Alternating Permutations

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M.I.T.

Definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots,$$

and **reverse alternating** if

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and **reverse alternating** if

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$\mathfrak{S}_n$: symmetric group of all permutations of $1, 2, \ldots, n$

E_n

$$\begin{aligned} E_n &= \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n : w \text{ is reverse alternating}\} \end{aligned}$$

via

$$a_1 a_2 \cdots a_n \mapsto n+1-a_1, n+1-a_2, \ldots, n+1-a_n$$

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E.g., $E_4 = 5$: 2143, 3142, 3241, 4132, 4231

André's theorem

Theorem (Désiré André, 1879)

$$y := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

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E_n is an **Euler number**,

E_{2n} a **secant number**,

E_{2n-1} a **tangent number**.

Naive proof

$$\underbrace{52439}_{\substack{\longleftarrow \\ alternating}}\mathbf{1}\underbrace{826}_{\substack{\longrightarrow \\ alternating}}$$

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Obtain w that is **either** alternating or reverse alternating.

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$$\Rightarrow 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k},\ n \geq 1$$

Completion of proof

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$$\begin{aligned} 2\sum_{n\geq 0} E_{n+1}\frac{x^n}{n!} &= 2y' \\ \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}E_k E_{n-k}\right)\frac{x^n}{n!} &= 1+y^2 \end{aligned}$$

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$$\Rightarrow 2y' = 1 + y^2, \ \text{etc.}$$

A generalization

$\exists$ more sophisticated approaches.

E.g., let

$$\boldsymbol{f_k(n)} = \#\{w \in \mathfrak{S}_n \,:\, w(r) < w(r+1) \;\Leftrightarrow k|r\}$$

$$f_2(n) = E_n$$

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Theorem.

$$\sum_{k\geq 0} f_k(kn)\frac{x^{kn}}{(kn)!} = \left(\sum_{n\geq 0}(-1)^n\frac{x^{kn}}{(kn)!}\right)^{-1}.$$

Combinatorial trigonometry

Define $\sec x$ and $\tan x$ by

$$\sec x + \tan x = \sum_{n\geq 0} E_n \frac{x^n}{n!}.$$

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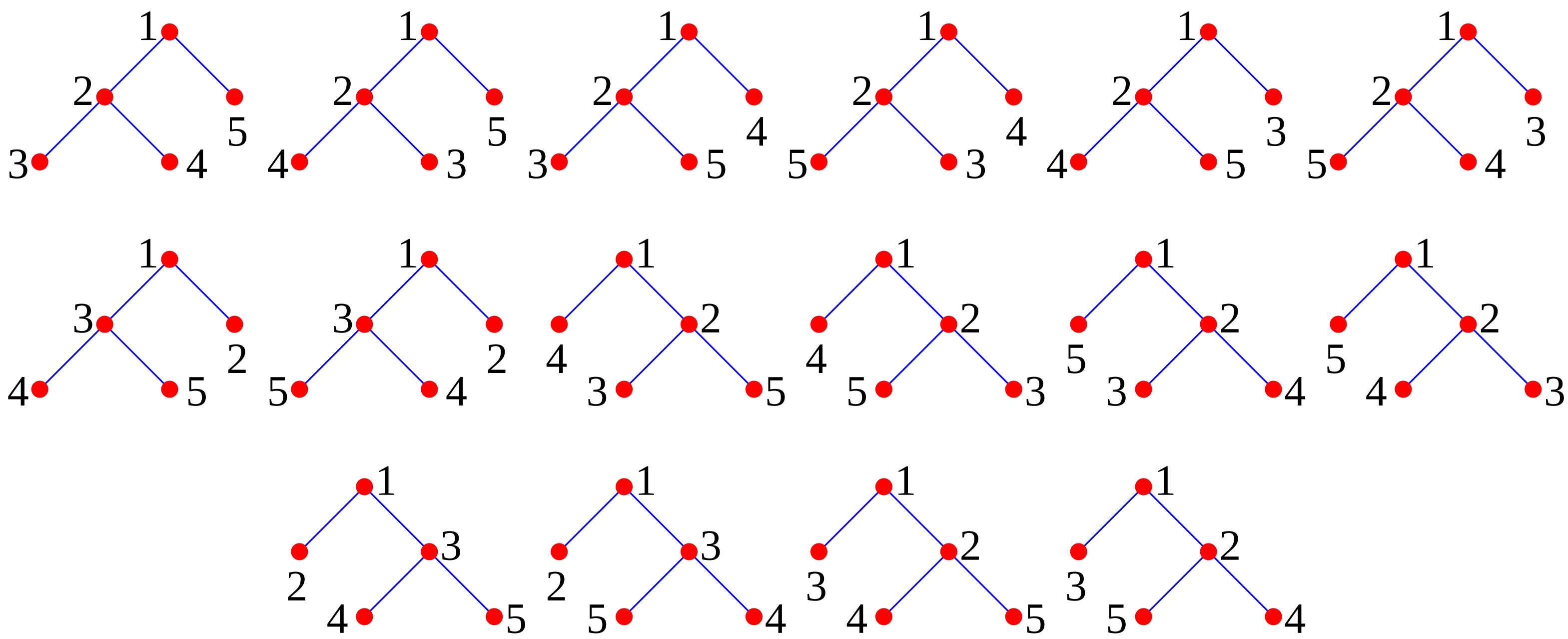
Exercise. $\sec^2 x = 1 + \tan^2 x$

