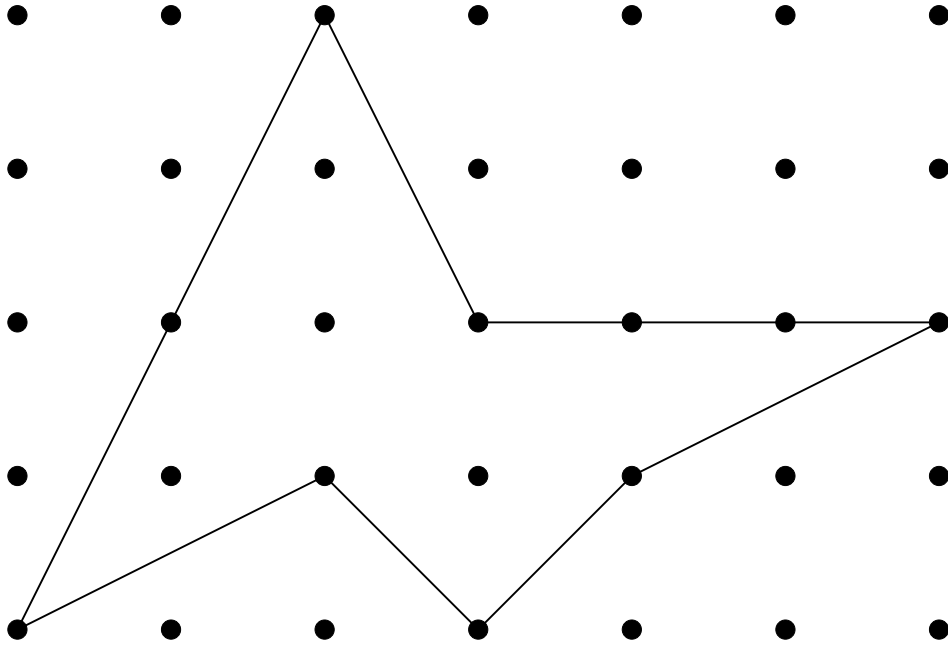
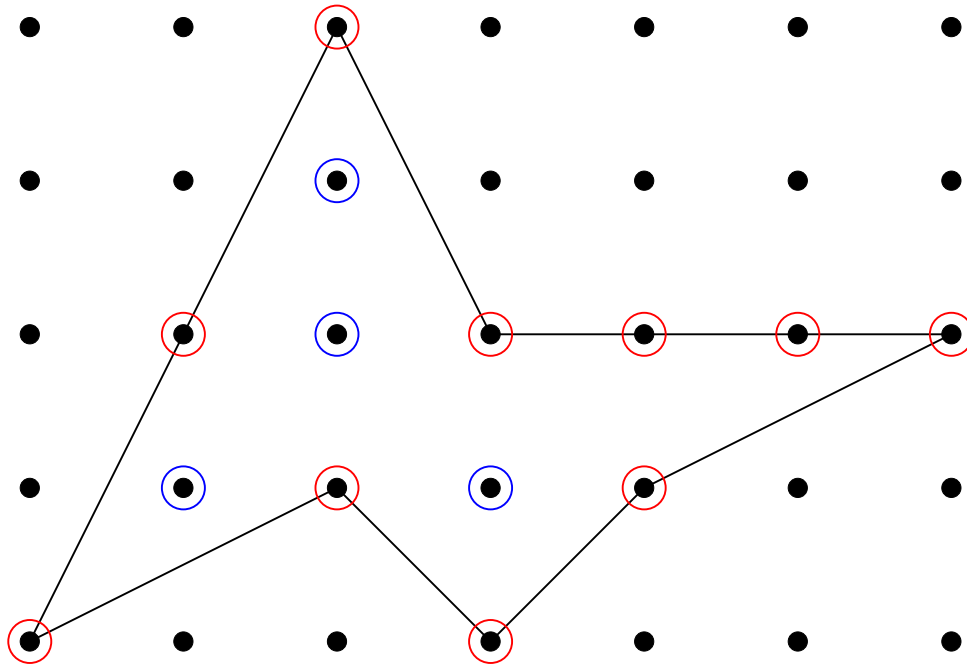


Georg Alexander Pick (1859–1942)

*P*: lattice polygon in  $\mathbb{R}^2$   
(vertices  $\in \mathbb{Z}^2$ , no self-intersections)





**A** = area of  $P$

**I** = # interior points of  $P$ (= 4)

**B** = #boundary points of  $P$ (= 10)

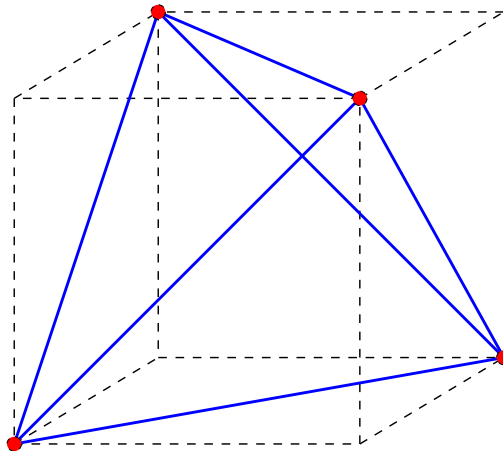
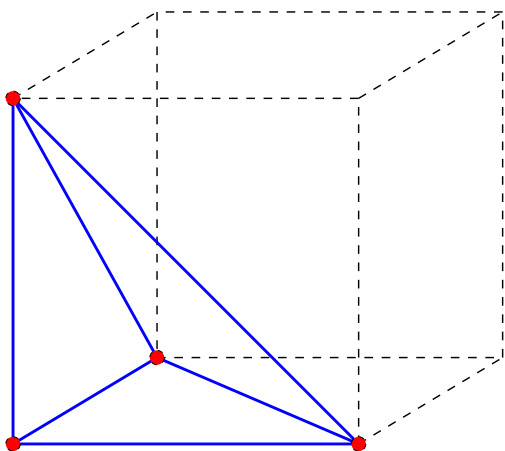
Then

$$\mathbf{A} = \frac{2\mathbf{I} + \mathbf{B} - 2}{2} = \frac{2 \cdot 4 + 10 - 2}{2} = 9.$$

