(with A. Postnikov)

$\boldsymbol{S_n}$: symmetric group on $\{1, 2, \ldots, n\}$

$\boldsymbol{\ell(w)} = \#\{(i, j) \ : \ i < j, w(i) > w(j)\}$

$\boldsymbol{s_i} = (i, i+1)$ (**adjacent transposition**)

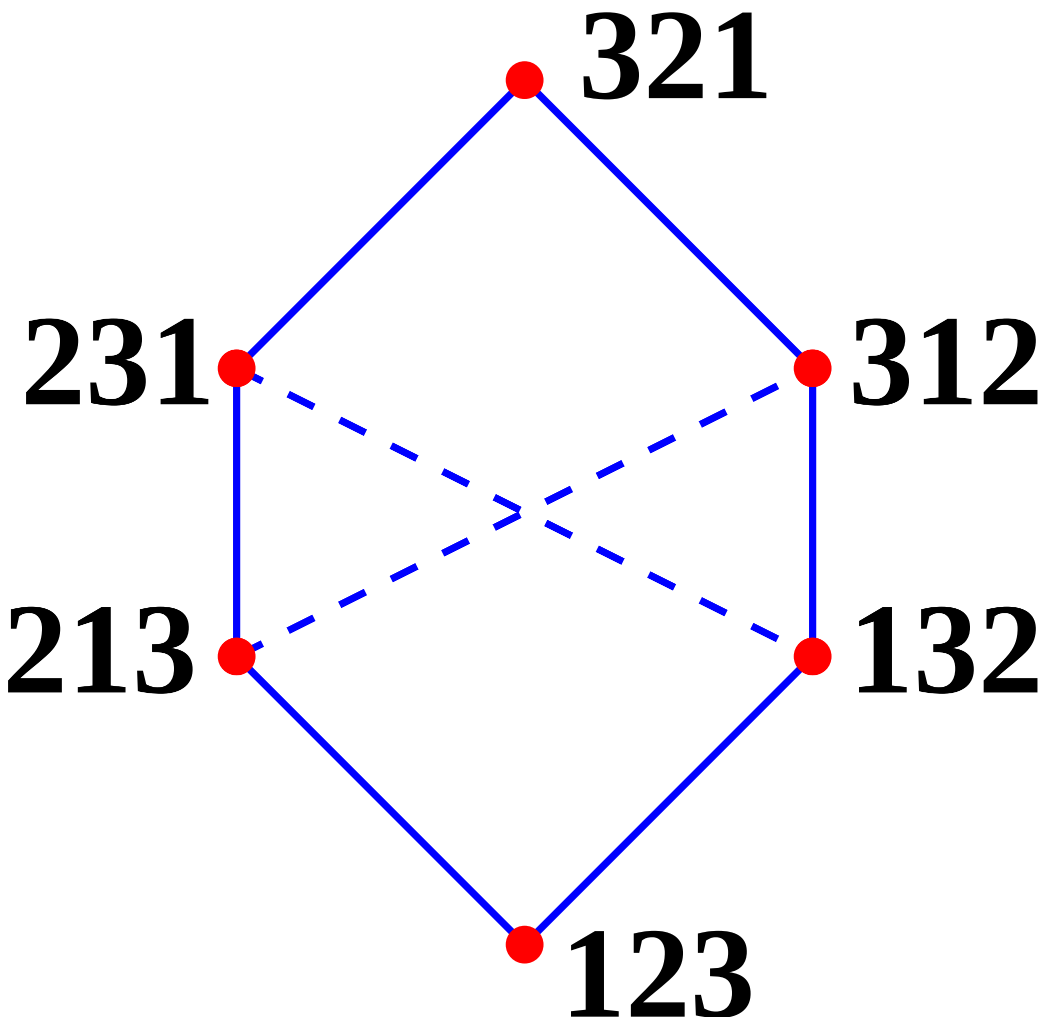
$\boldsymbol{W(S_n)}$: **weak (Bruhat) order** on S_n, with cover relations:

