Magic Squares and Syzygies

Richard P. Stanley
Traditional magic squares

Many elegant, ingenious, and beautiful constructions, but no general theory involving all of them.
Definition. An $n \times n$ magic square with line sum $r$ is an $n \times n$ matrix of nonnegative integers for which every row and column sums to $r$. 
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$$\begin{bmatrix}
2 & 0 & 5 \\
3 & 3 & 1 \\
2 & 4 & 1
\end{bmatrix}$$

$n = 3, \ r = 7$
Definition. An $n \times n$ magic square with line sum $r$ is an $n \times n$ matrix of nonnegative integers for which every row and column sums to $r$. 

\[
\begin{bmatrix}
2 & 0 & 5 \\
3 & 3 & 1 \\
2 & 4 & 1 \\
\end{bmatrix} \quad \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
\end{bmatrix}
\]

$n = 3, \ r = 7 \quad n = 3, \ r = 6$
A basic question

How many $n \times n$ magic squares have line sum $r$?

Call this number $H_n(r)$. 
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Trivial case: $H_n(0) = 1$. 

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
The cases $n = 1, n = 2$

$H_1(r) = 1$: $[r]$
The cases \( n = 1, n = 2 \)

\[ H_1(r) = 1: \quad [r] \]

What about \( n = 2 \)? Let \( 0 \leq i \leq r \).

\[
\begin{bmatrix}
i & ？ \\
？ & ？
\end{bmatrix}
\]
The cases $n = 1$, $n = 2$

$H_1(r) = 1$: $[r]$

What about $n = 2$? Let $0 \leq i \leq r$.

\[
\begin{bmatrix}
  i & r - i \\
  r - i & ?
\end{bmatrix}
\]
The cases $n = 1, n = 2$

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The cases $n = 1, n = 2$

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What about $n = 2$? Let $0 \leq i \leq r$.

$$\begin{bmatrix}
  i & r - i \\
  r - i & i
\end{bmatrix}$$

Hence $H_2(r) = r + 1$ ($r + 1$ choices for $0 \leq i \leq r$).
The case $r = 1$

If every row and column of an $n \times n$ magic square $M$ sums to 1, then $M$ has one 1 in every row and column and 0’s elsewhere, i.e., $M$ is a permutation matrix.
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If every row and column of an $n \times n$ magic square $M$ sums to 1, then $M$ has one 1 in every row and column and 0’s elsewhere, i.e., $M$ is a permutation matrix.

$n$ choices for where to put 1 in the first row. Then $n - 1$ choices for 1 in the second row. Then $n - 2$ choices for 1 in the third row. Etc.
**The case $r = 1$**

If every row and column of an $n \times n$ magic square $M$ sums to 1, then $M$ has one 1 in every row and column and 0’s elsewhere, i.e., $M$ is a **permutation matrix**.

$n$ choices for where to put 1 in the first row.
Then $n - 1$ choices for 1 in the second row.
Then $n - 2$ choices for 1 in the third row.
Etc.

Thus $H_n(1) = n(n - 1)(n - 2) \cdots 1 = n!$. 
Let $0 \leq k \leq n$. Define

$$\binom{n}{k} = \frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!}.$$
Binomial coefficients

Let \(0 \leq k \leq n\). Define

\[
\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.
\]

Examples.

\[
\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n)
\]

\[
\binom{n+2}{3} = \frac{(n+2)(n+1)n}{6} = \frac{1}{6}(n^3 + 3n^2 + 2n).
\]
Binomial coefficients

Let $0 \leq k \leq n$. Define

$$\binom{n}{k} = \frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!}.$$

Examples. \(\binom{n}{2} = \frac{n(n - 1)}{2} = \frac{1}{2}(n^2 - n)\)

\(\binom{n + 2}{3} = \frac{(n + 2)(n + 1)n}{6} = \frac{1}{6}(n^3 + 3n^2 + 2n)\).

