MATHEMATICAL GAMES

Transparencies available at:

www-math.mit.edu/~rstan/transparencies/games.ps

www-math.mit.edu/~rstan/transparencies/games.pdf

- Two players: **Blue** and **Red**.
- Perfect information.
- Players move alternately.
- First player unable to move loses.
- The game *must* terminate.

Outcomes of G (assuming perfect play):

- **Blue** wins (whoever moves first): $G > 0$
- **Red** wins (whoever moves first): $G < 0$
- Mover loses: $G = 0$
- Mover wins: $G \| 0$

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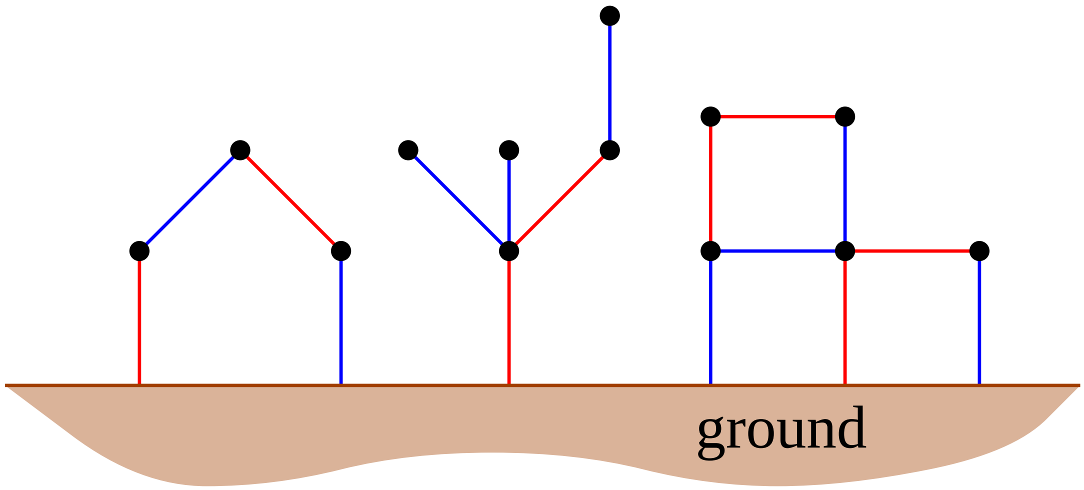
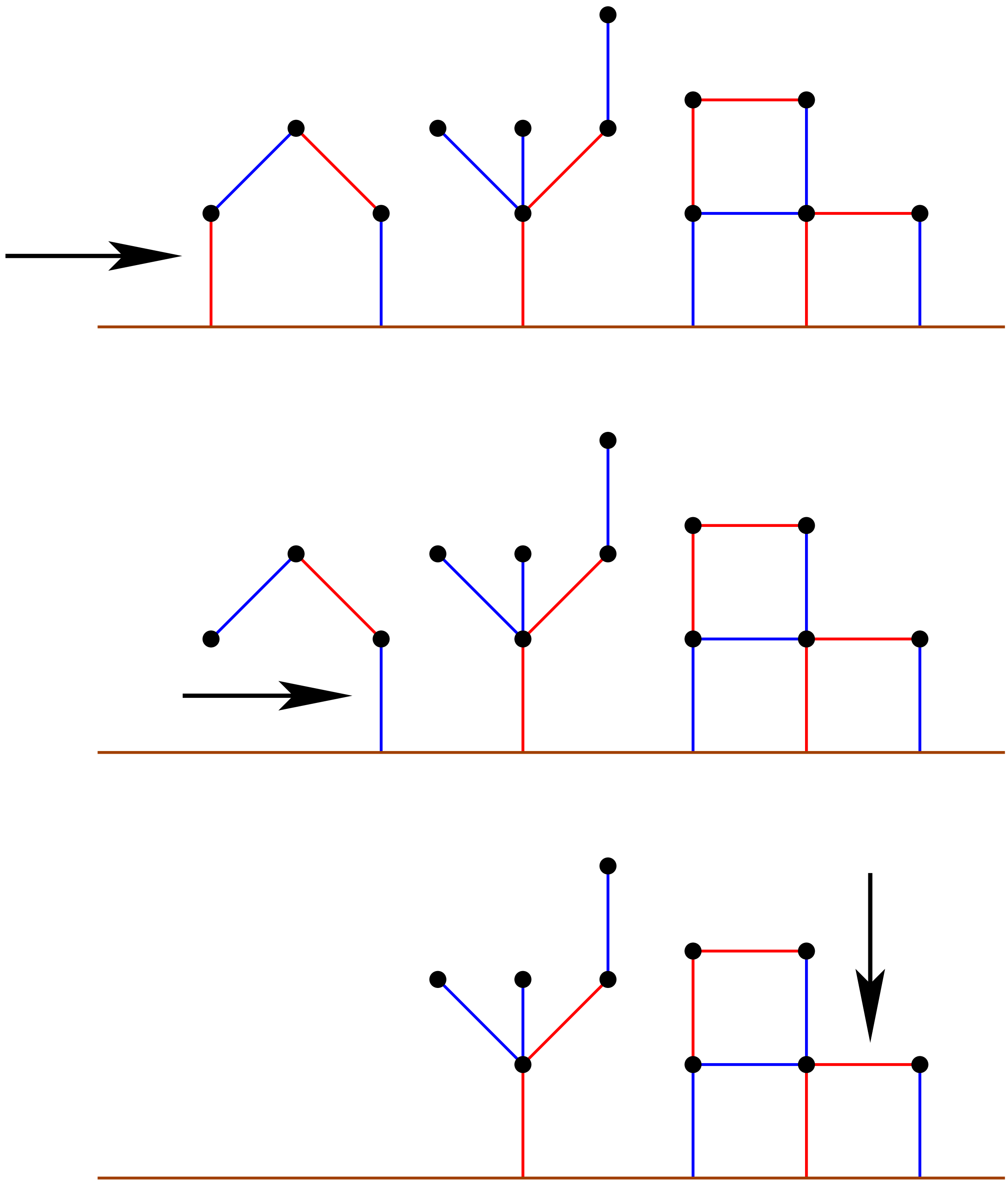
Two types of games with elegant theories:

