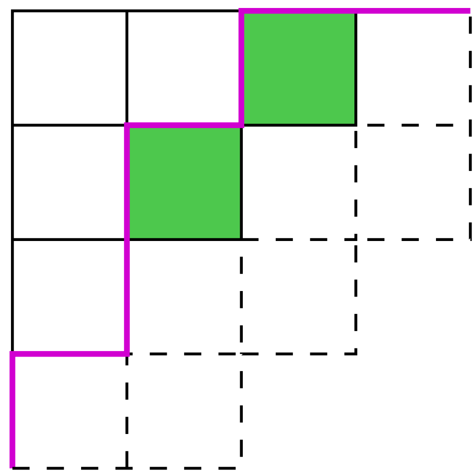
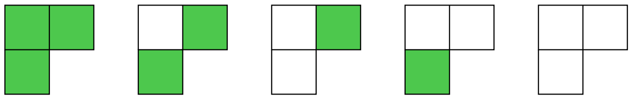


$u_{\delta_{n-1}}(x)\big|_{x_{ij}=q}$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $\boldsymbol{C_n(q)}$ of the Catalan number C_n.

A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

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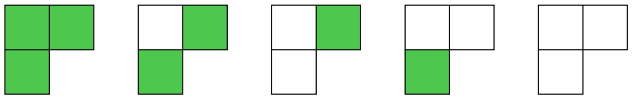

$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\left|\begin{array}{ccc} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{array}\right| \overset{\text{SNF}}{\sim} \mathrm{diag}(1, q, q^6)$$

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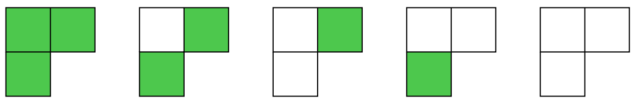

$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\begin{vmatrix} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{vmatrix} \overset{\mathrm{SNF}}{\sim} \mathrm{diag}(1, q, q^6)$$

- q-Catalan determinant previously known
- SNF is new

SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PIR.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$\boldsymbol{p_k(n, d)}$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \dots, e_n)$, then $e_1 = d$.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

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Theorem. $\lim_{k\to\infty} p_k(n, d) = 1/d^{n^2}\zeta(n^2)$

Work of Yinghui Wang

Work of Yinghui Wang (王颖慧)

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Sample result. $\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\boldsymbol{\mu(n)} = \lim_{k\to\infty} \mu_k(n).$$

Conclusion

$$\mu(n) = 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right)$$

$$\cdot\frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2$$

$$\cdot\prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).$$

A note on the proof

uses a 2014 result of **C. Feng**, **R. W. Nóbrega**, **F. R. Kschischang**, and **D. Silva**, Communication over finite-chain-ring matrix channels: number of $m \times n$ matrices over $\mathbb{Z}/p^s\mathbb{Z}$ with specified SNF

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Note. $\mathbb{Z}/p^s\mathbb{Z}$ is not a PID but is a PIR.

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF diag$(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

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Theorem. $\kappa(n) = \dfrac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$

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Corollary.
$$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)}$$
$$\approx 0.846936\cdots.$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

previous slide: Prob$(g = 1) = 0.846936 \cdots$

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Theorem. $\mathrm{Prob}(g \le \ell) =$

$$1 - (3.46275\cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$$

Jacobi-Trudi specialization

Jacobi-Trudi identity:

$$s_\lambda = \det[h_{\lambda_i - i + j}],$$

where s_λ is a **Schur function** and h_i is a **complete symmetric function**.

