FREE PROBABILITY FOR COMBINATORIALISTS

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Classical (commutative) probability: $b_i, i \in I$ random (complex) variables in a commutative $\mathbb{C}$-algebra $\mathcal{B}$ with 1

$\{b_i : i \in I\}$ are **independent** if

$E(b_{i_1}^{r_1} \cdots b_{i_n}^{r_n}) = E(b_{i_1}^{r_1}) \cdots E(b_{i_n}^{r_n})$ ***(s.t.a.)***

$\forall i_1, \ldots, i_n$ all distinct.

Equivalently, let $\mathcal{B}_i$, $i \in I$ be unital subalgebras, with $E(1) = 1$. Then $\mathcal{B}_i$, $i \in I$ are **independent** if

$$E(b_1 \cdots b_n) = 0$$

whenever $E(b_j) = 0$ $(1 \leq j \leq n)$ and $b_j \in \mathcal{B}_{i_j}$ with $i_1, \ldots, i_n$ all distinct.

Free (noncommutative) probability: $\mathcal{A} = \mathbb{C}$-algebra with 1; regard elements as random variables.

$$\mathcal{A}_i,\ i \in I : \text{unital subalgebras}$$

$$\varphi : \mathcal{A} \to \mathbb{C} \text{ linear } (\textbf{expectation})$$

$$\varphi(1) = 1$$

The $\mathcal{A}_i$'s are **free** if $\varphi(a_1 \cdots a_n) = 0$ whenever $\varphi(a_j) = 0$ $(1 \leq j \leq n)$ and $a_j \in \mathcal{A}_{i_j}$ with $i_j \neq i_{j+1}$ $(1 \leq j < n)$.

Note. Does **not** reduce to classical case when $\mathcal{A}$ is commutative. E.g., independent a, b need not satisfy $\varphi(aba) = 0$.

Let $\mathcal{B}, \mathcal{C}$ be free, $b, b_i \in \mathcal{B}$, etc. Formal computation gives:

$$\begin{aligned} \varphi(bc) &= \varphi(b)\varphi(c) \\ \varphi(b_1c_1b_2c_2) &= \varphi(b_1b_2)\varphi(c_1)\varphi(c_2) \\ &\quad +\varphi(b_1)\varphi(b_2)\varphi(c_1c_2) \\ &\quad -\varphi(b_1)\varphi(b_2)\varphi(c_1)\varphi(c_2), \end{aligned}$$

etc.

MOMENTS AND CUMULANTS

Classical case. Let x, y be independent random variables from the probability distributions (measures) μ_x, μ_y, say with compact support. Given the **moments** $E(x^n), E(y^n)$, we want

$$E((x+y)^n).$$

Let

$$F_\mu(t) = \sum_{n \geq 0} E(x^n) \frac{t^n}{n!},$$

essentially the **Fourier transform** of μ.

convolution: Classical convolution defined by

$$\mu_{x+y} = \mu_x * \mu_y$$

Then

$$\begin{aligned} F_{\mu*\nu}(t) &= F_\mu(t) F_\nu(t) \\ \log F_{\mu*\nu}(t) &= \log F_\mu(t) + \log F_\nu(t). \end{aligned}$$

($\log F$ **linearizes** $*$)

Hence if $m_n = E(x^n)$ and

$$F_\mu(t) = \sum_{n \geq 0} m_n \frac{t^n}{n!} = \exp \sum_{n \geq 1} \boldsymbol{c_n} \frac{t^n}{n!},$$

then c_n is a **cumulant** and

$$\begin{aligned} c_n(x+y) &= c_n(x) + c_n(y) \\ m_n &= \sum_{\{B_1, \ldots, B_r\} \in \Pi_n} c_{\#B_1} \cdots c_{\#B_r}. \end{aligned}$$

Now let a, b be free. Define

$$\mu_{a+b} = \mu_a \boxplus \mu_b$$

the **free convolution** of μ_a and μ_b.

Cauchy transform:

$$\begin{aligned} G_\mu(z) &= \int \frac{a}{z-a} d\mu(a) \\ &= z^{-1} + \sum_{n \geq 1} \varphi(a^n) z^{-n-1} \in \mathbb{C}[\![1/z]\!], \end{aligned}$$

the **ordinary** generating function for the moments $m_n = \varphi(a^n)$.