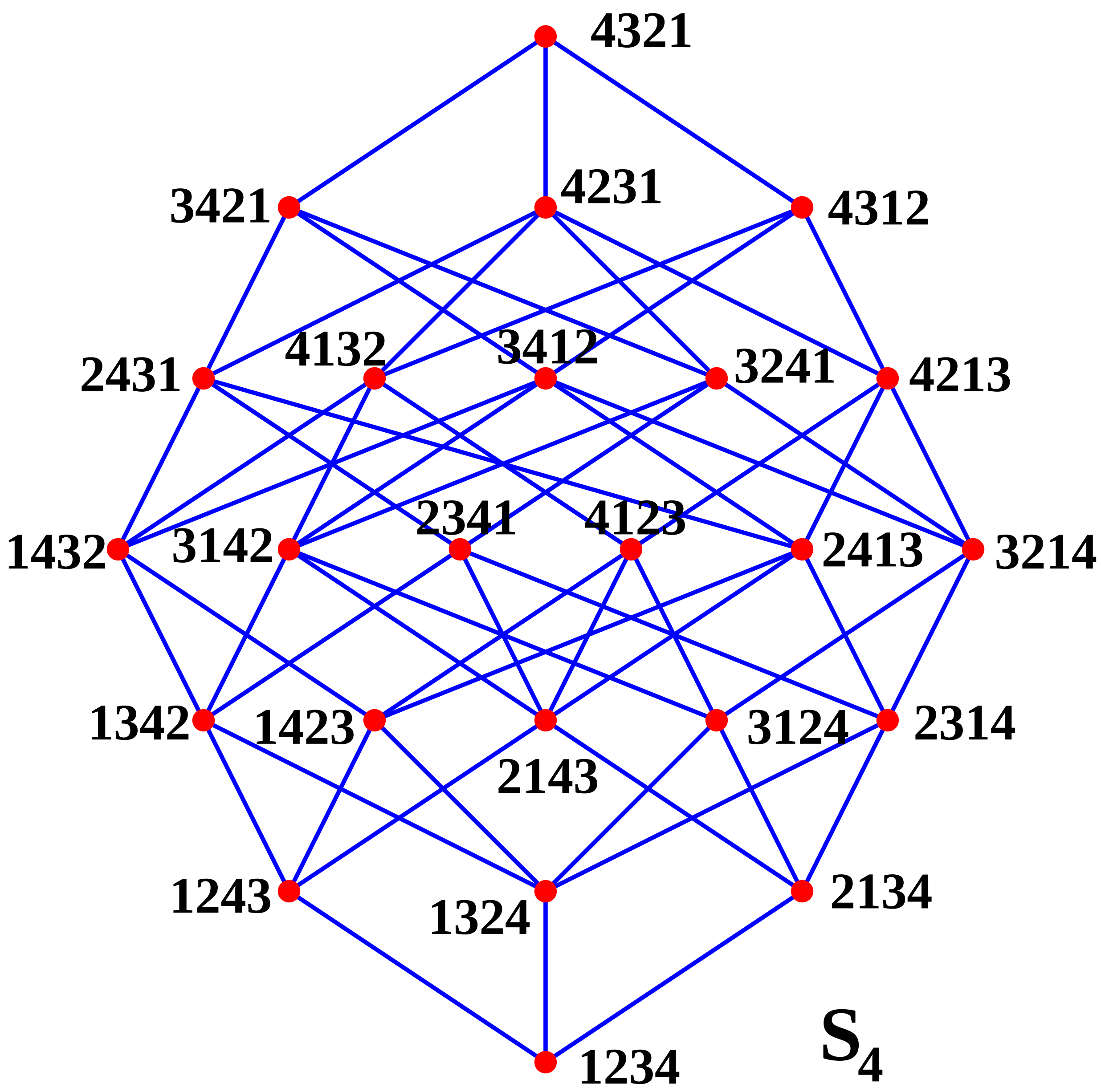
$$\boldsymbol{u \lessdot^* v} \text{ if } v = us_i, \ \ell(v) = 1 + \ell(u)$$

$\boldsymbol{S_n}$: **(strong) Bruhat order** on S_n, with cover relations:

$$\boldsymbol{u \lessdot v} \text{ if } v = u(i, j), \ \ell(v) = 1 + \ell(u)$$

$$u = 6\mathbf{2}\underbrace{718}_{\text{all } < 2 \text{ or } > 4}\mathbf{4}53 \lessdot 6\mathbf{4}718\mathbf{2}53 = u(2, 6)$$