Pick's theorem (seemingly) fails in higher dimensions. For example, let  $T_1$  and  $T_2$  be the tetrahedra with vertices

$$v(T_1) = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$v(T_2) = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$



Then

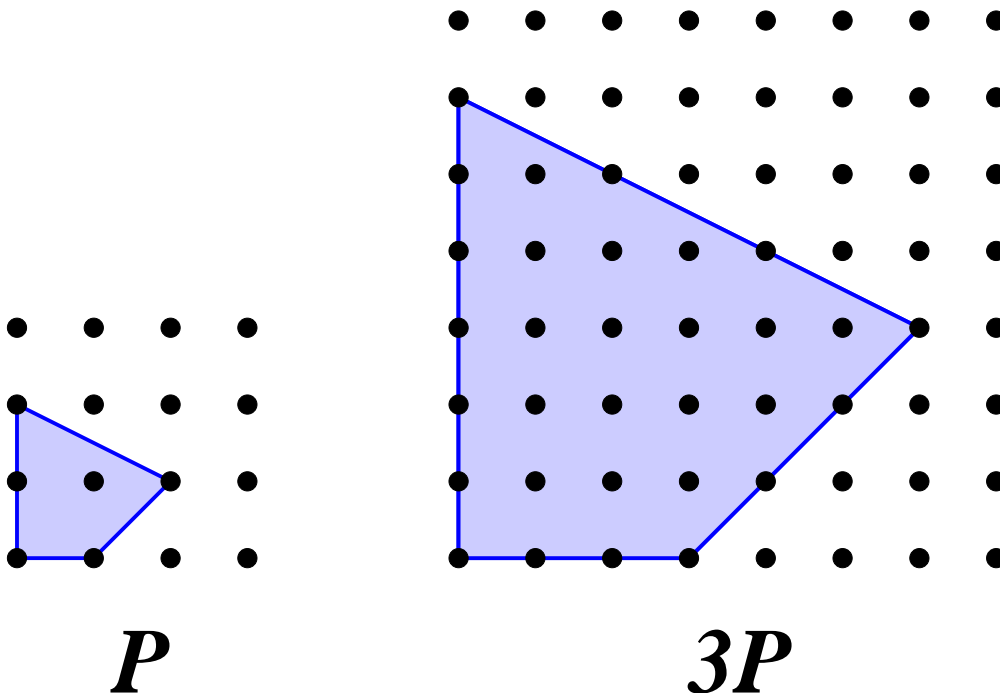
$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$A(T_1) = 1/6, \quad A(T_2) = 1/3.$$

Let  $\mathcal{P}$  be a convex polytope (convex hull of a finite set of points) in  $\mathbb{R}^d$ . For  $n \geq 1$ , let

$$n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}.$$



Let

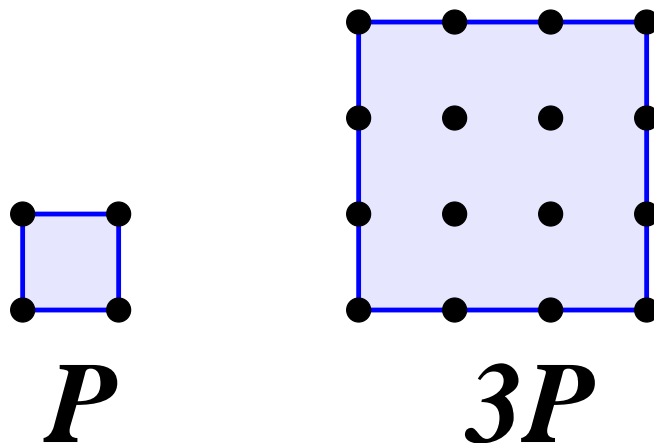
$$\begin{aligned} i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P} : n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in  $n\mathcal{P}$ .

Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

$$\begin{aligned} \bar{i}(\mathcal{P}, n) &= \#(n\mathcal{P}^\circ \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P}^\circ : n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$



$$i(\mathcal{P}, n) = (n + 1)^2$$

$$\bar{i}(\mathcal{P}, n) = (n - 1)^2 = i(\mathcal{P}, -n).$$

**lattice polytope:** polytope with integer vertices

**Theorem** (Reeve, 1957). *Let  $\mathcal{P}$  be a three-dimensional lattice polytope. Then the volume  $V(\mathcal{P})$  is a certain (explicit) function of  $i(\mathcal{P}, 1)$ ,  $\bar{i}(\mathcal{P}, 1)$ , and  $i(\mathcal{P}, 2)$ .*

**Theorem** (Ehrhart 1962, Macdonald 1963) *Let*

$\mathcal{P}$  = lattice polytope in  $\mathbb{R}^N$ ,  $\dim \mathcal{P} = d$ .

