Polynomial Sequences of Binomial Type

Richard P. Stanley

M.I.T.

我今天的 讲座要用中文说,

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除了接下来要说的以外。

Some motivation

Let $\boldsymbol{D} = \frac{d}{dn}$, acting on $f(n) \in \mathbb{C}[n]$. Then

$$\begin{aligned} Dn^k &= kn^{k-1} \\ f(n) &= \sum_{k\geq 0} D^k f(0)\frac{n^k}{k!} \text{ (Taylor series).} \end{aligned}$$

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Let $\boldsymbol{\Delta} f(n) = f(n+1) - f(n)$ and $\boldsymbol{(n)_k} = n(n-1)\cdots(n-k+1)$. Then

$$\begin{aligned} \Delta(n)_k &= k(n)_{k-1} \\ f(n) &= \sum_{k\geq 0} \Delta^k f(0) \frac{(n)_k}{k!}. \end{aligned}$$

Connection between D and Δ

By Taylor's theorem,

$$f(n+x) = \sum_{k\geq 0} D^k f(n) \frac{x^k}{k!}.$$

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Put $x = 1$:

Connection (continued)

$$f(n+1) = \left(\sum_{k\geq 0} \frac{D^k}{k!}\right) f(n)$$
$$= e^D f(n).$$

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$$f(n+1) = \left(\sum_{k\geq 0} \frac{D^k}{k!}\right) f(n) = e^D f(n).$$

$$\Rightarrow \Delta f(n) = (e^D - 1) f(n) \Rightarrow \boldsymbol{\Delta = e^D - 1}.$$

Thus also $D = \log(\Delta + 1)$.

Finite operator calculus

General theory developed by **G.-C. Rota** and collaborators, called **finite operator calculus**.

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G.-C. Rota, *Finite Operator Calculus*, Academic Press, 1976.

The shift operator

Define $\boldsymbol{E} : \mathbb{C}[n] \to \mathbb{C}[n]$ by

$$Ef(n) = f(n+1).$$

Main thm. of operator calculus

Theorem. Let $L\colon \mathbb{C}[n] \to \mathbb{C}[n]$ be linear (over $\mathbb{C}$) and satisfy $L(n) = 1$ and $L(\deg d) = \deg d - 1$. The following two conditions are equivalent.

- $LE = EL$

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Theorem. Let $L\colon \mathbb{C}[n] \to \mathbb{C}[n]$ be linear (over $\mathbb{C}$) and satisfy $L(n) = 1$ and $L(\deg d) = \deg d - 1$. The following two conditions are equivalent.

- $LE = EL$
- There exist polynomials $p_k(n)$, $k \geq 0$, such that $p_0(n) = 1$, $\deg p_k(n) = k$, and

$$\begin{aligned} Lp_k(n) &= kp_{k-1}(n) \\ \sum_{k\geq 0} p_k(n)\frac{x^k}{k!} &= \left(\sum_{k\geq 0} p_k(1)\frac{x^k}{k!}\right)^n. \end{aligned}$$

Binomial type

If $p_0(n), p_1(n), \ldots$ is a sequence of polynomials satisfying

$$\sum_{k\geq 0} p_k(n)\frac{x^k}{k!} = \left(\sum_{k\geq 0} p_k(1)\frac{x^k}{k!}\right)^n,$$

then we call $p_0(n), p_1(n), \ldots$ a **sequence of polynomials of binomial type**, or just **polynomials of binomial type**.

A characterization

Note. The condition $\deg p_k(n) = k$ is then equivalent to $p_1(n) \neq 0$ (or just $p_1(1) \neq 0$). Sometimes this extra condition is part of the definition of binomial type.

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Theorem. *A sequence* $p_0(n) = 1, p_1(n), \ldots$ *of polynomials is of binomial type if and only if*

$$p_k(m+n) = \sum_{i=0}^{k} \binom{k}{i} p_i(m) p_{k-i}(n), \ k \geq 0.$$

Some classical examples

- $p_k(n) = n^k$

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$$\sum_{k\geq 0} \boldsymbol{(n)_k} \frac{x^k}{k!} = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$$

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More on Abel polynomials

Binomial type is equivalent to **Abel's identity**:

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x(x-iz)^{i-1}(y+iz)^{k-i}.$$

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Closely related to tree enumeration.