321
231
312
213
132
123


4321
3421
4231
4312
2431
4132
3412
3241
4213
1432
3142
2341
4123
2413
3214
1342
1423
2143
3124
2314
1243
1324
2134
1234
S_4

S_n is a graded poset, where $\text{rank}(w) = \ell(w)$. Thus the **rank-generating function** of S_n is given by

$$\boldsymbol{F(S_n, q)} := \sum_{w \in S_n} q^{\text{rank}(w)}$$

$$= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Motivation. Let K be a field and

$$\boldsymbol{\mathcal{F}(K^n)} = \text{GL}(n, \mathbb{C})/B$$

the set of all (complete) **flags**

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = K^n$$

of subspaces of K^n (so $\dim V_i = i$).

For every such flag F, there are unique vectors $v_1, \ldots, v_n \in K^n$ such that:

- $\{v_1, \ldots, v_i\}$ is a basis for V_i
- The $n \times n$ matrix with rows $v_1, \ldots, v_n$ has the form

$$\begin{matrix} * & * & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{0} & * & * & \mathbf{1} \\ \mathbf{0} & * & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}.$$

The positions of the $\mathbf{1}$'s define a permutation $\boldsymbol{w_F} = 316452$. The number of $*$'s is $\ell(w_F)$.

For $w \in S_n$ define the **Bruhat cell**

$$\boldsymbol{\Omega_w} = \{F \in \mathcal{F}(K^n) \ : \ w = w_F\}.$$

Thus

$$\mathcal{F}(K^n) = \bigsqcup_{w \in S_n} \Omega_w,$$

the **Bruhat decomposition** of $\mathcal{F}(K^n)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\overline{\Omega}_w$: **closed** Bruhat cell

Theorem (Ehresmann, 1934)

$$\overline{\Omega}_v \subseteq \overline{\Omega}_w \Leftrightarrow v \leq w$$

(Bruhat order).

Example. $213 < 312$

$$\begin{bmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad < \quad \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} ax & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{x \to \infty} \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $u \lessdot^* us_i$ in $W(S_n)$. Define