Define the ***R*-transform** $R_\mu(z)$ by

$$\boldsymbol{R_\mu(z)} + \frac{1}{z} = G_\mu(z)^{\langle -1 \rangle}$$

(compositional inverse), so

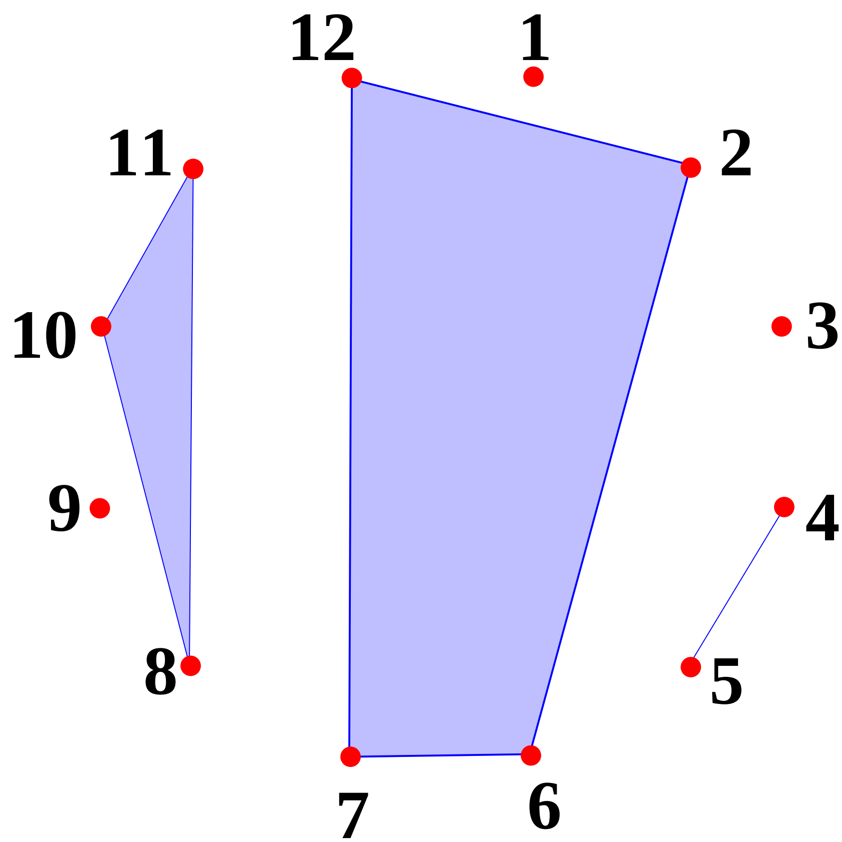
$$G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z.$$

Theorem. $R_{\mu \boxplus \nu} = R_\mu + R_\nu$

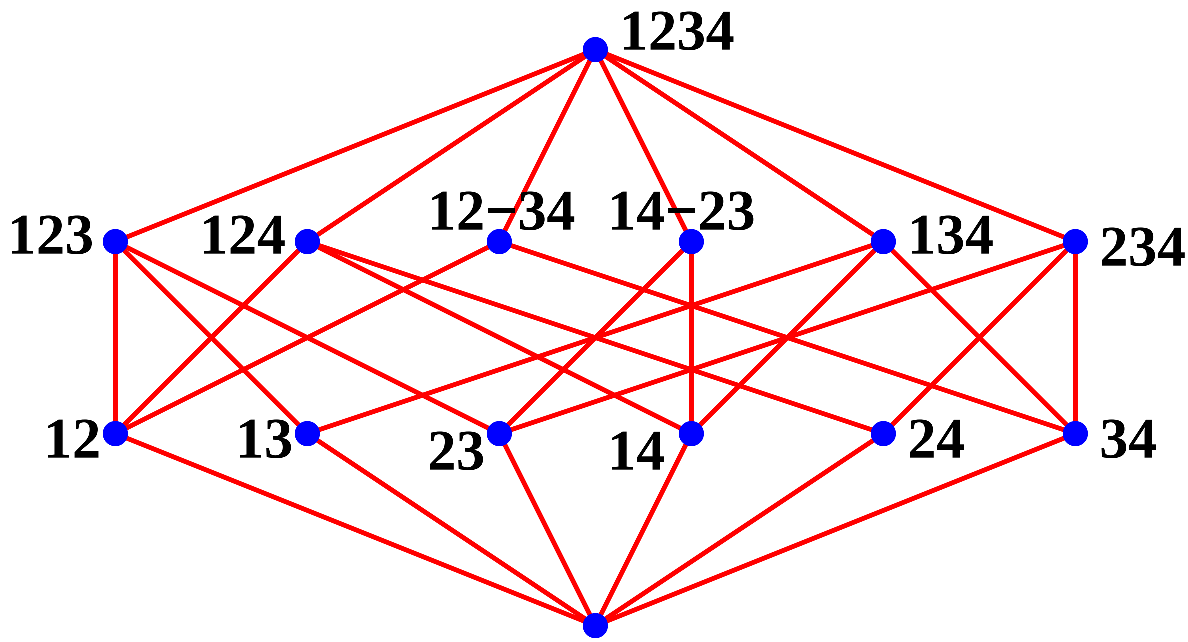
Set $R_\mu(z) = \dfrac{1}{z} + \sum_{n \geq 1} k_n(a) z^{-n-1}$, so