In general, \(\binom{n+j}{k}\) is a polynomial in \(n\), degree \(k\).
Let’s get serious

Theorem (Birkhoff-von Neumann, 1946, 1952). Every $n \times n$ magic square with row and column sum $r$ is a sum of $r$ permutation matrices (of size $n \times n$).
Let’s get serious

**Theorem (Birkhoff-von Neumann, 1946, 1952).**

Every $n \times n$ magic square with row and column sum $r$ is a sum of $r$ permutation matrices (of size $n \times n$).

\[
\begin{bmatrix}
2 & 0 & 1 \\
1 & 1 & 1 \\
0 & 2 & 1
\end{bmatrix}
= \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
+ \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
+ \\
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
Every $3 \times 3$ magic square with row and column sum $r$ is a sum of $r$ $3 \times 3$ permutation matrices (six such matrices).
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If all these sums were different, then from *Combinatorics 101* we would have

$$H_3(r) = \binom{r+5}{5}.$$
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If all these sums were different, then from Combinatorics 101 we would have

$$H_3(r) = \binom{r + 5}{5}.$$ 

However, the sums are not all different!
A relation

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]
Such a relation is called a **syzygy**.
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from suzugos (συζυγοζ), “yoked together”
Astronomical significance
The relation (syzygy) between the six $3 \times 3$ permutation matrices means that our first approximation $H_3(r) = \binom{r+5}{5}$ is an overcount. Get a “second approximation”

$$H_3(r) = \binom{r + 5}{5} - \binom{r + 2}{5}$$

$$= \frac{1}{8}(r^4 + 6r^3 + 15r^2 + 18r + 8)$$

$$= \frac{1}{8}(r + 1)(r + 2)(r^2 + 3r + 4).$$
No further corrections!

No further syzygies, so this is correct!
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No further syzygies, so this is correct!

First proved by Percy Alexander MacMahon (1854–1929) c. 1916, as part of his syzygetic method. First real result on $H_n(r)$. 
Conjecture (1966). For any $n$, $H_n(r)$ is a polynomial in $r$ of degree $(n - 1)^2$. Moreover,

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n + 1) = 0$$

$$H_n(r) = (-1)^{n-1} H_n(-n - r).$$
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\[
H_n(r) = (-1)^{n-1} H_n(-n - r).
\]

Example. \( H_3(r) = \frac{1}{8}(r + 1)(r + 2)(r^2 + 3r + 4) \)
What about $4 \times 4$ magic squares?
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There are $24$ $4 \times 4$ permutation matrices, giving a first approximation $\binom{r+23}{23}$. 
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Now, however, there are 178 “independent” syzygies with varying numbers of terms.
These 178 syzygies have relations among them, called second-order syzygies.
Not the end of the story · · ·

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There are 1837 second-order syzygies.
These 178 syzygies have relations among them, called **second-order syzygies**.

There are 1837 second-order syzygies.

Then 7416 third-order
These 178 syzygies have relations among them, called second-order syzygies.

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Not the end of the story · · ·

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These *178* syzygies have relations among them, called **second-order syzygies**.

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Then *7416* third-order, *16440* fourth order, *25144*, *35562*
These 178 syzygies have relations among them, called second-order syzygies.

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Then \textcolor{red}{7416} third-order, \textcolor{red}{16440} fourth order, \textcolor{red}{25144}, \textcolor{red}{35562}, \textcolor{red}{42204}, \textcolor{red}{35562}, \textcolor{red}{25144}, \textcolor{red}{16440}, \textcolor{red}{7416}, \textcolor{red}{1837}, \textcolor{red}{178}
These **178** syzygies have relations among them, called **second-order syzygies**.

There are **1837** second-order syzygies.

Then **7416** third-order, **16440** fourth order, **25144**, **35562**, **42204**, **35562**, **25144**, **16440**, **7416**, **1837**, **178**, and finally ending in **one** 14th order syzygy!
To define syzygies rigorously requires commutative algebra.

David Hilbert (1862–1943): proved famous Hilbert syzygy theorem (1890).
To define syzygies rigorously requires **commutative algebra**.

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For $n \times n$ magic squares, guarantees that the syzygy process will come to an end in at most $n!$ steps.
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**David Hilbert** (1862–1943): proved famous **Hilbert syzygy theorem** (1890).

For $n \times n$ magic squares, guarantees that the syzygy process will come to an end in at most $n!$ steps.

This implies that $H_n(r)$ is indeed a polynomial in $r$, at least for $r$ sufficiently large. Extra tweaking needed for all $r$. 
More refined results

More sophisticated algebra (Cohen-Macaulay rings, Auslander-Buchsbaum theorem): the chain of syzygies ends in exactly $n! - n^2 + 2n - 2$ steps.
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Number of syzygies at each step for $n = 4$:
1, 178, 1837, 7416, 16440, 25144, 35562, 42204, 35562, 25144, 16440, 7416, 1837, 178, 1
More refined results

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Number of syzygies at each step for \( n = 4 \):

1, 178, 1837, 7416, 16440, 25144, 35562, 42204, 35562, 25144, 16440, 7416, 1837, 178, 1

Note symmetry! (Sequence is palindromic.)
Key reason for symmetry: let $J$ be the $n \times n$ all 1’s matrix. Then $M$ is an $n \times n$ magic square with row and column sums $r$ if and only if $M + J$ is an $n \times n$ magic square with positive entries and row and column sums $r + n$. 
Key reason for symmetry: let $J$ be the $n \times n$ all 1’s matrix. Then $M$ is an $n \times n$ magic square with row and column sums $r$ if and only if $M + J$ is an $n \times n$ magic square with positive entries and row and column sums $r + n$.