$$\boldsymbol{m^*}(u, us_i) = i.$$

If

$$C : u_0 \lessdot^* u_1 \lessdot^* u_2 \lessdot^* \cdots \lessdot^* u_k$$

in $W(S_n)$, then define

$$\boldsymbol{m^*_C} = m^*(u_0, u_1)m^*(u_1, u_2) \cdots m^*(u_{k-1}, u_k).$$

Similarly let $u \lessdot u(i, j)$ in S_n, and define

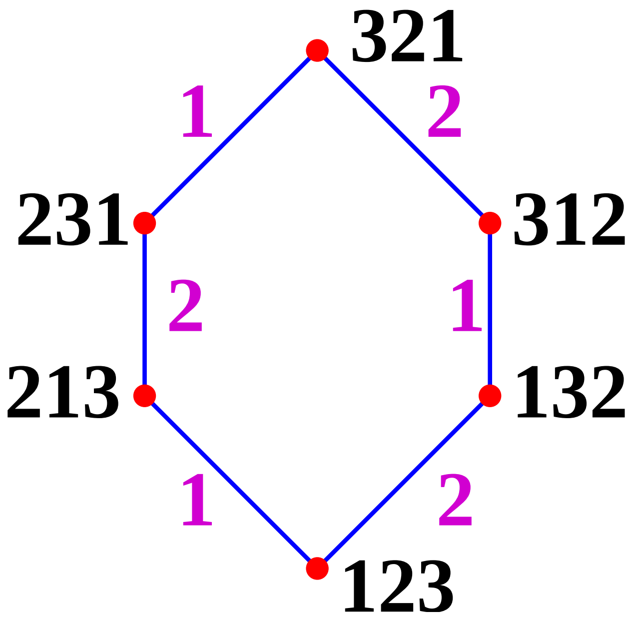
$$\boldsymbol{m}(u, u(i, j)) = j - i.$$

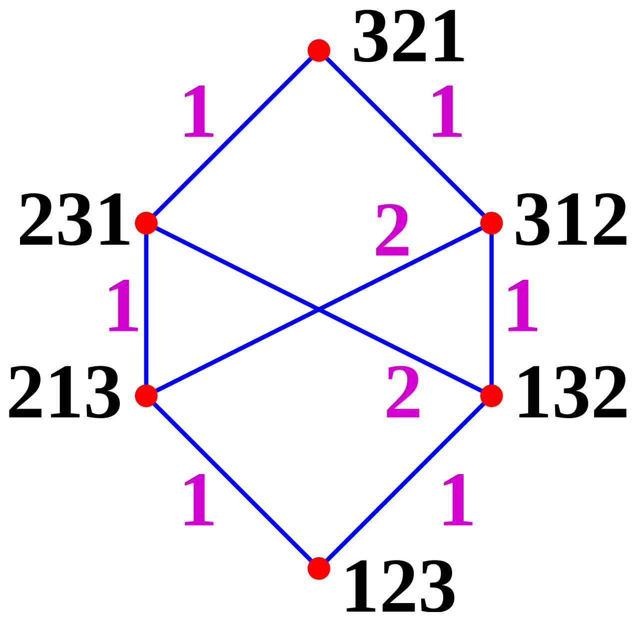
If

$$C : u_0 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k$$

in S_n, then define

$$\boldsymbol{m_C} = m(u_0, u_1)m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$





Let $\mathcal{M}(P)$ denote the set of maximal chains of the poset P. Thus

$$\sum_{C \in \mathcal{M}(W(S_3))} m_C^* = 1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2$$
$$= 6$$

$$\sum_{C \in \mathcal{M}(S_3)} m_C = 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1$$
$$+ 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 1$$
$$= 6.$$

Theorem. (a) (Macdonald; Fomin & RS)

$$\sum_{C \in \mathcal{M}(W(S_n))} m_C^* = \binom{n}{2}!$$

(b) (Stembridge (explicitly))

$$\sum_{C \in \mathcal{M}(S_n)} m_C = \binom{n}{2}!$$

Open. A bijective proof of (a) or (b), or a bijective proof that (a) = (b).

Generalize the definition $m(u, u(i,j))$ to

$$\boldsymbol{m}(u, u(i,j)) = \lambda_i - \lambda_j.$$

(Original definition corresponds to $\lambda_i = -i$.)
As before, if

$$C : u_0 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k$$

in S_n, then define

$$\boldsymbol{m_C}(\lambda) = m(u_0, u_1) m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$

If $u \leq v$ in S_n, define

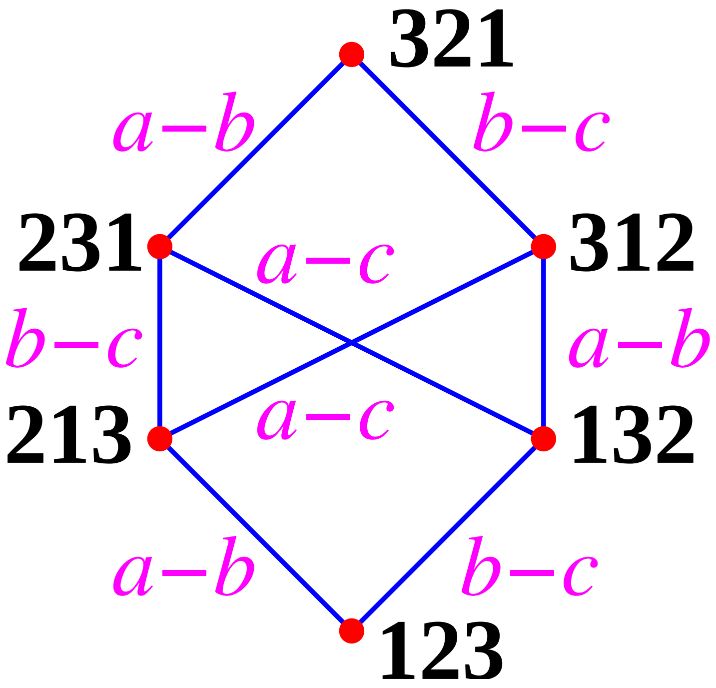
$$\boldsymbol{\mathcal{D}_{u,v}}(\lambda) = \frac{1}{(\ell(v) - \ell(u))!} \sum_C m_C(\lambda),$$

where C ranges over all saturated chains from u to v in S_n.