$$k_n(a+b) = k_n(a) + k_n(b)$$

when a, b are free. Call $k_n(a)$ a **free cumulant** of μ.



$\mathbf{NC}_{\boldsymbol{n}}$: lattice of noncrossing partitions of $\{1, 2, \ldots, n\}$.

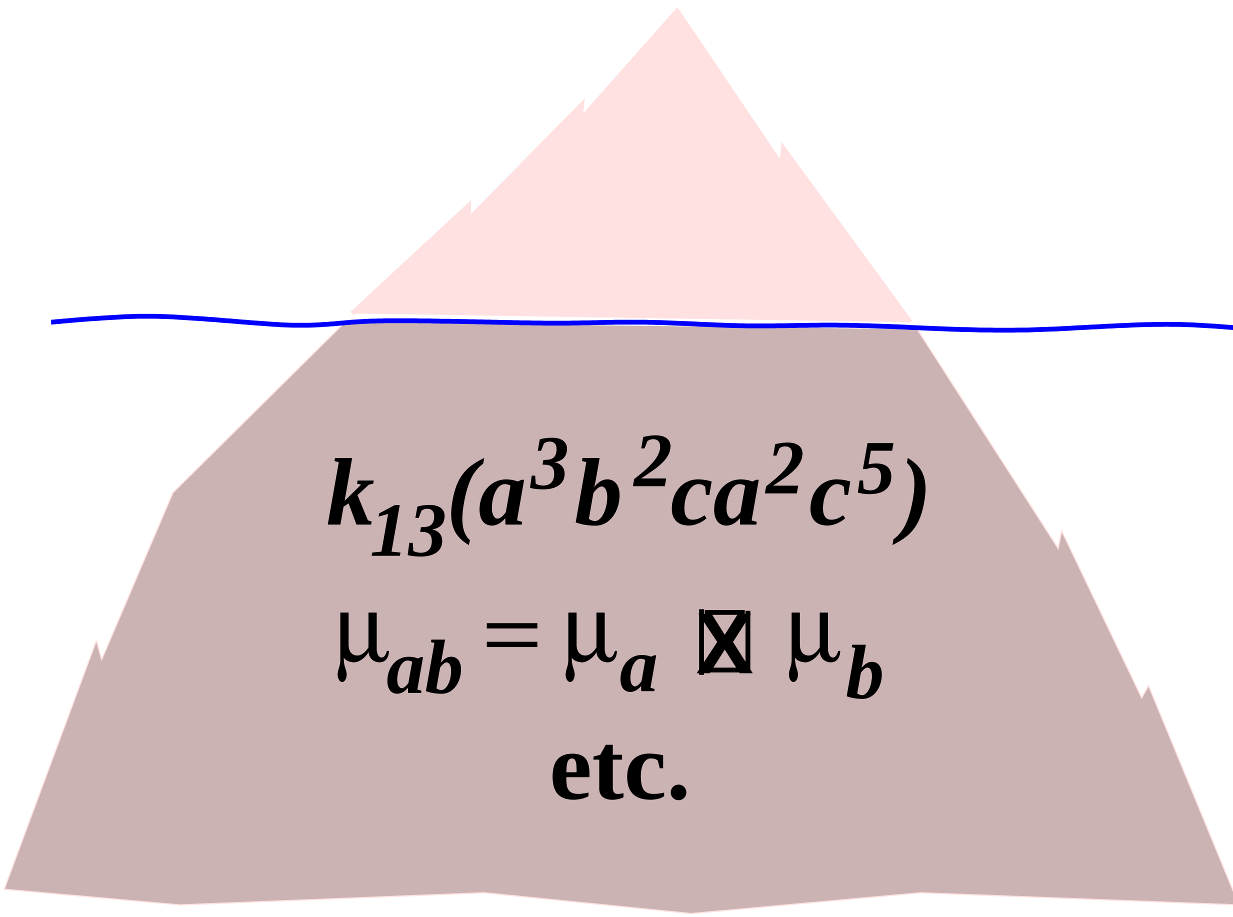


$\mathbf{NC_4}$

$$
\begin{aligned}
\#\mathrm{NC}_n \;&=\; C_n = \frac{1}{n+1}\binom{2n}{n} \\
\#(\text{max. chains}) \;&=\; n^{n-2} \\
\mu(\hat{0}, \hat{1}) \;&=\; (-1)^{n-1} C_{n-1}
\end{aligned}
$$