Symmetry implies the remainder of the Anand-Dumir-Gupta conjecture:

\[ H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0 \]

\[ H_n(r) = (-1)^{n-1} H_n(-n - r). \]
over $5.7 \times 10^{34}$ syzygies!
over $5.7 \times 10^{34}$ syzygies!

Mankind may never know the exact number.
Another approach

Every $3 \times 3$ magic square has exactly one of the forms:

$$\begin{bmatrix}
  a + e & b + d & c \\
  c + d & a & b + e \\
  b & c + e & a + d
\end{bmatrix}, \quad \begin{bmatrix}
  a & b + d & c + e + 1 \\
  c + d & a + e + 1 & b \\
  b + e + 1 & c & a + d
\end{bmatrix},$$

$$\begin{bmatrix}
  a + d + 1 & b & c + e + 1 \\
  c & a + e + 1 & b + d + 1 \\
  b + e + 1 & c + d + 1 & a
\end{bmatrix},$$

where $a, b, c, d, e \geq 0$. 

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This gives:

\[ H_3(r) = \binom{r + 4}{4} + \binom{r + 3}{4} + \binom{r + 2}{4}. \]
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This type of argument can be done for any \( n \), using a geometric approach (shellability).
Further formulas

This gives:

\[ H_3(r) = \binom{r + 4}{4} + \binom{r + 3}{4} + \binom{r + 2}{4}. \]

This type of argument can be done for any \( n \), using a geometric approach (shellability).

Regard an \( n \times n \) magic square as lying in the \( n^2 \)-dimensional space \( \mathbb{R}^{n^2} \).
For $4 \times 4$ magic squares we get

\[
1 \left( \begin{array}{c} r + 9 \\ 9 \end{array} \right) + 14 \left( \begin{array}{c} r + 8 \\ 9 \end{array} \right) + 87 \left( \begin{array}{c} r + 7 \\ 9 \end{array} \right) + 148 \left( \begin{array}{c} r + 6 \\ 9 \end{array} \right) + 87 \left( \begin{array}{c} r + 5 \\ 9 \end{array} \right) + 14 \left( \begin{array}{c} r + 4 \\ 9 \end{array} \right) + 1 \left( \begin{array}{c} r + 3 \\ 9 \end{array} \right)
\]
For $4 \times 4$ magic squares we get

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$$+ 87 \left( \begin{array}{c} r + 5 \\ 9 \end{array} \right) + 14 \left( \begin{array}{c} r + 4 \\ 9 \end{array} \right) + 1 \left( \begin{array}{c} r + 3 \\ 9 \end{array} \right)$$

(352 cases in all)
For $4 \times 4$ magic squares we get

\[1 \left( \binom{r+9}{9} \right) + 14 \left( \binom{r+8}{9} \right) + 87 \left( \binom{r+7}{9} \right) + 148 \left( \binom{r+6}{9} \right) + 87 \left( \binom{r+5}{9} \right) + 14 \left( \binom{r+4}{9} \right) + 1 \left( \binom{r+3}{9} \right)\]

(352 cases in all)

For $n = 5$ there are 4718075 cases!
For $4 \times 4$ magic squares we get

$$1 \binom{r + 9}{9} + 14 \binom{r + 8}{9} + 87 \binom{r + 7}{9} + 148 \binom{r + 6}{9}$$

$$+ 87 \binom{r + 5}{9} + 14 \binom{r + 4}{9} + 1 \binom{r + 3}{9}$$

(352 cases in all)

For $n = 5$ there are 4718075 cases!

Note symmetry of $(1, 14, 87, 148, 87, 14, 1)$. Equivalent to previous symmetry phenomena.
Symmetry property useful for computation.
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Example. $H_4(r)$ is a polynomial of degree 9. Ten values are needed to compute $H_4(r)$, say $H_4(0) = 1, H_4(1) = 24, H_4(2), \ldots, H_4(9)$. 
Symmetry property useful for computation.

**Example.** $H_4(r)$ is a polynomial of degree 9. Ten values are needed to compute $H_4(r)$, say $H_4(0) = 1$, $H_4(1) = 24$, $H_4(2)$, $\ldots$, $H_4(9)$.