*Then  $i(\mathcal{P}, n)$  is a polynomial (the **Ehrhart polynomial** of  $\mathcal{P}$ ) in  $n$  of degree  $d$ . Moreover,*

$$i(\mathcal{P}, 0) = 1$$

$$\bar{i}(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n), \quad n > 0$$

**(reciprocity).**

*If  $d = N$  then*

$i(\mathcal{P}, n) = V(\mathcal{P})n^d + \text{lower order terms,}$

*where  $V(\mathcal{P})$  is the volume of  $\mathcal{P}$ .*

**Corollary** (generalized Pick's theorem). *Let  $\mathcal{P} \subset \mathbb{R}^d$  and  $\dim \mathcal{P} = d$ . Knowing any  $d$  of  $i(\mathcal{P}, n)$  or  $\bar{i}(\mathcal{P}, n)$  for  $n > 0$  determines  $V(\mathcal{P})$ .*

**Proof.** Together with  $i(\mathcal{P}, 0) = 1$ , this data determines  $d + 1$  values of the polynomial  $i(\mathcal{P}, n)$  of degree  $d$ . This uniquely determines  $i(\mathcal{P}, n)$  and hence its leading coefficient  $V(\mathcal{P})$ .  $\square$

**Example.** When  $d = 3$ ,  $V(\mathcal{P})$  is determined by

$$\begin{aligned} i(\mathcal{P}, 1) &= \#(\mathcal{P} \cap \mathbb{Z}^3) \\ i(\mathcal{P}, 2) &= \#(2\mathcal{P} \cap \mathbb{Z}^3) \\ \bar{i}(\mathcal{P}, 1) &= \#(\mathcal{P}^\circ \cap \mathbb{Z}^3), \end{aligned}$$

which gives Reeve's theorem.

**Example** (magic squares). Let  $\mathcal{B}_M \subset \mathbb{R}^{M \times M}$  be the **Birkhoff polytope** of all  $M \times M$  **doubly-stochastic** matrices  $A = (a_{ij})$ , i.e.,

$$a_{ij} \geq 0$$

$$\sum_i a_{ij} = 1 \text{ (column sums 1)}$$

$$\sum_j a_{ij} = 1 \text{ (row sums 1)}.$$

**Note.**  $B = (b_{ij}) \in n\mathcal{B}_M \cap \mathbb{Z}^{M \times M}$   
if and only if

$$b_{ij} \in \mathbb{N} = \{0, 1, 2, \dots\}$$

$$\sum_i b_{ij} = n$$

$$\sum_j b_{ij} = n.$$

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \quad (M = 4, n = 7)$$

$$\begin{aligned}
 H_M(n) &:= \#\{M \times M \mathbb{N}\text{-matrices, line sums } n\} \\
 &= i(\mathcal{B}_M, n).
 \end{aligned}$$

E.g.,

$$H_1(n) = 1$$

$$H_2(n) = n + 1$$

$$\begin{bmatrix} a & n - a \\ n - a & a \end{bmatrix}, \quad 0 \leq a \leq n.$$

$$H_3(n) = \binom{n+2}{4} + \binom{n+3}{4} + \binom{n+4}{4}$$

(MacMahon)

$$H_M(0) = 1$$

$$H_M(1) = M! \text{ (permutation matrices)}$$

**Theorem** (Birkhoff-von Neumann) *The vertices of  $\mathcal{B}_M$  consist of the  $M!$   $M \times M$  permutation matrices. Hence  $\mathcal{B}_M$  is a lattice polytope.*

**Corollary** (Anand-Dumir-Gupta conjecture)  *$H_M(n)$  is a polynomial in  $n$  (of degree  $(M - 1)^2$ ).*

**Example.** 
$$H_4(n) = \frac{1}{11340} \left( 11n^9 + 198n^8 + 1596n^7 + 7560n^6 + 23289n^5 + 48762n^5 + 70234n^4 + 68220n^2 + 40950n + 11340 \right).$$

Reciprocity  $\Rightarrow$

$\pm H_M(-n) = \#\{M \times M \text{ matrices } B \text{ of}$   
**positive** integers, line sum  $n\}$ .