Yet another example

- $p_k(n) = \sum_{i=1}^{k} \underbrace{S(k,i)}_{\text{Stirling no. of 2nd kind}} n^i$

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$$\sum_{k\geq 0} \left(\sum_i S(k,i) n^i\right) \frac{x^k}{k!} = \left(\sum_{k\geq 0} \underbrace{B(k)}_{\text{Bell number}} \frac{x^k}{k!}\right)^n$$

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$$\sum_{k \geq 0} \left(\sum_{i} \binom{\boldsymbol{k}}{\boldsymbol{i}} \boldsymbol{i}^{\boldsymbol{k-i}} \boldsymbol{n}^{\boldsymbol{i}} \right) \frac{x^k}{k!} = \exp nxe^x$$

Binomial posets

Are there interesting examples of polynomials of binomial type for which explicit formulas don't exist?

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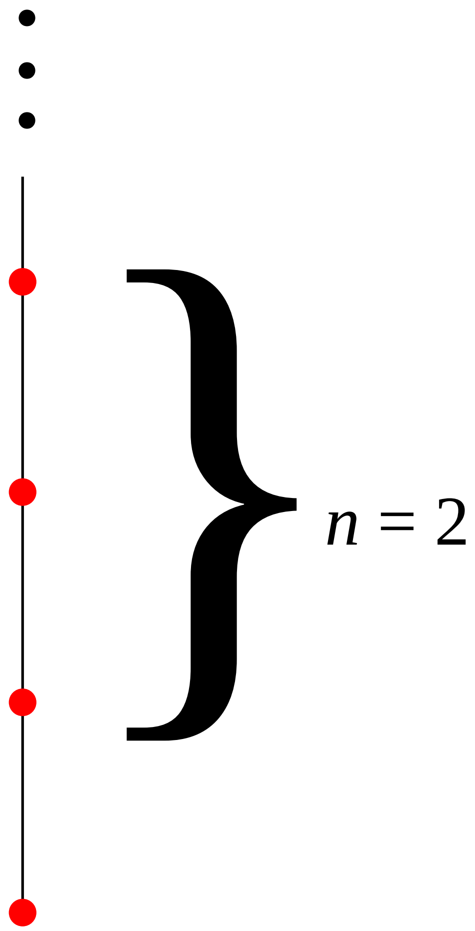
$\boldsymbol{P} = P_0 \cup P_1 \cup \cdots$ (disjoint union): a poset (partially ordered set) such that all maximal chains have the form $t_0 < t_1 < \dots$, where $t_i \in P_i$.

Write **rank**$(t_i) = i$. Then P is a **binomial poset** if for all $s \leq t$, where $\boldsymbol{k} = \mathrm{rank}(t) - \mathrm{rank}(s)$, the number of (saturated) chains $s = t_0 < t_1 < \cdots < t_k = t$ depends only on k. Call this number $\boldsymbol{B(k)}$.

Chains

- $P = \{0, 1, 2, \dots\}$ (a chain): $B(k) = 1$.



Two further examples

- P is the set of all finite subsets of $\{1, 2, \ldots\}$, ordered by inclusion: $B(k) = k!$.

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- P is the set of all finite-dimensional subspaces of an infinite-dimensional vector space over the finite field $\mathbb{F}_q$:

$$B(k) = \boldsymbol{(k)!} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1}).$$

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Many other examples ...

Multichains

Theorem. *Let P be a binomial poset. Let $p_k(n)$ be the number of multichains*

$$s = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = t,$$

where $\mathrm{rank}(t) - \mathrm{rank}(s) = k$. *Then*

$$\sum_{k \geq 0} p_k(n) \frac{x^k}{B(k)} = \left(\sum_{k \geq 0} \frac{x^k}{B(k)} \right)^n .$$

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$$\sum_{k \geq 0} p_k(n) \frac{x^k}{B(k)} = \left(\sum_{k \geq 0} \frac{x^k}{B(k)} \right)^n .$$

Corollary. $k!\, p_k(n)/B(k)$, $k \geq 0$, *is a sequence of polynomials of binomial type.*

Work of Jon Schneider

$\mathbb{Z}_{\boldsymbol{n}}^{\boldsymbol{d}}$: the $n \times n \times \cdots \times n$ (d times) d-dimensional toroidal graph.