Set

$$\boldsymbol{\mathcal{D}_w}(\lambda) = \mathcal{D}_{\mathrm{id},w}(\lambda).$$

Write $a = \lambda_1$, $b = \lambda_2$, $c = \lambda_3$.



$$u = 123 = \mathrm{id}, \quad v = 321 = w_0$$

$$\begin{aligned}
\mathcal{D}_{321}(\lambda) &= \frac{1}{3!}((a-b)(b-c)(a-b) \\
&\quad +(a-b)(a-c)(b-c) \\
&\quad +(b-c)(a-c)(a-b) \\
&\quad +(b-c)(a-b)(b-c)) \\
&= \frac{1}{2}(a-b)(a-c)(b-c)
\end{aligned}$$

Schubert polynomials. Define the **divided difference operator ∂_i** by

$$\partial_i f(x_i, x_{i+1}) = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}.$$

Let $(a_1, a_2, \cdots, a_p)$ be a **reduced decomposition** of $w^{-1}w_0 \in S_n$, i.e.,

$$w^{-1}w_0 = s_{a_1} \cdots s_{a_p}, \quad p = \ell(w^{-1})w_0.$$

Schubert polynomial:

$$\mathfrak{S}_w = \partial_{a_1} \cdots \partial_{a_p} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Example. (a) $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$
(b) $(s_2)^{-1} w_0 = s_1 s_2$, so

$$\begin{aligned} \mathfrak{S}_{s_2} &= \partial_1 \partial_2 \, x_1^2 x_2 \\ &= x_1 + x_2. \end{aligned}$$

Theorem (B-J-S).

$$\mathfrak{S}_w = \sum_{(a_1,\ldots,a_p)} \sum_{(i_1,\ldots,i_p)} x_{i_1} \cdots x_{i_p},$$

where

- $(a_1, \ldots, a_p)$ ranges over all reduced decompositions of w
- $1 \le i_1 \le \cdots \le i_p$
- $i_j < i_{j+1}$ if $a_j < a_{j+1}$
- $i_j \le a_j$

Example. $w = 2143 = s_1 s_3 = s_3 s_1$

$$(a_1, a_2) = (1, 3) \Rightarrow (i_1, i_2) = (1, 2), (1, 3)$$
$$(a_1, a_2) = (3, 1) \Rightarrow (i_1, i_2) = (1, 1),$$

so $\mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3.$

Regard $S_n \subset S_{n+1}$ via $w(n+1) = n+1$ for $w \in S_n$. Let

$$\boldsymbol{S_\infty} = \bigcup S_n,$$

the permutations of $\{1, 2, \ldots\}$ moving finitely many letters. Then $\{\mathfrak{S}_w : w \in S_\infty\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, x_2, \ldots]$.

Monk's rule for $(x_1 + x_2 + \cdots + x_i)\mathfrak{S}_w$ gives:

$$(\lambda_1 x_1 + \lambda_2 x_2 + \cdots)^k \mathfrak{S}_u = k! \sum_{\ell(v)=k+\ell(u)} \mathcal{D}_{u,v}(\lambda)\mathfrak{S}_v.$$

Geometric interpretation of $\mathfrak{S}_w$.

$$\boldsymbol{H^*(\mathcal{F}(\mathbb{C}^n);\mathbb{R})}\ :\ \text{cohomology ring}$$

$$\text{basis}:\ \{[\bar{\Omega}_w]\ :\ w\in S_n\}$$

Let

$$e_j=\sum_{1\le i_1<\cdots<i_j\le n} x_{i_1}\cdots x_{i_j}.$$

Theorem.

$$H^*(\mathcal{F}(\mathbb{C}^n);\mathbb{R})\ \cong\ \mathbb{R}[x_1,\ldots,x_n]/(e_1,\ldots,e_n)$$
$$[\bar{\Omega}_w]\leftrightarrow\mathfrak{S}_w$$

Geometric interpretation of $\mathcal{D}_w(\lambda)$.