- **partizan game**: always disadvantageous to move (so never $G \| 0$)
- **impartial game**: same moves always available to each player (so never $G > 0$, $G < 0$)



prototypical partizan game:

Blue-Red Hackenbush: A player removes one edge of his or her color. Any edges not connected to the ground are also removed. First person unable to move loses.

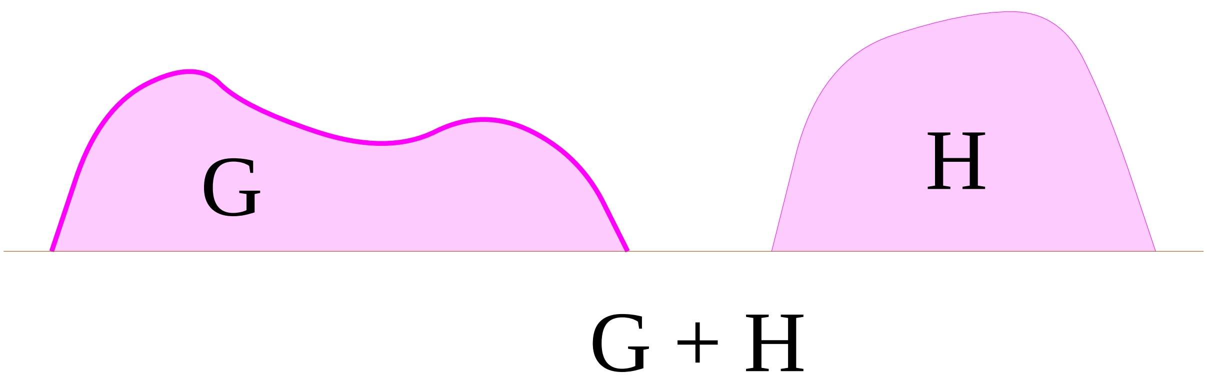


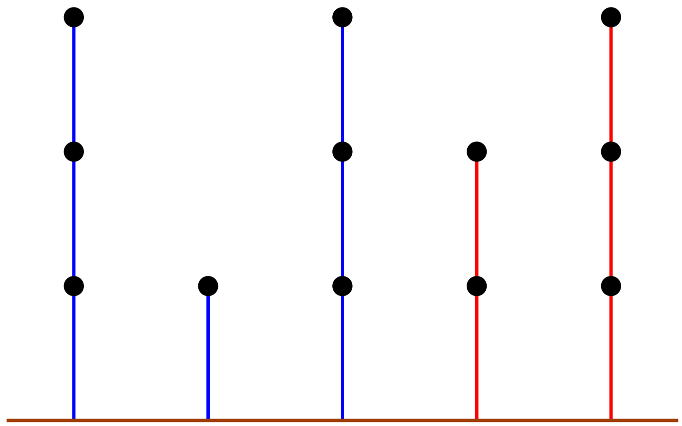
Why is Blue-Red Hackenbush a partizan game, i.e., why is it always disadvantageous to move?

Why isn't it an advantage for Blue to move first and remove many Red moves? Because Blue can always do this after any Red move on the string, so Blue is no worse off than without the string. But with the string, Blue might want to remove it before Red has played on it.

Let G be a Blue-Red Hackenbush position (or any game). Recall:

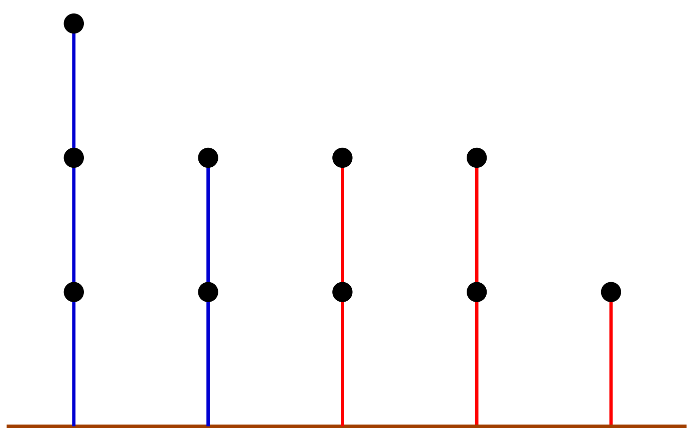
- **Blue** wins: $\boldsymbol{G > 0}$
- **Red** wins: $\boldsymbol{G < 0}$
- Mover loses: $\boldsymbol{G = 0}$





value (to Blue): 3 1 3 −2 −3

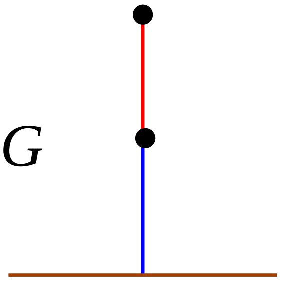
sum: 2 (Blue is two moves ahead), G> 0



3 2 −2 −2 −1

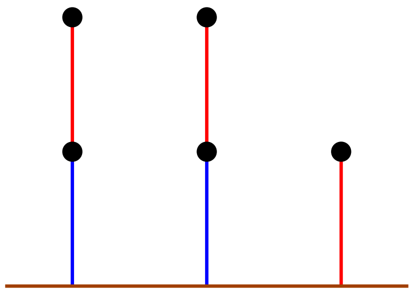
sum: 0 (mover loses), G= 0

G



value = ?

clearly >0: Blue wins

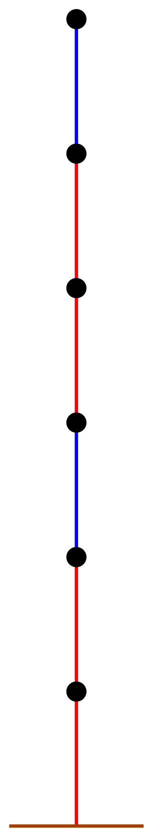


mover loses!