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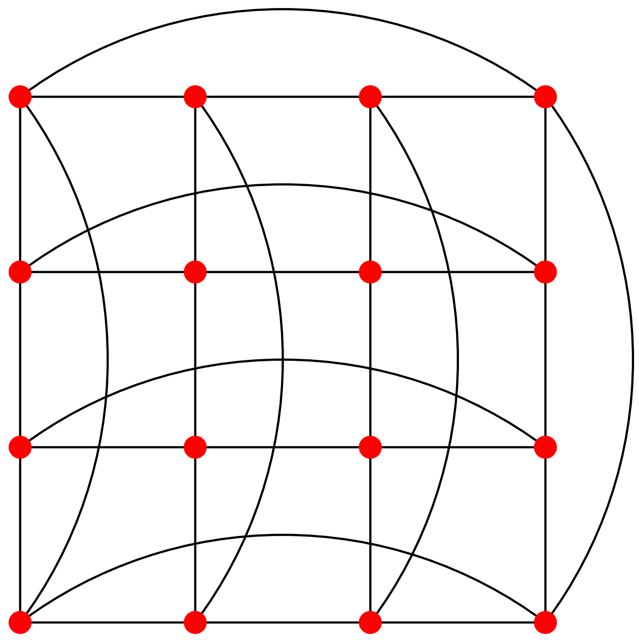

$\mathbb{Z}_4^2$

Work of Jon Schneider

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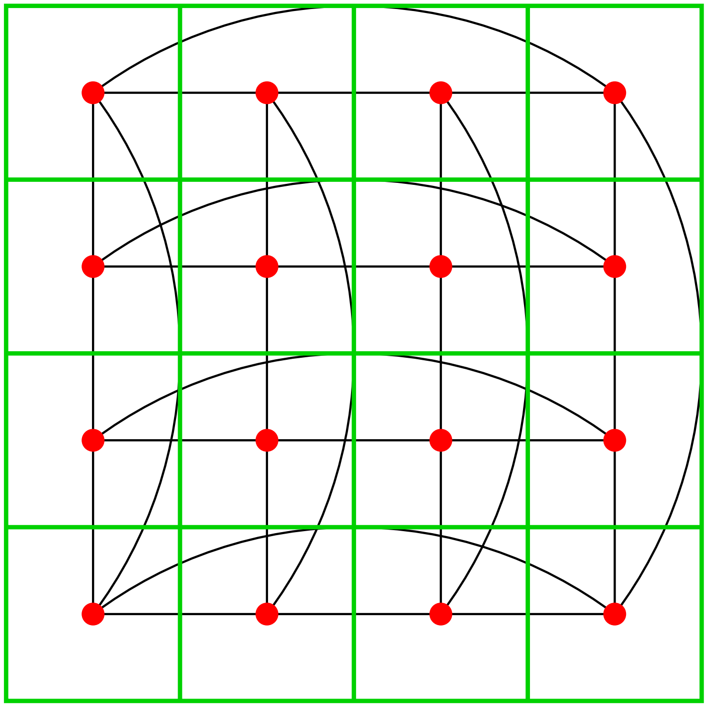


The green squares are the vertices of $\mathbb{Z}_4^2$.

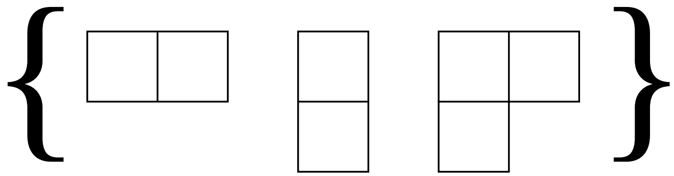
Algebraic definition

$\boldsymbol{\mathbb{Z}_n}$: integers modulo n

$\boldsymbol{\mathbb{Z}_n^d}$: $\{(a_1, \ldots, a_d) \ : \ a_i \in \mathbb{Z}_n\}$ (vertex set)

$\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_d)$ are **adjacent** if $\alpha - \beta$ has one nonzero coordinate, which is equal to ± 1 (modulo n).