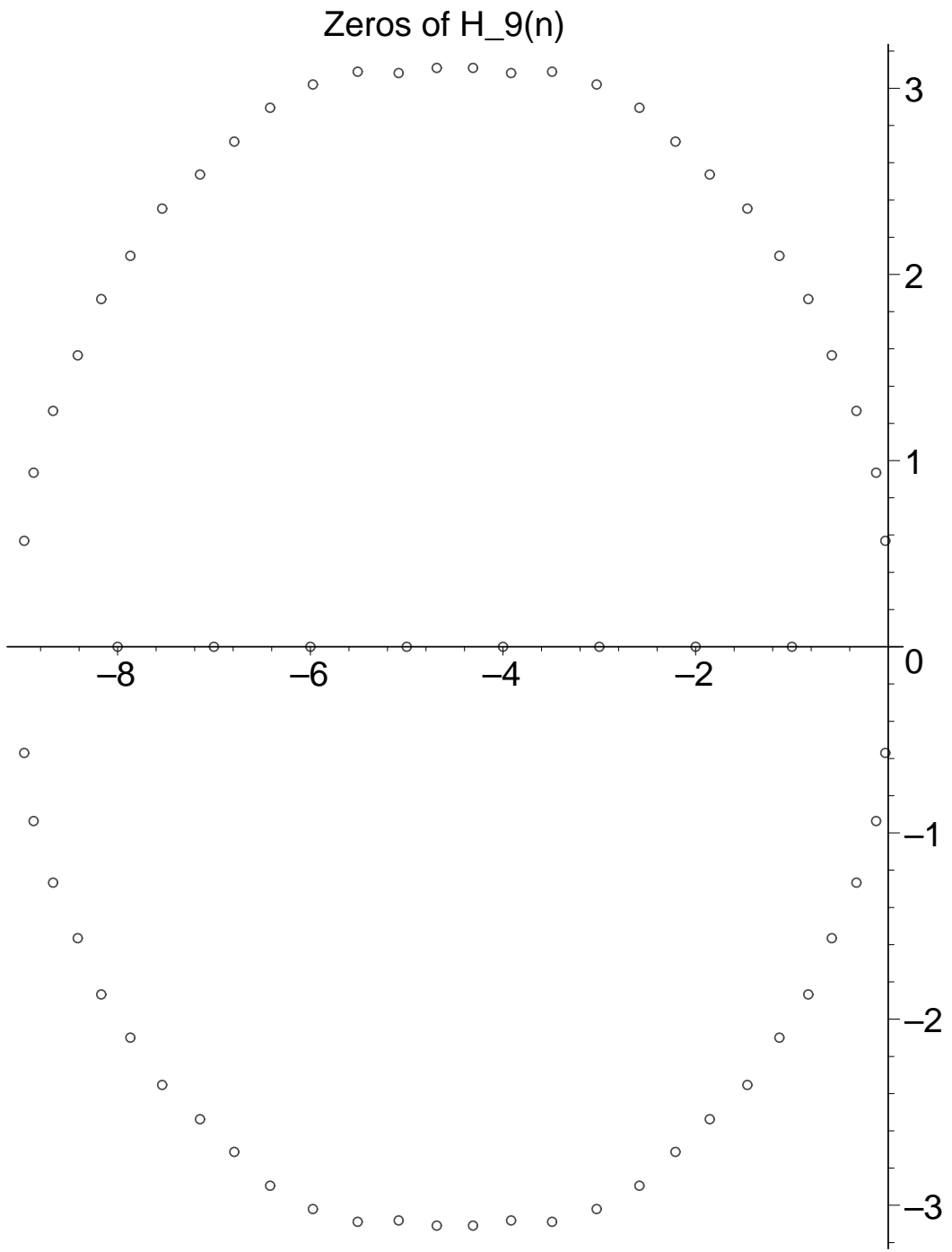
But every such  $B$  can be obtained from an  $M \times M$  matrix  $A$  of **nonnegative** integers by adding 1 to each entry.

**Corollary.**  $H_M(-1) = H_M(-2) = \dots = H_M(-M + 1) = 0$

$$H_M(-M - n) = (-1)^{M-1} H_M(n)$$

(greatly reduces computation)

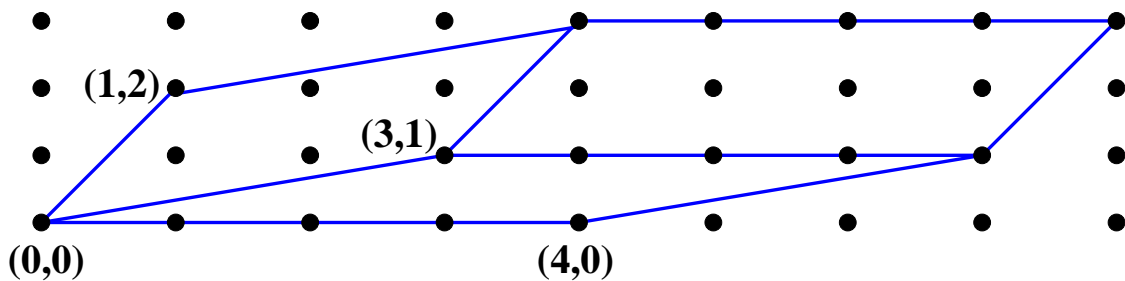
Applications e.g. to statistics (contingency tables).



**Zonotopes.** Let  $v_1, \dots, v_k \in \mathbb{R}^d$ .  
 The **zonotope**  $Z(v_1, \dots, v_k)$  generated  
 by  $v_1, \dots, v_k$ :

$$Z(v_1, \dots, v_k) = \{\lambda_1 v_1 + \dots + \lambda_k v_k : 0 \leq \lambda_i \leq 1\}$$

**Example.**  $v_1 = (4, 0)$ ,  $v_2 = (3, 1)$ ,  
 $v_3 = (1, 2)$



**Theorem.** *Let*

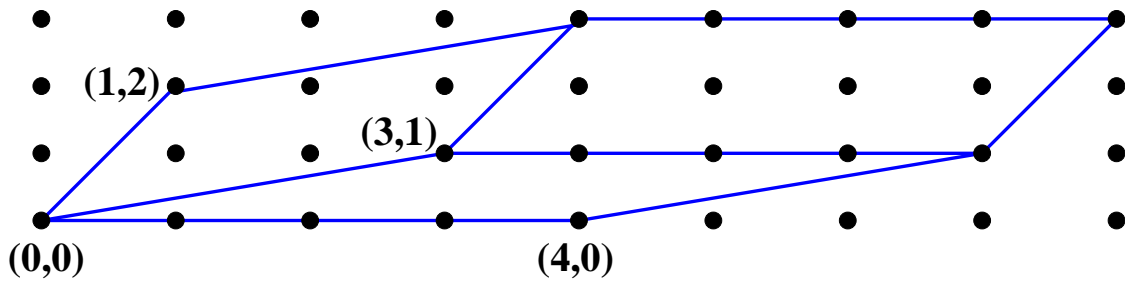
$$Z = Z(v_1, \dots, v_k) \subset \mathbb{R}^d,$$