But $H_4(-1) = H_4(-2) = H_4(-3) = 0$, $H_4(0) = -H_4(-4)$, $H_4(1) = -H_4(-5)$, $H_4(2) = -H_4(-6)$, $H_4(3) = -H_4(-7)$.

Thus only need to compute $H_4(0) = 1$, $H_4(1) = 24$, $H_4(2)$, $H_4(3)$. 
(1, 14, 87, 148, 87, 14, 1)

Note $1 \leq 14 \leq 87 \leq 148$ (unimodality).
Unimodality

\[ (1, 14, 87, 148, 87, 14, 1) \]

Note \( 1 \leq 14 \leq 87 \leq 148 \) (unimodality).

True for any \( n \).

Very deep proof: toric varieties, intersection homology, hard Lefschetz theorem, . . . .
$H_9(r)$ is a polynomial of degree 64 and thus has 64 complex zeros (or roots) $z$, i.e., $H_9(z) = 0$. 
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Zeros of $H_9(n)$
Holey magic squares: some entries (or holes) are specified to be 0.
**First variation**

**Holey magic squares:** some entries (or holes) are specified to be 0.

Example.

\[
\begin{bmatrix}
* & * & 0 & * \\
* & * & 0 & * \\
0 & 0 & * & * \\
* & * & * & * \\
\end{bmatrix}
\]
$H_{n,S}(r)$: number of $n \times n$ magic squares with line sum $r$ and hole set $S$. 
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Example.

$$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$
$H_{n,S}(r)$: number of $n \times n$ magic squares with line sum $r$ and hole set $S$.

Example. 

\[
\begin{bmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{bmatrix}
\]

$0 \leq i \leq r: \begin{bmatrix}
0 & i & r-i \\
r-i & 0 & i \\
i & r-i & 0
\end{bmatrix}$,

so $H_{3,S}(r) = r + 1$. 
If a magic square with holes is written as a sum of permutation matrices $P$, then each $P$ still has these holes.
If a magic square with holes is written as a sum of permutation matrices $P$, then each $P$ still has these holes.

Thus Birkhoff-von Neumann, syzygy, and shellability arguments still apply:

**Theorem.** $H_{n,S}(r)$ is a polynomial in $r$. 
An example

\[ M = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix} \]
An example

\[ M = \begin{bmatrix}
  \ast & \ast & 0 & \ast \\
  \ast & \ast & 0 & \ast \\
  0 & 0 & \ast & \ast \\
  \ast & \ast & \ast & \ast 
\end{bmatrix} \]

\[ H_{4, S}(r) = \frac{1}{24} (r + 1)(r + 2)(r + 3)(r^2 + 3r + 4) \]

\[ = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]
An example

\[ M = \begin{bmatrix}
  * & * & 0 & * \\
  * & * & 0 & * \\
  0 & 0 & * & * \\
  * & * & * & * 
\end{bmatrix} \]

\[ H_{4, S}(r) = \frac{1}{24} (r + 1)(r + 2)(r + 3)(r^2 + 3r + 4) \]

\[ = \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]

**Note.** The sequence \((1, 2, 2)\) is not symmetric!
Minimal positive magic squares

\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]
Minimal positive magic squares

\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]

There are two minimal positive magic squares:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
2 & 1 & 0 & 1 \\
0 & 0 & 3 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & 0 & 1 \\
1 & 2 & 0 & 1 \\
0 & 0 & 3 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]
\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]
Reciprocity

\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]

In fact: let \( \bar{H}_{4,S}(r) \) be the number of positive (except for the holes \( S \)) \( 4 \times 4 \) magic squares with hole set \( S \) and line sum \( r \).
Reciprocity

\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5}. \]

In fact: let \( \bar{H}_{4,S}(r) \) be the number of positive (except for the holes \( S \)) 4 \times 4 magic squares with hole set \( S \) and line sum \( r \).

\[ \bar{H}_{4,S}(r) = 2 \binom{r + 1}{5} + 2 \binom{r}{5} + 1 \binom{r - 1}{5}. \]
Reciprocity

\[ H_{4,S}(r) = 1 \binom{r + 5}{5} + 2 \binom{r + 4}{5} + 2 \binom{r + 3}{5} . \]

**In fact:** let \( \bar{H}_{4,S}(r) \) be the number of **positive** (except for the holes \( S \)) \( 4 \times 4 \) magic squares with hole set \( S \) and line sum \( r \).