Figures

a set S of **figures**: 

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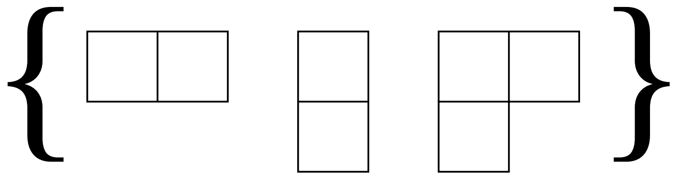
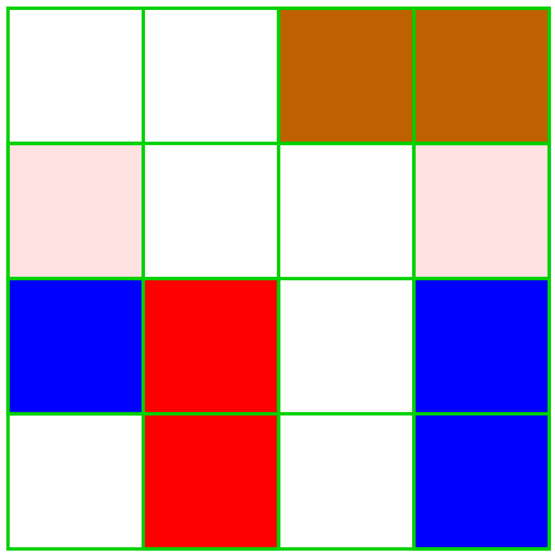


A **placement** of S on $\mathbb{Z}_4^2$:



The function $f_k(n^d)$

Fix d and a finite set S of tiles.

$f_k(n^d)$: number of placements of S on $\mathbb{Z}_n^d$ using a total of k $1 \times 1 \times \cdots \times 1$ boxes.

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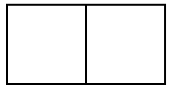
Example. $S = \{\square\}$. Then $f_k(n^2) = \binom{n^2}{k}$

Another example

Example. $S = \{$  $\}$

$$\begin{aligned} f_{2j+1}(n^2) &= 0 \\ f_2(n^2) &= n^2 \\ f_4(n^2) &= \frac{1}{2}n^2(n^2-3) \end{aligned}$$

Still another example

Example. $S = \{\square \quad \square\!\square\}$

$$
\begin{aligned}
f_1(n^2) &= n^2 \\
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\end{aligned}
$$

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Note that these are **polynomials** in n^2.

A main result of Schneider

Theorem.

(a) *For* $n \gg 0$ *(so all tiles fit on* $\mathbb{Z}_d^n$*), there is a polynomial* p_k *for which* $p_k(n) = f_k(n^d)$.

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(b) $p_0, 1!\,p_1, 2!\,p_2, \ldots$ *is a sequence of polynomials of binomial type.*

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Recall: $S = \{\square \quad \square\!\square\}$

$$f_1(n^2) = n^2, \quad f_2(n^2) = \binom{n^2}{2} + n^2$$

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$$1 + nx + \left(\binom{n}{2} + n\right)x^2 + \left(\binom{n}{3} + n(n-2)\right)x^3$$

$$+ \cdots = (1 + x + x^2 - x^3 + \cdots)^n$$

Chromatic polynomials

G: finite graph with vertex set V, $q \geq 1$

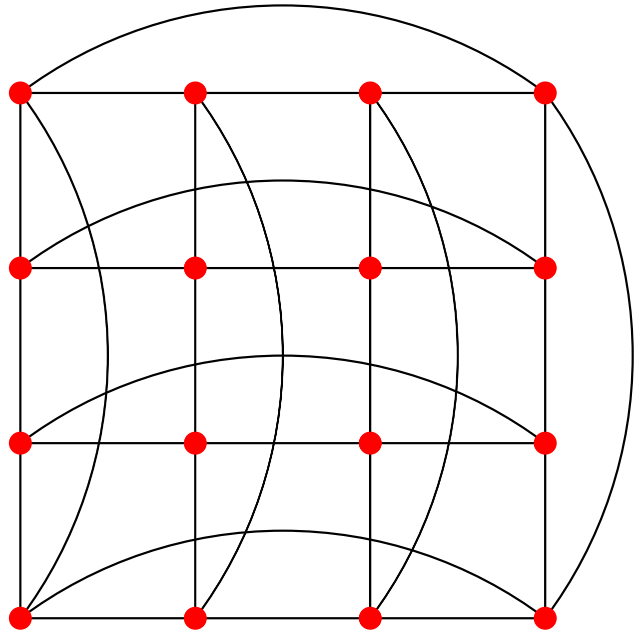
$\chi_G(q)$: number of **proper** colorings

$$f : V \to \{1, \ldots, q\},$$

i.e., adjacent vertices get different colors

The graph $\mathbb{Z}_n^d$

Recall $\mathbb{Z}_n^d$ is a graph:



$\mathbb{Z}_4^2$

Much interest from physicists in the chromatic polynomial $\chi_{\mathbb{Z}_n^d}(q)$.