*where*  $v_i \in \mathbb{Z}^d$ . *Then*

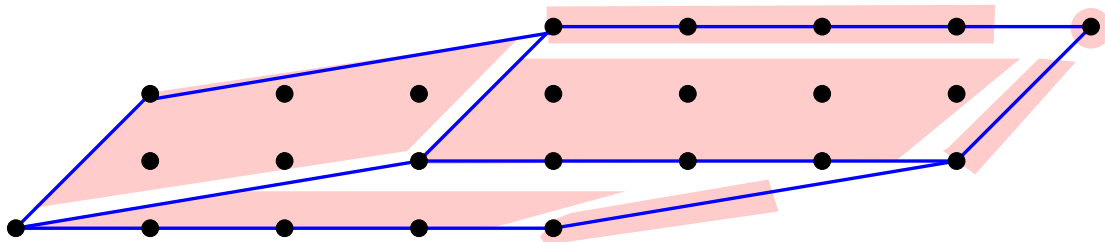
$$i(Z, 1) = \sum_X h(X),$$

*where*  $X$  *ranges over all linearly independent subsets of*  $\{v_1, \dots, v_k\}$ , *and*  $h(X)$  *is the gcd of all*  $j \times j$  *minors* ( $j = \#X$ ) *of the matrix whose rows are the elements of*  $X$ .

**Example.**  $v_1 = (4, 0)$ ,  $v_2 = (3, 1)$ ,  
 $v_3 = (1, 2)$



$$\begin{aligned}
 i(Z, 1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\
 &\quad + \gcd(4, 0) + \gcd(3, 1) \\
 &\quad + \gcd(1, 2) + \det(\emptyset) \\
 &= 4 + 8 + 5 + 4 + 1 + 1 + 1 \\
 &= 24.
 \end{aligned}$$



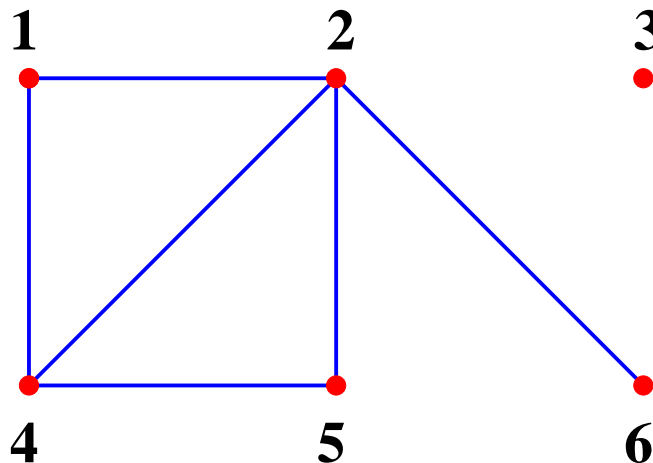
Let  $G$  be a graph (with no loops or multiple edges) on the vertex set  $V(G) = \{1, 2, \dots, n\}$ . Let

$d_i$  = degree (# incident edges) of vertex  $i$ .