$x + x - 1 = 0$, so $x = 1/2$

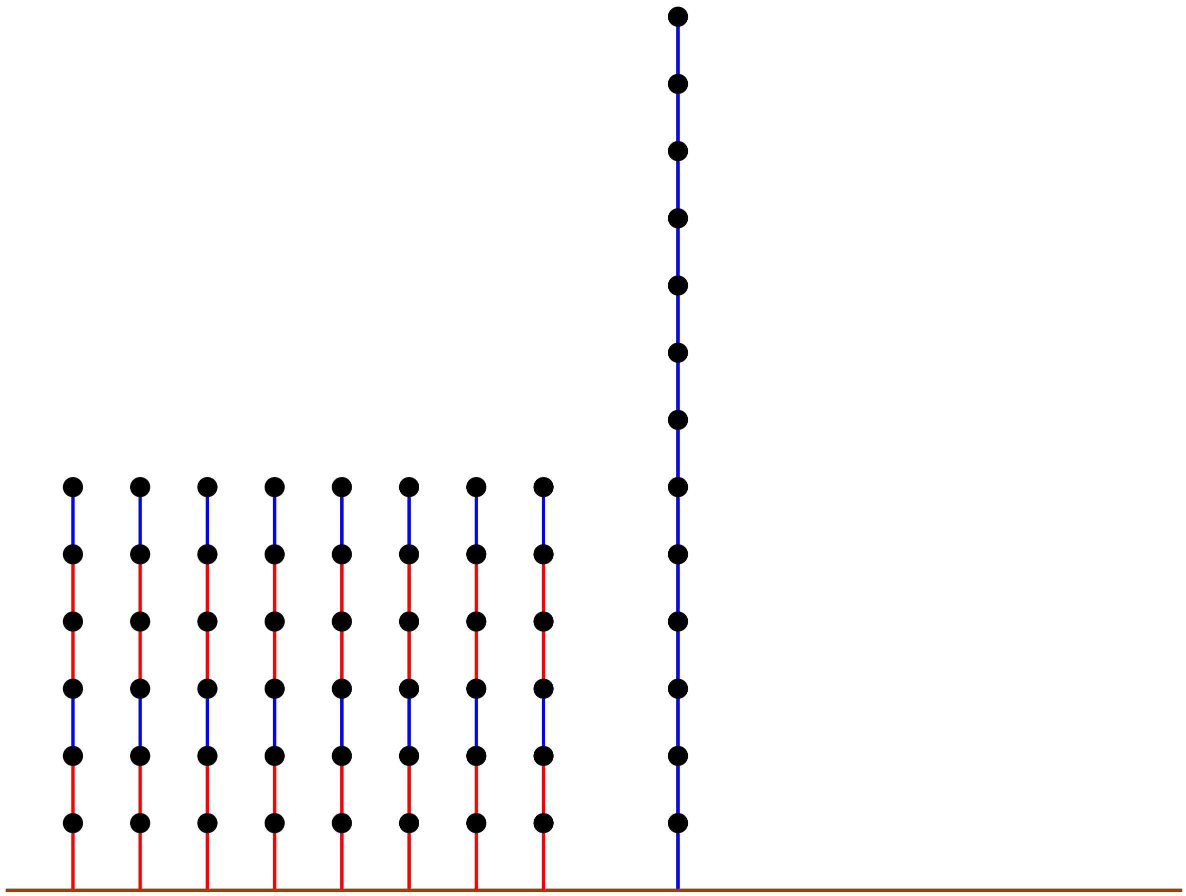
Blue is 1/2 move ahead in *G*.

What about



?

Clearly $G < 0$.



8x + 13 = 0 (mover loses!)

$x = -13/8$

How to compute the value $\boldsymbol{v(G)}$ of any Blue-Red Hackenbush position G?

Let b be the **largest** value of any position to which **Blue** can move. Let r be the **smallest** value of any position to which **Red** can move. (We will always have $b < r$.)

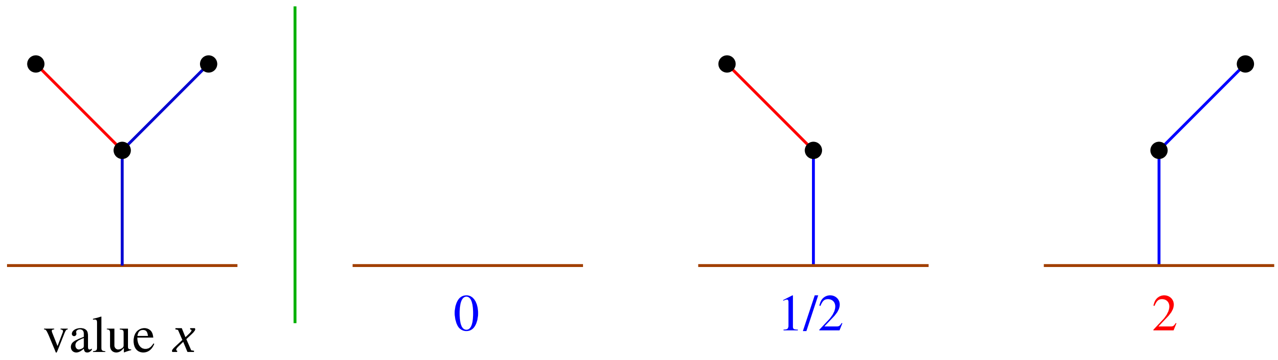
The Simplicity Rule. (a) *If there is an integer n satisfying $b < n < r$, then $v(G)$ is the closest such integer to 0.*