A trivial and nontrivial result

$$\chi_{\mathbb{Z}_n^d}(2) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

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Theorem (**E. Lieb**, 1967)

$$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^2}(3)^{1/n^2} = \left(\frac{4}{3}\right)^{3/2} = 1.5396\cdots$$

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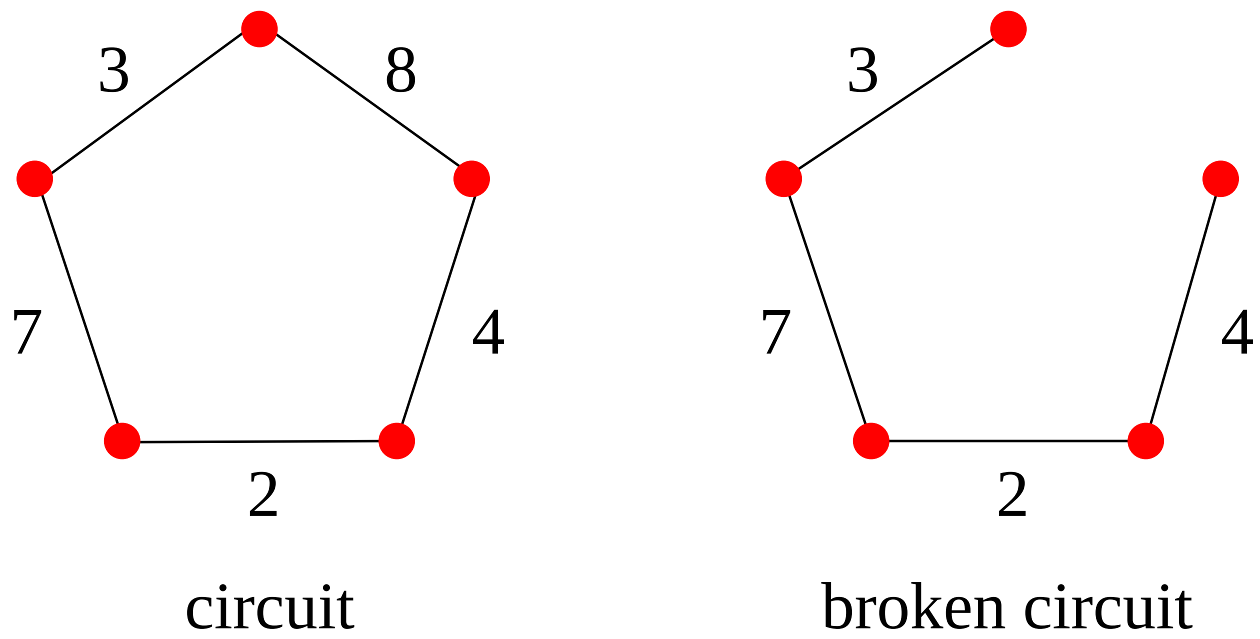
$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^2}(4)^{1/n^2}$: not known

$\lim_{n\to\infty} \chi_{\mathbb{Z}_n^3}(3)^{1/n^3}$: not known

Broken circuits

Label the edges of the graph G as $1, 2, \ldots, m$.

broken circuit: a circuit with its largest edge removed



The broken circuit theorem

Theorem (**H. Whitney**, 1932) *Let* G *have* N *vertices. Write*

$$\chi_G(q) = a_0 q^N - a_1 q^{N-1} + a_2 q^{N-2} - \cdots .$$

Then a_i *is the number of* i*-element sets of edges of* G *that contain no broken circuit.*

An example

Example. If G is a 4-cycle, then no 0-element, 1-element, or 2-element set of edges contains a broken circuit. One 3-element set contains (in fact, is) a broken circuit, and all four edges contain a broken circuit. Hence

$$\begin{aligned}\chi_G(q) &= q^4 - \binom{4}{1} q^3 + \binom{4}{2} q^2 - \left(\binom{4}{3} - 1\right) q \\ &= q^4 - 4q^3 + 6q^2 - 3.\end{aligned}$$