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We consider the specialization $x_1 = x_2 = \cdots = x_n = 1$, other $x_i = 0$. Then

$$h_i \to \binom{n+i-1}{i}.$$

Specialized Schur function

$$s_\lambda \to \prod_{u\in\lambda} \frac{n+c(u)}{h(u)}.$$

$\boldsymbol{c(u)}$: **content** of the square u

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

$$\lambda = (5,4,4,2)$$

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
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D_1

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
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−3	−2			

D_2

Diagonal hooks $D_1, \ldots, D_m$

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

D_3

SNF result

$$\boldsymbol{R} = \mathbb{Q}[n]$$

Let

$$\mathrm{SNF}\left[\binom{n+\lambda_i-i+j-1}{\lambda_i-i+j}\right] = \mathrm{diag}(e_1,\ldots,e_m).$$

Then

$$e_i = \prod_{u\in D_{m-i+1}} \frac{n+c(u)}{h(u)}.$$

Idea of proof

We will use the fact that if

$$\mathrm{SNF}(A) = \mathrm{diag}(e_1, e_2, \ldots, e_n),$$

then $e_1 e_2 \cdots e_i$ is the gcd of the $i \times i$ minors of A.

Idea of proof (cont.)

$$\boldsymbol{f_i} = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}$$

Then $f_1 f_2 \cdots f_i$ is the value of the lower-left $i \times i$ minor. (Special argument for 0 minors.)

Idea of proof (cont.)

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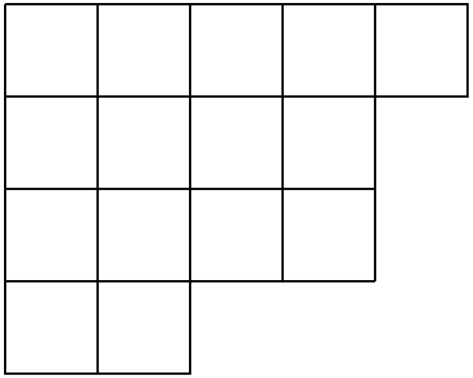
Then $f_1 f_2 \cdots f_i$ is the value of the lower-left $i \times i$ minor. (Special argument for 0 minors.)

Every $i \times i$ minor is a specialized skew Schur function $s_{\mu/\nu}$. Let $\boldsymbol{s_\alpha}$ correspond to the lower left $i \times i$ minor.

An example



$$s_{5442} = \begin{bmatrix} h_5 & h_6 & h_7 & h_9 \\ h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 & h_5 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$$

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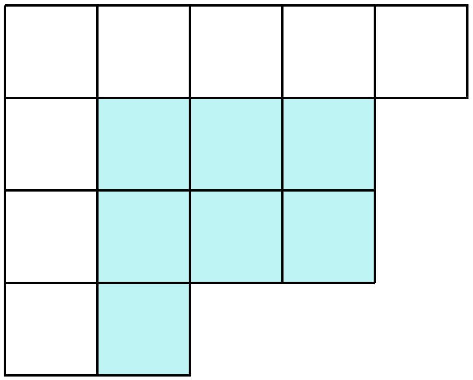

$$s_{5442} = \begin{vmatrix} h_5 & h_6 & h_7 & h_9 \\ \boldsymbol{h_3} & \boldsymbol{h_4} & \boldsymbol{h_5} & h_6 \\ \boldsymbol{h_2} & \boldsymbol{h_3} & \boldsymbol{h_4} & h_5 \\ \mathbf{0} & \mathbf{1} & \boldsymbol{h_1} & h_2 \end{vmatrix}$$

$$s_{331} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 0 & 1 & h_1 \end{vmatrix}$$

Conclusion of proof

Let

$$s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}.$$

By Littlewood-Richardson rule,

$$c^{\mu}_{\nu\rho} \neq 0 \;\Rightarrow\; \alpha \subseteq \rho.$$

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Hence

$$f_1 \cdots f_i = \gcd(i \times i \text{ minors}) = e_1 \cdots e_i.$$

$\lrcorner$

A generalization?

What about the specialization $x_i = q^{i-1}$, $1 \leq i \leq n$, other $x_i = 0$?

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Now it seems the ring should be $\mathbb{Q}[q]$. Looks difficult.

The last slide

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