\[ \bar{H}_{4,S}(r) = 2 \binom{r + 1}{5} + 2 \binom{r}{5} + 1 \binom{r - 1}{5} . \]

(1, 2, 2) and (2, 2, 1) are reverses!
Second variation

\( S_n(r) \): number of \( n \times n \) symmetric magic squares with line sums equal to \( r \)
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$S_n(r)$: number of $n \times n$ symmetric magic squares with line sums equal to $r$

\[
\begin{bmatrix}
1 & 2 & 0 & 5 \\
2 & 2 & 3 & 1 \\
0 & 3 & 4 & 1 \\
5 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Second variation

\[ S_n(r) \]: number of \( n \times n \) symmetric magic squares with line sums equal to \( r \)

\[
\begin{bmatrix}
1 & 2 & 0 & 5 \\
2 & 2 & 3 & 1 \\
0 & 3 & 4 & 1 \\
5 & 1 & 1 & 1
\end{bmatrix}
\]
Not a polynomial!

\[ S_3(r) = \begin{cases} \frac{1}{8}(2r^3 + 9r^2 + 14r + 8), & r \text{ even} \\ \frac{1}{8}(2r^3 + 9r^2 + 14r + 7), & r \text{ odd} \end{cases} \]
Not a polynomial!

\[ S_3(r) = \begin{cases} 
\frac{1}{8}(2r^3 + 9r^2 + 14r + 8), & r \text{ even} \\
\frac{1}{8}(2r^3 + 9r^2 + 14r + 7), & r \text{ odd}
\end{cases} \]

**Theorem.** For any \( n \geq 1 \) there exist polynomials \( P_n(r) \) and \( Q_n(r) \) such that

\[ S_n(r) = \begin{cases} 
P_n(r), & r \text{ even} \\
Q_n(r), & r \text{ odd.}
\end{cases} \]
Birkhoff-von Neumann fails

**False** that every symmetric magic square $M$ is a sum of symmetric permutation matrices:

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} = ?
$$
Why $r$ even and $r$ odd

However $2M$ is sum of symmetric magic squares of row and column sum two:

$$M = P_1 + \cdots + P_k \quad \text{(permutation matrices)}$$

$$\Rightarrow 2M = M + M^t$$

$$= (P_1 + P_1^t) + \cdots + (P_k + P_k^t).$$
Third variation

\( I_n(r) \): number of \( n \times n \times n \) magic cubes (rows, columns, and pillars sum to \( r \)
$I_n(r)$: number of $n \times n \times n$ magic cubes (rows, columns, and pillars sum to $r$)

No analogue of Birkhoff-von Neumann theorem. Very general results give:

Theorem. For every $n \geq 1$ there is a $p$ (depending on $n$) and polynomials
$T_{n,0}(r), \ldots, T_{n,p-1}(r)$ such that

$$I_n(r) = T_{n,i}(r) \text{ if } r \equiv i \pmod{p}.$$
Third variation

\( I_n(r) \): number of \( n \times n \times n \) magic cubes (rows, columns, and pillars sum to \( r \))

No analogue of Birkhoff-von Neumann theorem. Very general results give:

**Theorem.** For every \( n \geq 1 \) there is a \( p \) (depending on \( n \)) and polynomials \( T_{n,0}(r), \ldots, T_{n,p-1}(r) \) such that

\[
I_n(r) = T_{n,i}(r) \text{ if } r \equiv i \pmod{p}.
\]

Value of \( p \) not known in general.
$I_3(r) = \frac{1}{4480} \left( 18r^8 + 216r^7 + 1218r^6 + 4158r^5 + 9387r^4 + 14364r^3 + 14612r^2 + 10032r + 4480 \right),$

$r$ even
$$I_3(r) = \frac{1}{4480} \left( 18r^8 + 216r^7 + 1218r^6 + 4158r^5 ight.$$

$$+ 9387r^4 + 14364r^3 + 14612r^2 + 10032r + 4480 \right), \quad r \text{ even}$$

$$I_3(r) = \frac{1}{4480} \left( \cdots + 8142r + 1645 \right), \quad r \text{ odd}$$
$I_3(r) = \frac{1}{4480} (18r^8 + 216r^7 + 1218r^6 + 4158r^5$

$+ 9387r^4 + 14364r^3 + 14612r^2 + 10032r + 4480), \quad r \text{ even}$

$I_3(r) = \frac{1}{4480} (\cdots + 8142r + 1645), \quad r \text{ odd}$

$4 \times 4 \times 4: \ p = ?$
The last slide 😞

That’s all Folks!