Broken circuits in $\mathbb{Z}_n^2$

Let $\boldsymbol{G} = \mathbb{Z}_n^2$ and $\boldsymbol{N} = n^2$ (number of vertices), so $2N$ edges. The smallest cycle in G has length four. There are N such cycles, so N 3-element sets of edges containing (in fact, equal to) a broken circuit. Hence

$$\chi_{\mathbb{Z}_n^2}(q) = q^N - \binom{2N}{1} q^{N-1} + \binom{2N}{2} q^{N-2} - \left(\binom{2N}{3} - N\right) q^{N-3} + \cdots$$

Chromatic polynomial of $\mathbb{Z}_n^d$

Theorem (J. Schneider). *Let* $N = n^d$*, the number of vertices of* $\mathbb{Z}_n^d$*. Write*

$$\chi_{\mathbb{Z}_n^d}(q) = c_0(N)q^N - c_1(N)q^{N-1} + c_2(N)q^{N-2} - \cdots .$$

Then for $N \gg 0$*,* $c_k(N)$ *agrees with a polynomial* $p_k(N)$*. Moreover,* $p_0, 1!\,p_1, 2!\,p_2, \ldots$ *is a sequence of polynomials of binomial type.*

Chromatic polynomial of $\mathbb{Z}_n^d$

Theorem (**J. Schneider**). *Let* $\boldsymbol{N} = n^d$, *the number of vertices of* $\mathbb{Z}_n^d$. *Write*

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Then for $N \gg 0$, $c_k(N)$ *agrees with a polynomial* $p_k(N)$. *Moreover,* $p_0, 1!\, p_1, 2!\, p_2, \ldots$ *is a sequence of polynomials of binomial type.*

Proof uses a variant of Schneider's previous result on placing tiles on $\mathbb{Z}_n^d$.

Computations

Let $d = 2$. **D. Kim** and **I. G. Enting** made a computation (1979) equivalent to

$$\sum_{k\geq 0} p_k(N)x^k = (1 + 2x + x^2 - x^3 + x^4 - x^5 + x^6 \\ -2x^7 + 9x^8 - 38x^9 + 130x^{10} \\ -378x^{11} + 987x^{12} - 2436x^{13} \\ +5927x^{14} - 14438x^{15} + 34359x^{16} \\ -75058x^{17} + 134146x^{18} + \cdots)^N.$$

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Can anything be said about these numbers?

Does the series converge for small x?

Some small values

$$
\begin{aligned}
p_1(N) &= 2N \\
p_2(N) &= 2N(2N-1) \\
p_3(N) &= 2N(4N^2-6N-1) \\
p_4(N) &= 4N(N+1)(2N-3)(2N-5) \\
p_5(N) &= 8N(N+2)(N-2)(2N-3)(2N-7) \\
p_6(N) &= 8N(8N^5-60N^4+50N^3+495N^2 \\
&\quad -1228N+825) \\
p_7(N) &= 8N(16N^6-168N^5+280N^4+2310N^3 \\
&\quad -10241N^2+14553N-8010)
\end{aligned}
$$

Further directions

What about $n_1 \times n_2 \times \cdots \times n_d$ tori?

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Nothing new: simply replace $N = n^d$ with $N = n_1 n_2 \cdots n_d$.

Tutte polynomials

What about replacing chromatic polynomials with Tutte polynomials?

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Currently under investigation. No known satisfactory generalization of broken circuit theorem.

Multi-indexed polynomials

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Interesting example. $\boldsymbol{K_{jk}}$: complete bipartite graph

Theorem (EC2, Exercise 5.6).

$$\sum_{j,k\geq 0} \chi_{K_{jk}}(n) \frac{x^j}{j!} \frac{y^k}{k!} = (e^x + e^y - 1)^n$$

Continuous variant

Theorem (**Schneider**). *Let $\mathcal{S}$ be a bounded measurable set in d-dimensional Euclidean space. Let $P_k(n^d)$ be the probability that no two copies intersect when we place k copies of $\mathcal{S}$ independently and uniformly at random inside a d-dimensional torus of side length n. Then $n^{dk}P_k(n^d)$ is eventually a polynomial $p_k(n)$ for each k, and these polynomials form a sequence of binomial type.*

A reference

arXiv:1206.6174

The last slide

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The last slide