$$
\varphi(a^n) = m_n = \sum_{\{B_1,\ldots,B_r\} \in \mathrm{NC}_n} k_{\#B_1} \cdots k_{\#B_r},
$$

where $k_i = k_i(a)$.


$k_{13}(a^3b^2ca^2c^5)$
$\mu_{ab} = \mu_a \otimes \mu_b$
etc.

RANDOM MATRICES

Let A be a random $n \times n$ hermitian matrix, chosen from the **Gaussian unitary ensemble** (**GUE**). Let

$$\boldsymbol{\theta_1 > \cdots > \theta_n}$$

be the spectrum of A.

Choose $w \in \mathfrak{S}_n$ uniformly, and let $\boldsymbol{(\lambda_1, \ldots, \lambda_n)}$ be the shape of w under the RSK algorithm.

Theorem. *θ_i and λ_i have the same distribution (after rescaling) as $n \to \infty$.*

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be $n \times n$ hermitian matrices with eigenvalues $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$.

Horn's conjecture. Characterization of (α, β, γ) when $A + B = C$.

$$s_\mu s_\nu = \sum_\lambda \boldsymbol{c^\lambda_{\mu\nu}} s_\lambda$$

$c^\lambda_{\mu\nu}$: **Littlewood-Richardson coefficient**

Saturation conjecture.

$$c^{k\lambda}_{k\mu,k\nu} \neq 0 \Rightarrow c^\lambda_{\mu\nu} \neq 0$$

Klyachko. Saturation $\Rightarrow$ Horn.

Knutson-Tao. Saturation conjecture is true.

$$\boldsymbol{A} = n \times n \text{ hermitian matrix}$$

$$\text{spectrum}: \ \boldsymbol{\theta_1, \ldots, \theta_n}$$

$$\boldsymbol{\rho_A} = \text{“natural” measure on } n \times n$$
$$\text{hermitian matrices } A' \text{ with}$$
$$\operatorname{spec}(A) = \operatorname{spec}(A')$$

$$\boldsymbol{\nu_A} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\theta_j}$$

(mass $1/n$ at each eigenvalue).

$\boldsymbol{A_n, B_n}$: $n \times n$ hermitian **(s.t.a.)**

Let $\boldsymbol{\nu_1, \nu_2}$ be probability measures with compact support on $\mathbb{R}$ such that

$$\nu_{A_n} \to \nu_1, \quad \nu_{B_n} \to \nu_2$$

(weakly).

Theorem. *Choose* A'_n *and* B'_n *from* ρ_{A_n}, ρ_{B_n}. *Then*

$$\nu_{A'_n+B'_n} \to \nu_1 \boxplus \nu_2$$

(weakly, in probability) as $n \to \infty$.

I.e., if we know spec(A) and spec(B) (n large) then we can bet with a good chance to win, that

$$\nu_{A+B} \approx \nu_A \boxplus \nu_B.$$

Thus spec(A) and spec(B) determine spec($A+B$) almost surely as $n \to \infty$.

Equivalently, A'_n and B'_n become free in the limit $n \to \infty$ with respect to