(b) *Otherwise $v(G)$ is the (unique) rational number x satisfying $b < x < r$ whose denominator is the smallest possible power of 2.*

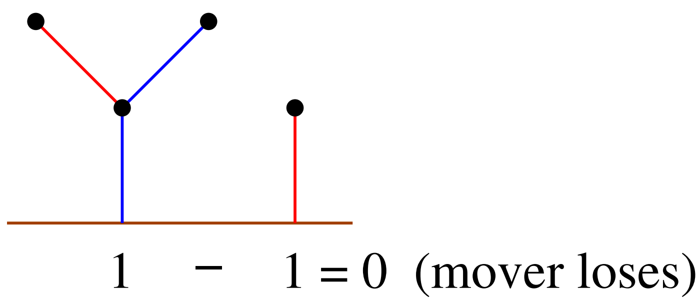
Moreover, $v(G+H) = v(G) + v(H)$.

Examples.

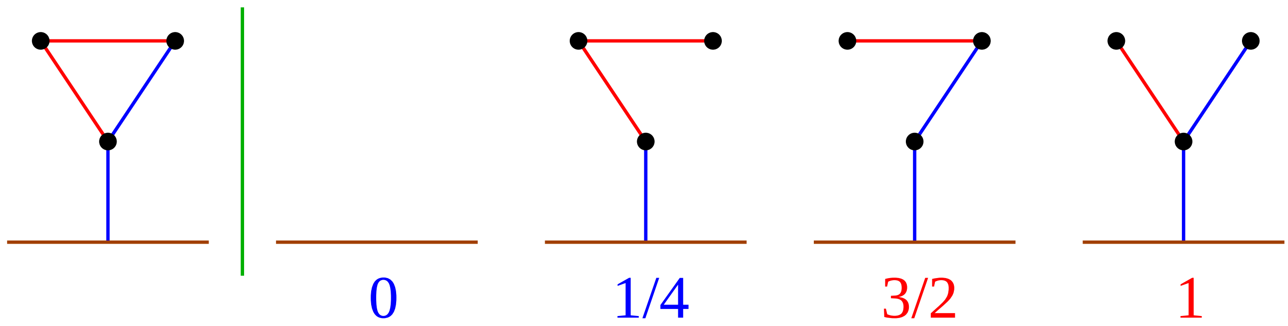
b	r	x
$2\frac{3}{4}$	$6\frac{1}{2}$	3
-5	$2\frac{5}{8}$	0
0	1	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{16}$	$\frac{9}{32}$
$\frac{1}{4}$	$\frac{7}{16}$	$\frac{3}{8}$
$-2\frac{7}{8}$	$-2\frac{3}{32}$	$-2\frac{1}{2}$



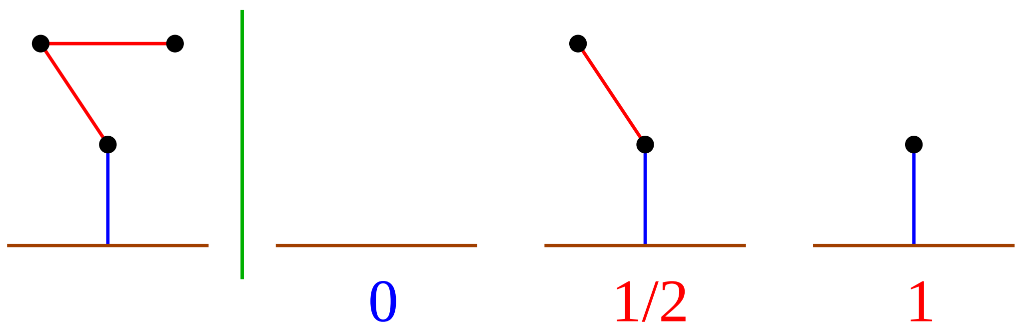
b = 1/2, r = 2, x = 1



Another example:

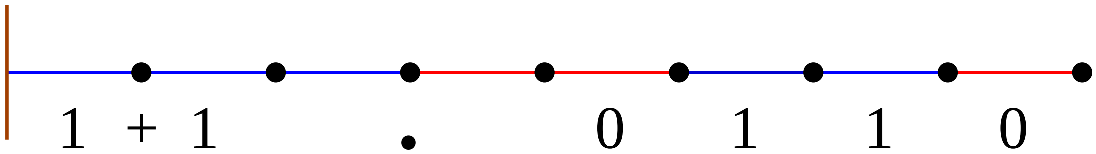


< 0, 1/4 | 3/2, 1 > = 1/2



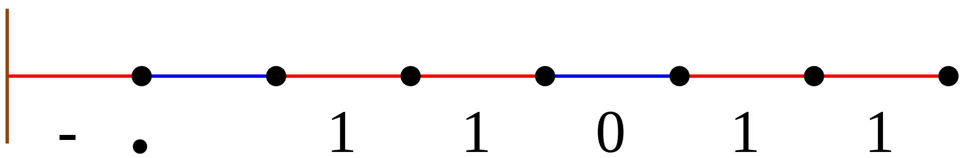
< 0 | 1/2, 1 > = 1/4

Value of Blue-Red strings:



1 + 1 . 0 1 1 0 1

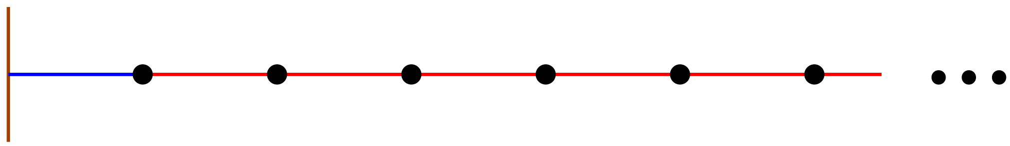
= 2 + 1/4 + 1/8 + 1/32 = 2 13/32



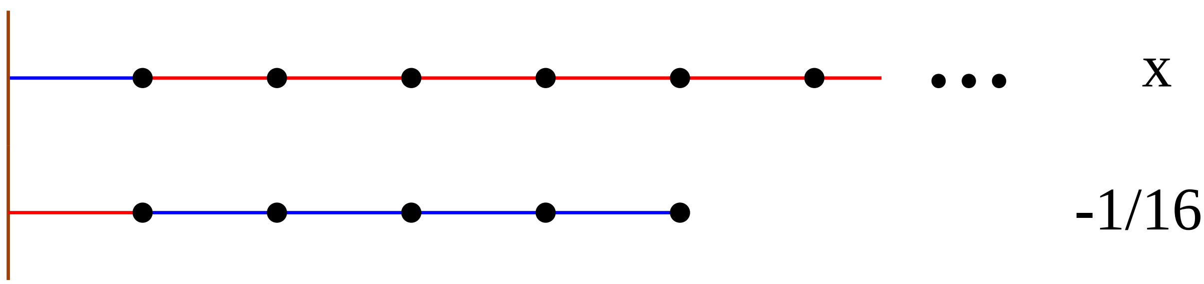
- . 1 1 0 1 1 1

= - (1/2 + 1/4 + 1/16 + 1/32 + 1/64) = - 55/64

Infinite Strings:



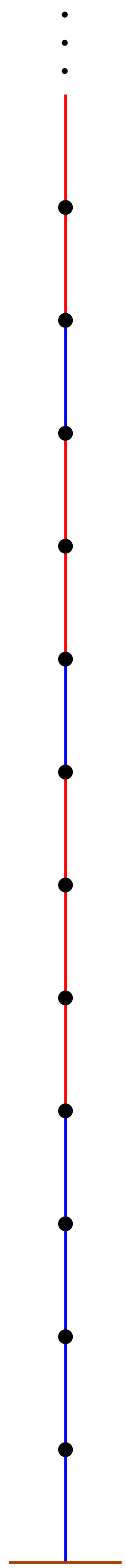
$x > 0$, but $x < 1/2^n$ for all $n > 0$