Define the **ordered degree sequence**  $d(G)$  of  $G$  by

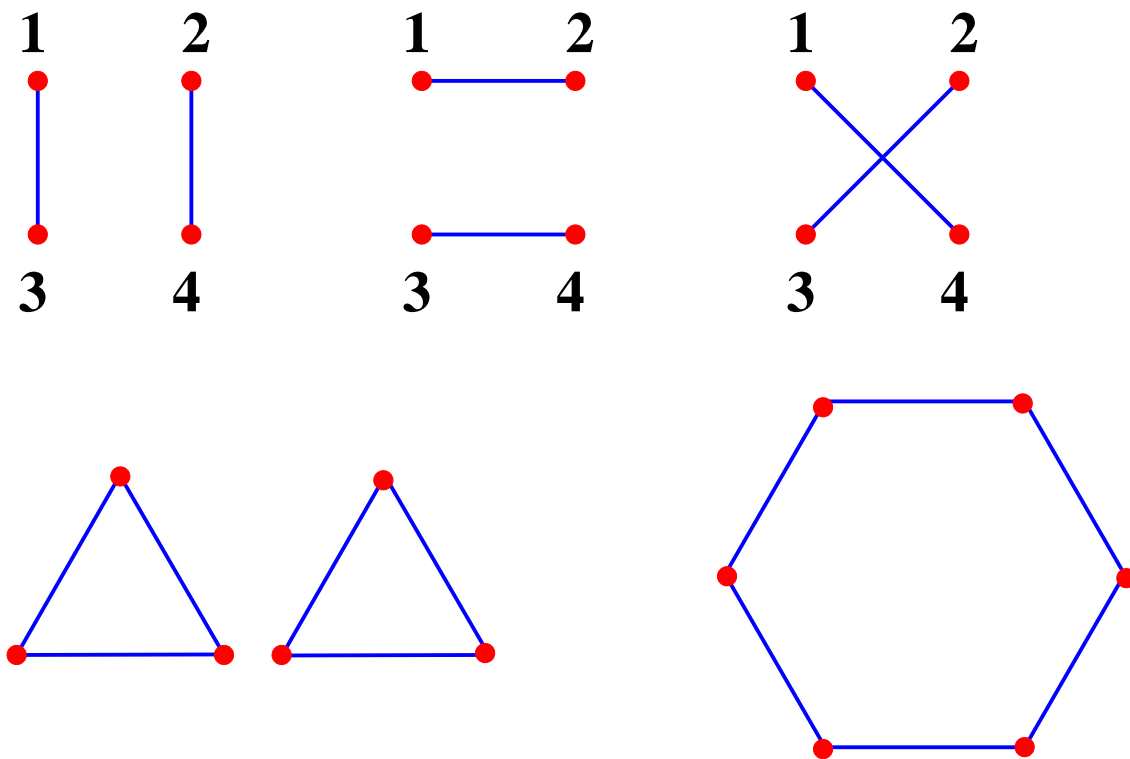
$$d(G) = (d_1, \dots, d_n).$$

**Example.**  $d(G) = (2, 4, 0, 3, 2, 1)$



Let  $f(n)$  be the number of distinct  $d(G)$ , where  $V(G) = \{1, 2, \dots, n\}$ .

**Example.** If  $n \leq 3$ , all  $d(G)$  are distinct, so  $f(1) = 1$ ,  $f(2) = 2^1 = 2$ ,  $f(3) = 2^3 = 8$ . For  $n \geq 4$  we can have  $G \neq H$  but  $d(G) = d(H)$ , e.g.,



In fact,  $f(4) = 54 < 2^6 = 64$ .

Let **conv** denote convex hull, and

$$\mathcal{D}_n = \text{conv}\{d(G) : V(G) = \{1, \dots, n\}\},$$

the **polytope of degree sequences** (Perles, Koren).

**Easy fact.** Let  $e_i$  be the  $i$ th unit coordinate vector in  $\mathbb{R}^n$ . E.g., if  $n = 5$  then  $e_2 = (0, 1, 0, 0, 0)$ . Then

$$\mathcal{D}_n = Z(e_i + e_j : 1 \leq i < j \leq n).$$

**Theorem** (Erdős-Gallai). *Let  $\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n$ . Then  $\alpha = d(G)$  for some  $G$  if and only if*

- $\alpha \in \mathcal{D}_n$
- $a_1 + a_2 + \dots + a_n$  is even.

“Fiddling around” leads to:

**Theorem.** *Let*

$$\begin{aligned} F(x) &= \sum_{n \geq 0} f(n) \frac{x^n}{n!} \\ &= 1 + x + 2 \frac{x^2}{2!} + 8 \frac{x^3}{3!} + 54 \frac{x^4}{4!} + \dots \end{aligned}$$

*Then*

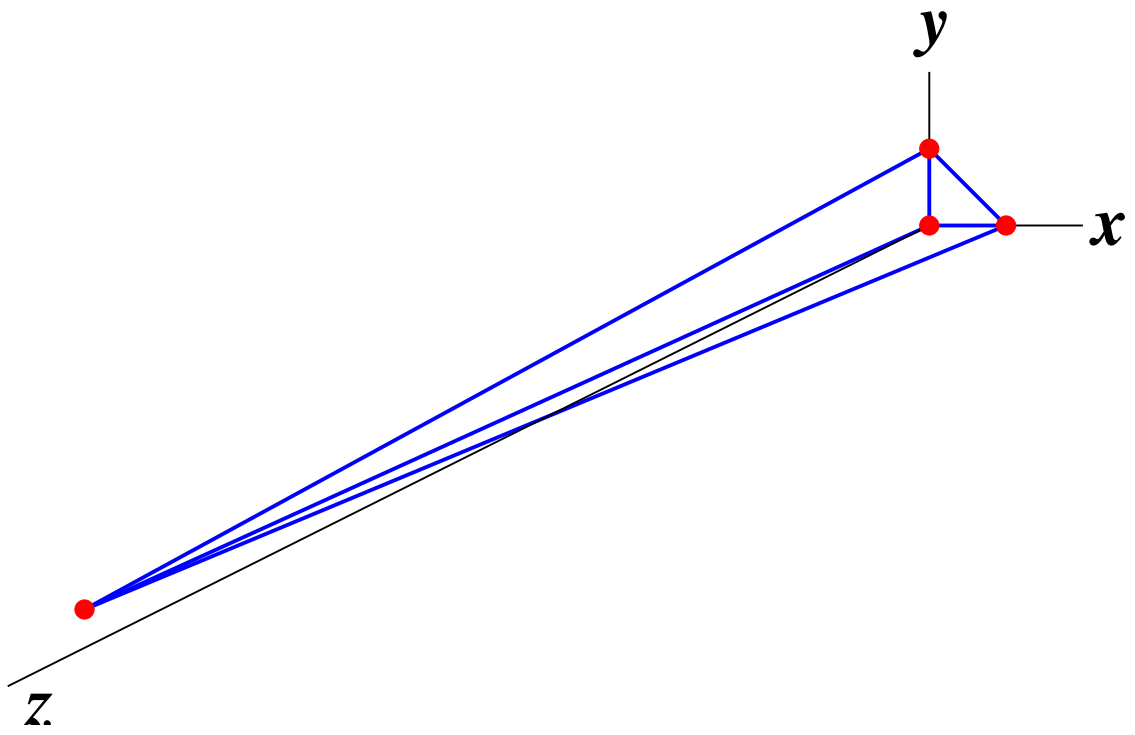
$$\begin{aligned} F(x) &= \frac{1}{2} \left[ \left( 1 + 2 \sum_{n \geq 1} n^n \frac{x^n}{n!} \right)^{1/2} \right. \\ &\quad \times \left. \left( 1 - \sum_{n \geq 1} (n-1)^{n-1} \frac{x^n}{n!} \right) + 1 \right] \\ &\quad \times \exp \sum_{n \geq 1} n^{n-2} \frac{x^n}{n!}. \end{aligned}$$

## The $h$ -vector of $i(\mathcal{P}, n)$

Let  $\mathcal{P}$  denote the tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(1, 1, 13)$ . Then

$$i(\mathcal{P}, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

Thus in general the coefficients of Ehrhart polynomials are not “nice.” Is there a “better” basis?