Exercise. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - (\tan x)(\tan y)}$

Some occurences of Euler numbers

(1) E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

Five vertices



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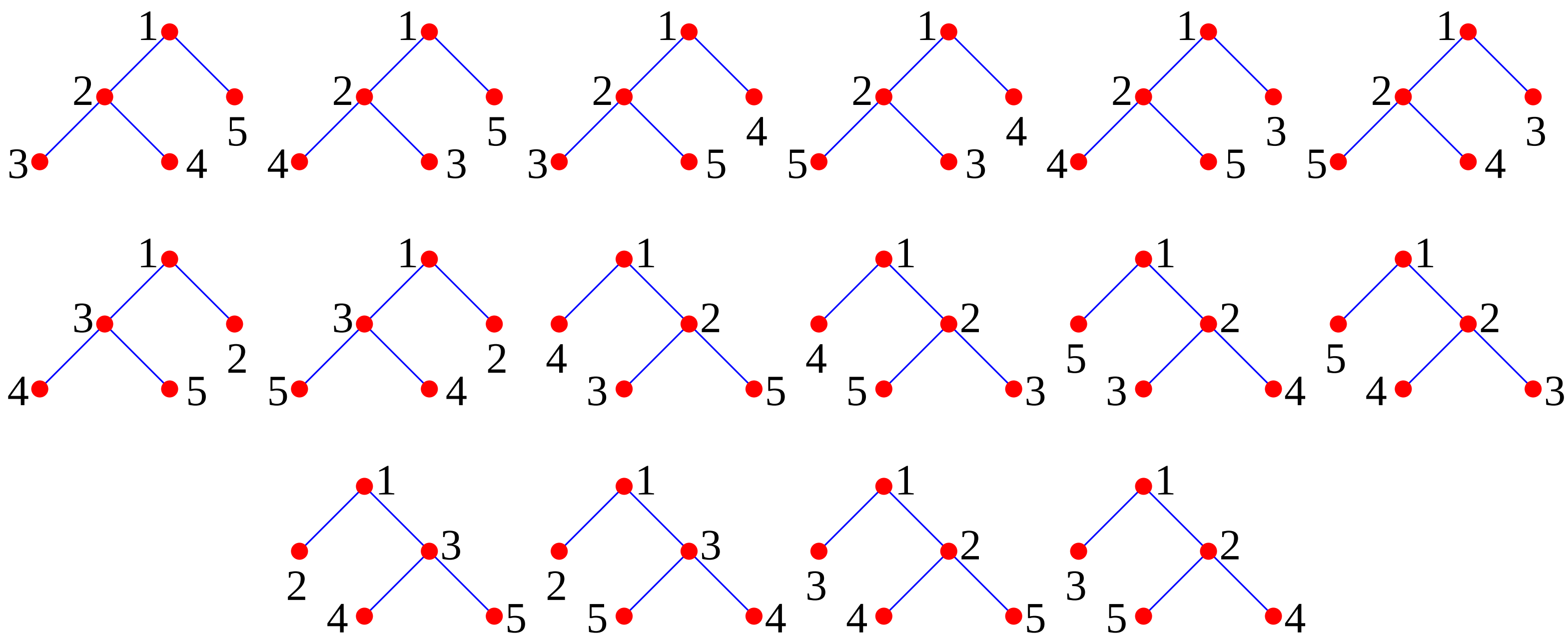