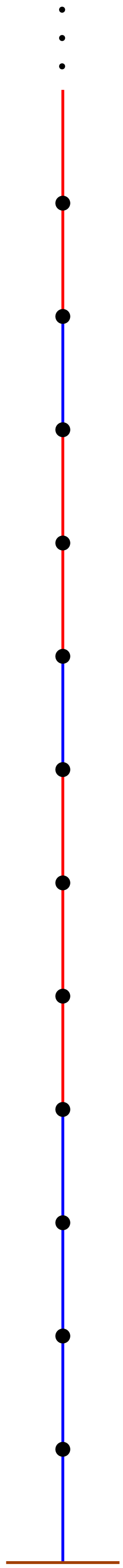


Red wins, so $x - 1/16 < 0$ or $x < 1/16$

x is an infinitesimal!

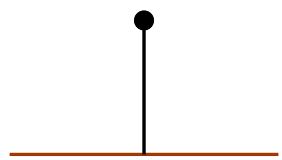
NOTE: Game still terminates after finitely many moves.





$$(11.001001000011111101\cdots)_2$$
$$= 3.1415926535\cdots$$

Now suppose there are also **black** edges, which either player can remove. A game with all **black** edges is called an **impartial** (Hackenbush) game. At any stage of such a game, the two players always have the same available moves.



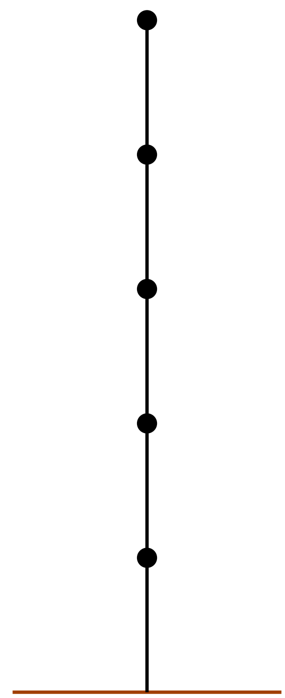
*1

Mover wins! Not a partizan game.

Neither $= 0$, < 0, or > 0: $*1 || 0$.

Two outcomes of any impartial game: mover wins or mover loses.

Denote by $*n$ (star n) the impartial game with one chain of length n.



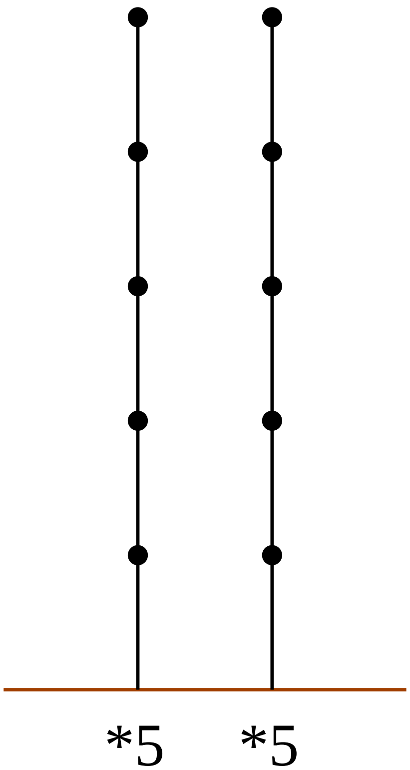
We can still assign a number with useful properties to an impartial game, based on the following fact.

Fact. Given any (finite) impartial game G, there is a unique integer $n \geq 0$ such that mover loses in the sum of G and $*n$, i.e.,

$$G + *n = 0.$$

Denote this integer by $N(G)$, the **Sprague-Grundy** number of G.

NOTE: Mover loses (i.e., $G = 0$) if and only if $N(G) = 0$.



Mover loses (second player copies first player).

N (*5) = 5

*5 + *5 = 0

In general, $N(*n) = n$.

Nim addition. Define $m \oplus n$ by writing m and n in binary, adding **without carrying** (mod 2 addition in each column), and reading result in binary.