Let  $\mathcal{P}$  be a lattice polytope of dimension  $d$ . Since  $i(\mathcal{P}, n)$  is a polynomial of degree  $d$ ,  $\exists h_i \in \mathbb{Z}$  such that

$$\sum_{n \geq 0} i(\mathcal{P}, n)x^n = \frac{h_0 + h_1x + \cdots + h_dx^d}{(1-x)^{d+1}}.$$

**Definition.** Define

$$h(\mathcal{P}) = (h_0, h_1, \dots, h_d),$$

the  **$h$ -vector** of  $\mathcal{P}$ .

**Example.** Recall

$$\begin{aligned} i(\mathcal{B}_4, n) = & \frac{1}{11340} (11n^9 \\ & + 198n^8 + 1596n^7 + 7560n^6 + 23289n^5 \\ & + 48762n^5 + 70234n^4 + 68220n^2 \\ & + 40950n + 11340). \end{aligned}$$

Then

$$h(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

## Elementary properties of

$$h(\mathcal{P}) = (h_0, \dots, h_d):$$

- $h_0 = 1$
- $h_d = (-1)^{\dim \mathcal{P}} i(\mathcal{P}, -1) = I(\mathcal{P})$
- $\max\{i : h_i \neq 0\} = \min\{j \geq 0 : i(\mathcal{P}, -1) = i(\mathcal{P}, -2) = \dots = i(\mathcal{P}, -(d-j)) = 0\}$

E.g.,  $h(\mathcal{P}) = (h_0, \dots, h_{d-2}, 0, 0) \Leftrightarrow i(\mathcal{P}, -1) = i(\mathcal{P}, -2) = 0.$

- $i(\mathcal{P}, -n-k) = (-1)^d i(\mathcal{P}, n) \forall n \Leftrightarrow$

$$h_i = h_{d+1-k-i} \forall i, \text{ and}$$

$$h_{d+2-k-i} = h_{d+3-k-i} = \dots = h_d = 0$$

Recall:

$$h(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

Thus

$$i(\mathcal{B}_4, -1) = i(\mathcal{B}_4, -2) = i(\mathcal{B}_4, -3) = 0$$

$$i(\mathcal{B}_4, -n - 4) = -i(\mathcal{B}_4, n).$$

**Theorem A** (nonnegativity). (McMullen, RS)  $h_i \geq 0$ .

**Theorem B** (monotonicity). (RS)  
*If  $\mathcal{P}$  and  $\mathcal{Q}$  are lattice polytopes and  $\mathcal{Q} \subseteq \mathcal{P}$ , then  $h_i(\mathcal{Q}) \leq h_i(\mathcal{P}) \forall i$ .*

B  $\Rightarrow$  A: take  $\mathcal{Q} = \emptyset$ .

Theorem A can be proved geometrically, but Theorem B requires **commutative algebra**.

$\mathcal{P}$  = lattice polytope in  $\mathbb{R}^d$

$R = R_{\mathcal{P}}$  = vector space over  $K$  with basis  $\{x^{\alpha}y^n : \alpha \in \mathbb{Z}^d, n \in \mathbb{P}, \alpha/n \in \mathcal{P}\} \cup \{1\}$ ,  
where if  $\alpha = (\alpha_1, \dots, \alpha_d)$  then

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

If  $\alpha/m, \beta/n \in \mathcal{P}$ , then

$$(\alpha + \beta)/(m + n) \in \mathcal{P}$$

by convexity. Hence  $R_{\mathcal{P}}$  is a **subalgebra** of the polynomial ring  $K[x_1, \dots, x_d, y]$ .

**Example.** (a) Let

$$\mathcal{P} = \text{conv}\{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Then

$$R_{\mathcal{P}} = K[y, x_1 y, x_2 y, x_1 x_2 y].$$

(b) Let

$$\mathcal{P} = \text{conv}\{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$

Then

$$R_{\mathcal{P}} = K[y, x_1 x_2 y, x_1 x_3 y, x_2 x_3 y, x_1 x_2 x_3 y^2].$$

Let

$$R_n = \text{span}_K \{x^\alpha y^n : \alpha/n \in \mathcal{P}\},$$

with  $R_0 = \text{span}_K \{1\} = K$ . Then

$$R = R_0 \oplus R_1 \oplus \cdots \quad (\text{vector space } \oplus)$$

$$R_i R_j \subseteq R_{i+j}.$$

Thus  $R$  is a **graded algebra**. Moreover,

$$\begin{aligned} \dim_K R_n &= \#\{x^\alpha y^n : \alpha/n \in \mathcal{P}\} \\ &= i(\mathcal{P}, n). \end{aligned}$$

Thus  $i(\mathcal{P}, n)$  is the **Hilbert function** of  $R$ . Moreover,

$$F(\mathcal{P}, x) := \sum_{n \geq 0} i(\mathcal{P}, n) x^n$$

is the **Hilbert series** of  $R_{\mathcal{P}}$ .