$$\varphi(\cdot) = \frac{1}{n}\mathrm{tr}(\cdot).$$

ASYMPTOTIC REPRESENTATION THEORY OF $\mathfrak{S}_n$

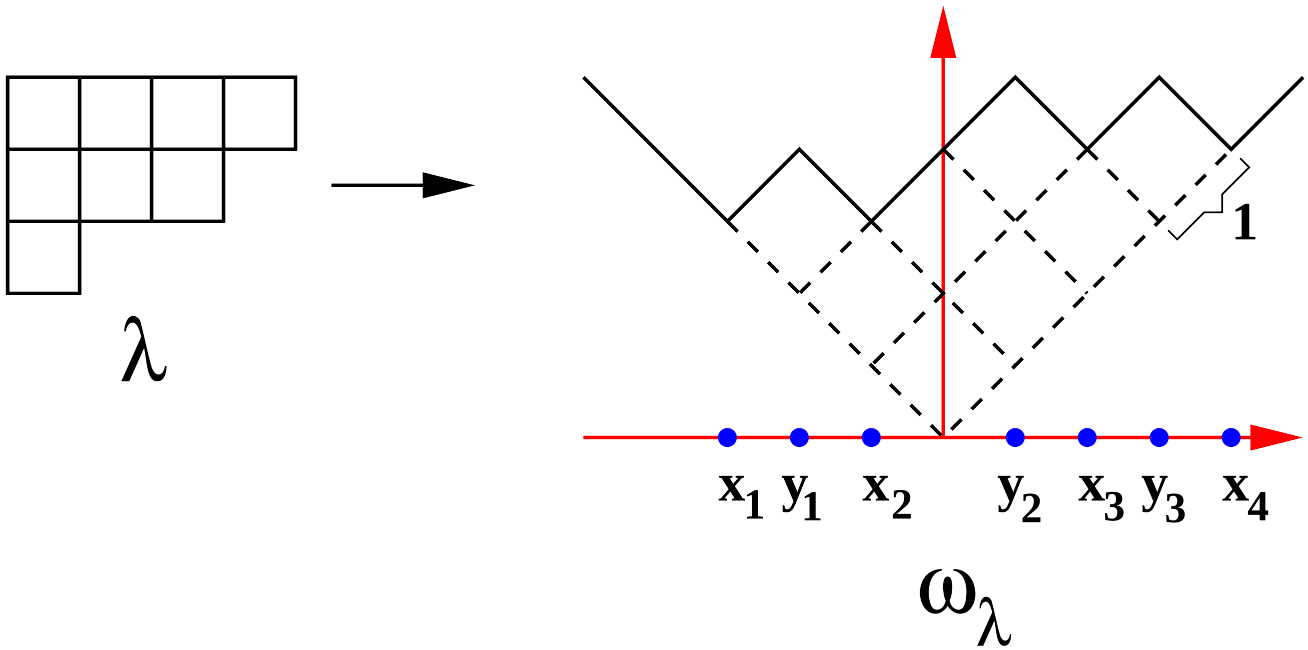
Fix $\mu \vdash k$. Let $\lambda \vdash n$,

$$\chi^\lambda = \text{irred. char. of } \mathfrak{S}_n \text{ indexed by } \lambda$$

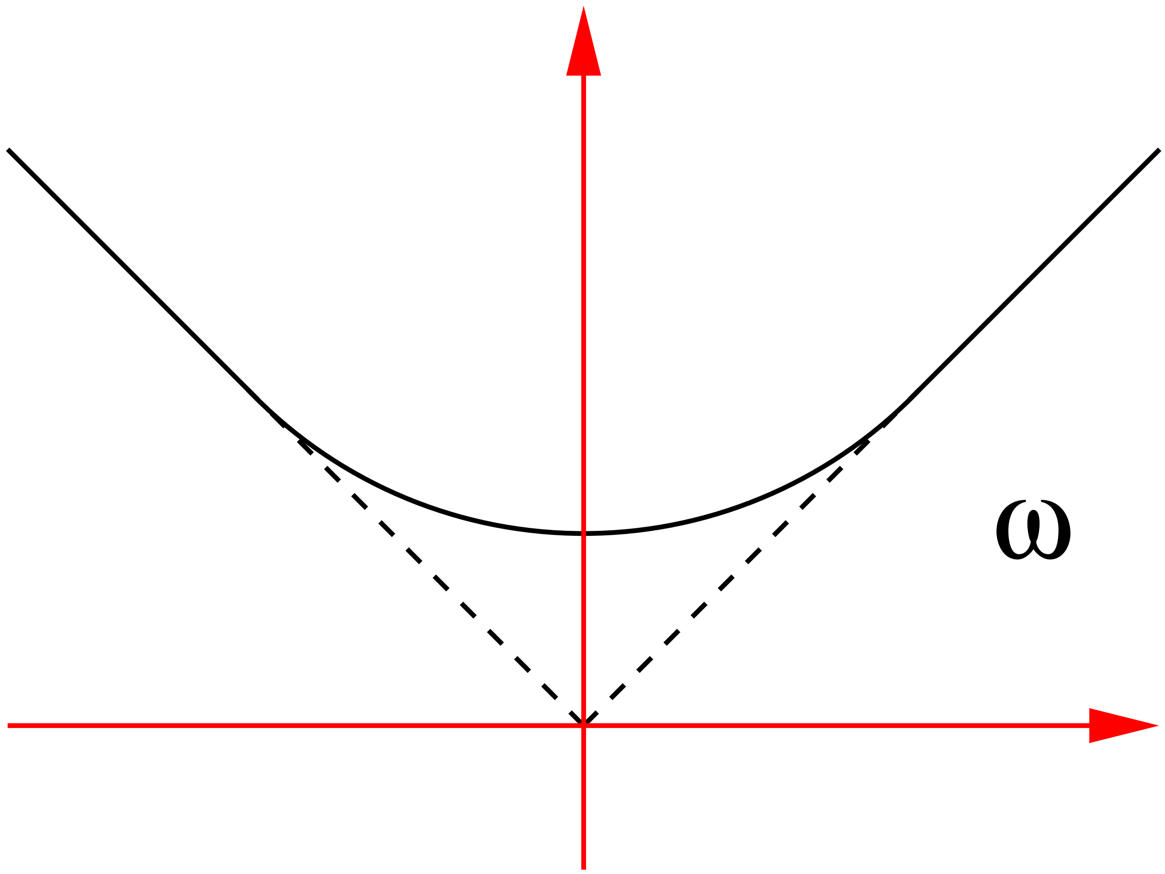
$$\chi^\lambda(\mu) := \chi^\lambda(\mu 1^{n-k}),$$