Example. $13 \oplus 11 \oplus 7 \oplus 4 = 5$

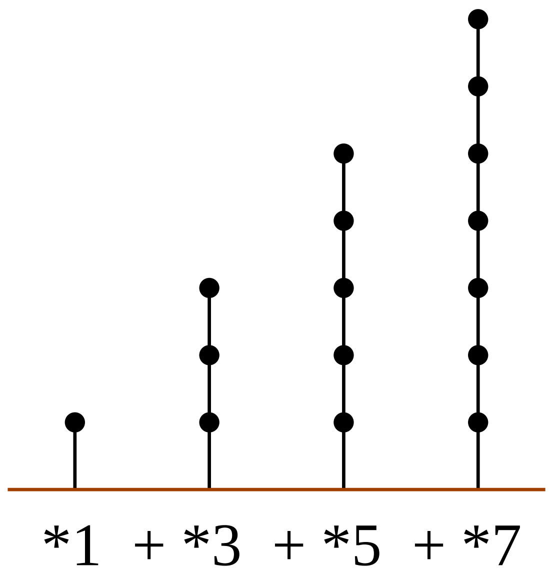
$$\begin{array}{rcl} & & \underline{8\,4\,2\,1} \\ 13 & = & 1\,1\,0\,1 \\ 11 & = & 1\,0\,1\,1 \\ 7 & = & 1\,1\,1 \\ 4 & = & 1\,0\,0 \\ \hline 5 & = & 0\,1\,0\,1 \end{array}$$

Nim-sum Theorem. Let G and H be inpartial games. Then

$$N(G + H) = N(G) \oplus N(H).$$

Nim: sum of $*n$'s.

Last Year at Marienbad:



$$
\begin{array}{rcr}
 & & \underline{4\,2\,1} \\
1 & = & 1 \\
3 & = & 1\,1 \\
5 & = & 1\,0\,1 \\
7 & = & 1\,1\,1 \\
\hline
0 & = & 0\,0\,0,
\end{array}
$$

so mover loses!

How to play Nim:

$$G = *23 + *18 + *13 + *7 + *5$$

23 = 10111
18 = 10010
13 = 1101 → 0111 = 7
7 = 111
5 = 101

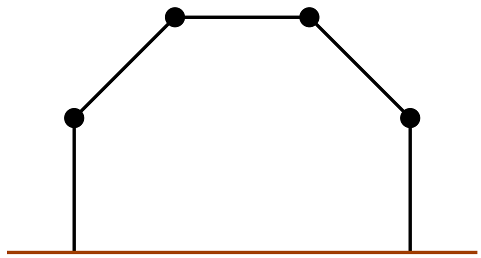
Only winning move is to change $*13$ to $*7$.

How to compute $N(G)$ in general: If S is a set of nonnegative integers, let **mex**(S) (the **minimal excludant** of S) be the least nonnegative integer not in S.

$$\begin{aligned} \text{mex}\{0,1,2,5,6,8\} &= 3 \\ \text{mex}\{4,7,8,12\} &= 0. \end{aligned}$$

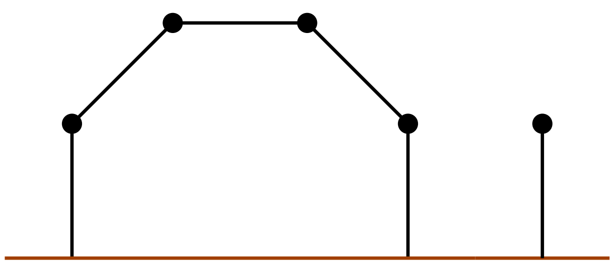
Mex Rule (analogue of Simplicity Rule). Let S be the set of all Sprague-Grundy numbers of positions that can be reached in one move from the impartial game G. Then

$$N(G) = \text{mex}(S).$$



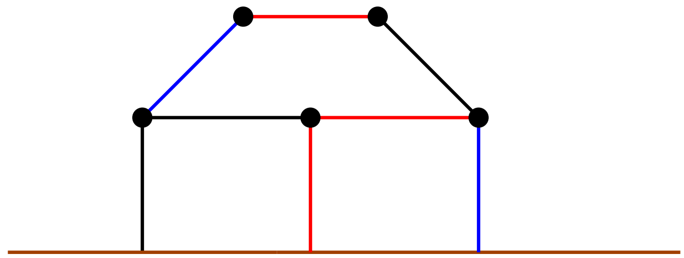
Can move to 4, $1 \oplus 3 = 2$, and $2 \oplus 2 = 0$. Thus

$$N(G) = \text{mex}\{0, 2, 4\} = 1.$$



mover loses!

Mixed games.



Much more complicated!

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