**Theorem** (Hochster). *Let  $\mathcal{P}$  be a lattice polytope of dimension  $d$ . Then  $R_{\mathcal{P}}$  is a **Cohen-Macaulay** ring.*

This means:  $\exists$  algebraically independent  $\theta_1, \dots, \theta_{d+1} \in R_1$  (called a **homogeneous system of parameters** or **h.s.o.p.**) such that  $R_{\mathcal{P}}$  is a finitely generated free module over

$$S = K[\theta_1, \dots, \theta_{d+1}].$$

Thus  $\exists \eta_1, \dots, \eta_s$  ( $\eta_i \in R_{e_i}$ ) such that

$$R_{\mathcal{P}} = \bigoplus_{i=1}^s \eta_i S$$

and  $\eta_i S \cong S$  (as  $S$ -modules).

Now

$$\begin{aligned} F(R_{\mathcal{P}}, x) &:= \sum_{n \geq 0} i(\mathcal{P}, n) x^n \\ &= \sum_{i=1}^s x^{e_i} F(S, x) \\ &= \frac{\sum_{i=1}^s x^{e_i}}{(1-x)^{d+1}}. \end{aligned}$$

Compare with

$$F(R_{\mathcal{P}}, x) = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{d+1}}$$

to conclude:

**Corollary.**  $\sum_{i=1}^s x^{e_i} = \sum_{j=0}^d h_j x^j$ . In particular,  $h_i \geq 0$ .

Now suppose:

$\mathcal{P}, \mathcal{Q}$  : lattice polytopes in  $\mathbb{R}^N$

$$\dim \mathcal{P} = d, \quad \dim \mathcal{Q} = e$$

$$\mathcal{Q} \subseteq \mathcal{P}.$$

Let

$$I = \text{span}_K \{x^\alpha y^n : \alpha \in \mathbb{Z}^N, \alpha/n \in \mathcal{P} - \mathcal{Q}\}.$$

**Easy:**  $I$  is an ideal of  $R_{\mathcal{P}}$  and

$$R_{\mathcal{P}}/I \cong R_{\mathcal{Q}}.$$

**Lemma.**  $\exists$  an h.s.o.p.  $\theta_1, \dots, \theta_{d+1}$  for  $R_{\mathcal{P}}$  such that  $\theta_1, \dots, \theta_{e+1}$  is an h.s.o.p. for  $R_{\mathcal{Q}}$  and

$$\theta_{e+2}, \dots, \theta_{d+1} \in I.$$

Thus

$$R_{\mathcal{Q}}/(\theta_1, \dots, \theta_{e+1}) \cong R_{\mathcal{Q}}/(\theta_1, \dots, \theta_{d+1}),$$

so the natural surjection  $f : R_{\mathcal{P}} \rightarrow R_{\mathcal{Q}}$  induces a (degree-preserving) surjection

$$\begin{aligned} \bar{f} : A_{\mathcal{P}} &:= R_{\mathcal{P}}/(\theta_1, \dots, \theta_{d+1}) \\ &\rightarrow A_{\mathcal{Q}} := R_{\mathcal{Q}}/(\theta_1, \dots, \theta_{e+1}). \end{aligned}$$

Since  $R_{\mathcal{P}}$  and  $R_{\mathcal{Q}}$  are Cohen-Macaulay,

$$\dim(A_{\mathcal{P}})_i = h_i(\mathcal{P}), \quad \dim(A_{\mathcal{Q}})_i = h_i(\mathcal{Q}).$$

The surjection

$$(A_{\mathcal{P}})_i \rightarrow (A_{\mathcal{Q}})_i$$

gives  $h_i(\mathcal{P}) \geq h_i(\mathcal{Q})$ .  $\square$

## Zeros of Ehrhart polynomials.

**Sample theorem** (de Loera, Develin, Pfeifle, RS) *Let  $\mathcal{P}$  be a lattice  $d$ -polytope. Then*

$$i(\mathcal{P}, \alpha) = 0, \alpha \in \mathbb{R} \Rightarrow -d \leq \alpha \leq \lfloor d/2 \rfloor.$$

**Theorem.** *Let  $d$  be odd. There exists a  $0/1$   $d$ -polytope  $\mathcal{P}_d$  and a real zero  $\alpha_d$  of  $i(\mathcal{P}_d, n)$  such that*

$$\lim_{\substack{d \rightarrow \infty \\ d \text{ odd}}} \frac{\alpha_d}{d} = \frac{1}{2\pi e} = 0.0585 \dots$$

**Open.** Is the set of all complex zeros of all Ehrhart polynomials of lattice polytopes dense in  $\mathbb{C}$ ? (True for chromatic polynomials of graphs.)

## Further directions

- $R_{\mathcal{P}}$  is the coordinate ring of a projective algebraic variety  $X_{\mathcal{P}}$ , a **toric variety**. Leads to deep connections with toric geometry, including new formulas for  $i(\mathcal{P}, n)$ .
- **Complexity.** Computing  $i(\mathcal{P}, n)$ , or even  $i(\mathcal{P}, 1)$  is **#P-complete**. Thus an “efficient” (polynomial time) algorithm is extremely unlikely. However:

**Theorem** (A. Barvinok, 1994). *For fixed  $\dim \mathcal{P}$ ,  $\exists$  polynomial-time algorithm for computing  $i(\mathcal{P}, n)$ .*

**Reference.** M. Barvinok and J. Pommersheim, An algorithmic theory of lattice points in polyhedra, in *New Perspectives in Algebraic Combinatorics*, MSRI Publications, vol. 38, 1999, pp. 91–147.