Let $\mathbf{\Phi} \subset \mathfrak{h}^*$ denote the root lattice for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of $G = \mathrm{SL}(n, \mathbb{C})$. Let

$$\mathbf{\Lambda} = \{\lambda \in \mathfrak{h}* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\},$$

the weight lattice of $\mathfrak{g}$. Let $\lambda \in \Lambda^+$ be a dominant weight. Let

$$\boldsymbol{V_\lambda} = \lambda\text{-weight space}$$

$$\boldsymbol{v_\lambda} \in V_\lambda : \text{ highest weight vector}$$

$$\mathbb{P}(\boldsymbol{V_\lambda}) = \text{projectivization of } V_\lambda$$

$$\boldsymbol{e} : G/B \to \mathbb{P}(V_\lambda)$$

$$e(gB) = g(v_\lambda)$$

$$\boldsymbol{\bar\Omega_w} \subset G/B \quad \text{(Schubert variety)}$$

Thus e is a projective embedding $G/B \hookrightarrow \mathbb{P}(V_\lambda)$. Define the **$\boldsymbol\lambda$-degree** of $\bar\Omega_w$ by:

$$\mathbf{deg}_{\boldsymbol\lambda}(\bar\Omega_w) = \#(e(\bar\Omega_w) \cap L),$$

where $\boldsymbol{L}$ is a generic linear subspace of $\mathbb{P}(V_\lambda)$ of complex codimension $\ell(w)$.

Theorem. $\deg_\lambda(\bar\Omega_w) = \ell(w)!\,\mathcal{D}_w(\lambda)$

An expression for $\mathcal{D}_{u,v}(\lambda)$.

Theorem. *Let $w \in S_n$ and*

$$V_n = \frac{1}{1!\,2!\,\cdots(n-1)!}\prod_{1\le i<j\le n}(\lambda_i - \lambda_j).$$

Then

$$\mathcal{D}_{u,v} =$$

$$\mathfrak{S}_u\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots\right)\mathfrak{S}_{w_0 v}\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots\right)\cdot V_n.$$

In particular,

$$\mathcal{D}_w = \mathcal{D}_{\mathrm{id},w} = \mathfrak{S}_{w_0 w}\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots\right)\cdot V_n.$$

$$\mathcal{D}_{w_0} = V_n \quad (\dots,\text{ Stembridge})$$

Corollary. *$\{\mathcal{D}_w : w \in S_\infty\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\lambda_1, \lambda_2, \ldots]$. Let*

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w \boldsymbol{c^w_{u,v}} \mathfrak{S}_w.$$

Then

$$\mathcal{D}_{u,w} = \sum_v c^w_{u,v} \mathcal{D}_v.$$

Note. (1) $\{\mathcal{D}_w : w \in S_n\}$ is a $\mathbb{Z}$-basis for Har_n, the harmonic polynomials in $\mathbb{Z}[\lambda_1, \ldots, \lambda_n]$.

(2) $\langle \mathfrak{S}_u, \mathcal{D}_v \rangle = \delta_{uv}$ under the "D-pairing"

$$\langle f, g \rangle = f\left(\frac{\partial}{\partial x_1}, \ldots\right) g(x_1, x_2, \ldots)\bigg|_{x_i=0}.$$

Corollary. *Let $w \in S_n$ be 312-avoiding, i.e.,*

$$a < b < c \Rightarrow \textbf{not } w(b) < w(c) < w(a).$$

Let **code**$(w_0 w) = (c_1, c_2, \ldots)$, *where*

$$c_i = \#\{j \,:\, i < j,\ w(i) > w(j)\}.$$

Then

$$\mathcal{D}_w = \det\left(\frac{\lambda_i^{n-c_i-j}}{(n-c_i-j)!}\right)_{i,j=1}^{n},$$

where $\alpha^k/k! = 0$ *if* $k < 0$.

Idea of proof.

w 312-avoiding $\Rightarrow$ $w_0 w$ 132-avoiding

(dominant)

$\Rightarrow \mathfrak{S}_{w_0 w} = x_1^{c_1} x_2^{c_2} \cdots$, etc.

Other special values are interesting, e.g.,

$$\begin{aligned}\mathcal{D}_{41532} = {} & \frac{1}{12}(f(5,4,2) - f(5,4,1) - f(5,3,2) \\ & +f(5,3,1) + f(4,3,2) - f(4,3,1)),\end{aligned}$$

where

$$\boldsymbol{f(i,j,k)} = (x_i - x_j)(x_i - x_k)(x_j - x_k).$$

Note that $\mathrm{code}(4,1,5,3,2) = (3,0,2,1,0)$.

Further connections:

- Demazure characters (key polynomials)
- Gelfand-Tsetlin polytopes
- inverse "Schubert Kostka" matrix