value at $w \in \mathfrak{S}_n$ of cycle type $(\mu, 1^{n-k})$.

Asymptotics of $\chi^\lambda(\mu)$ for μ fixed, n large?



λ
1
x1
y1
x2
y2
x3
y3
x4
ω
λ



$$|\omega(u) - \omega(v)| \leq |u - v|, \ \omega(u) = |u| \text{ for } |u| >> 0$$

$$\boldsymbol{\sigma(u)} = \frac{1}{2}(\omega(u) - |u|) \text{ (compact support)}$$

$$\boldsymbol{G_\omega(z)} := \frac{1}{z}\exp\int_{\mathbb{R}}\frac{1}{x-z}\sigma'(x)dx$$

$$G_{\omega_\lambda} = \frac{\prod(z-y_i)}{\prod(z-x_i)}$$

$$G_\omega(z) = \frac{1}{z} + \sum_{n\geq 1}\boldsymbol{a_n}(\omega)z^{-n-1}$$

$$\begin{aligned}\boldsymbol{K_\omega(z)} &= G_\omega(z)^{\langle -1\rangle}\\ &= \frac{1}{z}\sum_{n\geq 1}\boldsymbol{C_n}(\omega)z^{n-1}\end{aligned}$$

$$C_1(\omega) = 0\ \forall\omega,\quad C_2(\omega_\lambda) = |\lambda|$$

$a_n(\omega)$ is a moment and $C_n(\omega)$ a free cumulant of some probability measure m_ω.

Consequences.

Theorem. *Let* $\lambda^n \vdash n$ *and*

$$\omega_{\hat{\lambda}^n}(u) = n^{-1/2}\omega_{\lambda^n}(n^{1/2}u)$$

(so diagram of λ *is rescaled to have area 1). Supppose that*

$$\hat{\omega}_{\lambda^n} \to \omega$$

(so $C_2(\omega) = 1$*). Then* $\chi^{\lambda^n}(\nu, 1^{n-k})/f^{\lambda^n}$

$$= \left(\prod_{i=1}^{\ell} C_{\nu_i+1}(\omega)\right) n^{(k-\ell)/2}\left(1 + O\left(\frac{1}{n}\right)\right),$$

where

$$\begin{aligned} \nu &= (\nu_1, \ldots, \nu_\ell) \vdash k \\ f^{\lambda^n} &= \chi^{\lambda^n}(1^n) \\ &= \#\ \text{SYT of shape } \lambda^n. \end{aligned}$$

Theorem (asymptotic Littlewood-Richardson rule). *Let*

$$\hat{\omega}_{\mu^n} \to \omega, \quad \hat{\omega}_{\nu^n} \to \omega'.$$

Almost all shapes (counting multiplicity) appearing in $s_{\mu^n} s_{\nu^n}$ *are "near" a shape* λ^n *such that*

$$\hat{\omega}_{\lambda^n} \to \omega \boxplus \omega',$$

where $C_n(\omega \boxplus \omega') = C_n(\omega) + C_n(\omega')$.