Slightly more complicated for E_{2n}

Chains of partitions

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$$1{-}2{-}3{-}4{-}5{-}6,\quad 12{-}3{-}4{-}5{-}6,\quad 12{-}34{-}5{-}6$$

$$125{-}34{-}6,\quad 125{-}346,\quad 123456$$

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Theorem. *The number of $\mathfrak{S}_n$-orbits is E_{n-1}.*

Orbit representatives for $n = 5$

$12{-}3{-}4{-}5$	$123{-}4{-}5$	$1234{-}5$
$12{-}3{-}4{-}5$	$123{-}4{-}5$	$123{-}45$
$12{-}3{-}4{-}5$	$12{-}34{-}5$	$125{-}34$
$12{-}3{-}4{-}5$	$12{-}34{-}5$	$12{-}345$
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Ulam's problem

Let $E(n)$ be the expected length is(w) of the longest **increasing** subsequence of w.

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$$w = 5\mathbf{24}1\mathbf{7}36 \;\Rightarrow\; \text{is}(w) = 3$$

Two highlights

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Baik-Deift-Johansson:

$$\mathrm{Prob}_{w \in \mathfrak{S}_n} \left(\frac{\mathrm{is}(w) - 2\sqrt{n}}{n^{1/6}} \le t \right) = F(t),$$

the **Tracy-Widom distribution**.

Alternating subsequences?

$\mathbf{as}(\boldsymbol{w})$ = length of longest alternating subseq. of w

$$w = \mathbf{5}6\mathbf{2}1\mathbf{8}\mathbf{3}\mathbf{4}7 \Rightarrow \mathrm{as}(w) = 5$$

The main lemma

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$$\begin{aligned} \boldsymbol{a_k(n)} &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\} \\ \boldsymbol{b_k(n)} &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) \le k\}. \end{aligned}$$

Recurrence for $a_k(n)$

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

The main generating function

$$\text{Define } \boldsymbol{B(x,t)} = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\begin{aligned} \Rightarrow \quad b_1(n) &= 1 \\ b_2(n) &= n \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\ b_4(n) &= \frac{1}{8}(4^n - (2n-4)2^n) \\ &\vdots \end{aligned}$$

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no such formulas for longest increasing subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$A(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w),$$

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Recall $E(n) \sim 2\sqrt{n}$.

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Recall $E(n) \sim 2\sqrt{n}$.

Corollary.

$$A(n) = \frac{4n+1}{6},\ n \geq 2$$

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\left(\mathrm{as}(w) - \frac{4n+1}{6}\right)^2,\ n \geq 2$$

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Corollary.

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similar results for higher moments

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \leq t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \leq t\right)$$

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Umbral enumeration

Umbral formula: involves E^k, where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k. (Technique from 19th century, modernized by Rota et al.)

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Example.

$$\begin{aligned}(1+E^2)^3 &= 1+3E^2+3E^4+E^6\\ &= 1+3E_2+3E_4+E_6\\ &= 1+3\cdot 1+3\cdot 5+61\\ &= 80\end{aligned}$$

Another example

$$\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \cdots \\
&= 1 + Et + \frac{1}{2}E(E-1)t^2 + \cdots \\
&= 1 + E_1 t + \frac{1}{2}(E_2 - E_1))t^2 + \cdots \\
&= 1 + t + \frac{1}{2}(1-1)t^2 + \cdots \\
&= 1 + t + O(t^3).
\end{aligned}$$

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$$\begin{aligned} n = 3: \quad 214365 &= (1,2)(3,4)(5,6) \\ 645231 &= (1,6)(2,4)(3,5) \end{aligned}$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$\boldsymbol{F(x)} = \sum_{n \geq 0} f(n) x^n$$

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Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

Ramanujan's Lost Notebook

Theorem (Ramanujan, Berndt, implicitly) *As* $x \to 0+$,

$$2\sum_{n\geq 0}\left(\frac{1-x}{1+x}\right)^{n(n+1)} \sim \sum_{k\geq 0} f(k)x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2\sum_{n\geq 0} q^n \frac{\prod_{j=1}^{n}(1-q^{2j-1})}{\prod_{j=1}^{2n+1}(1+q^j)},$$

where $\boldsymbol{q} = \left(\frac{1-x}{1+x}\right)^{2/3}$*, a* ***formal*** *identity.*

Simple result, hard proof

Recall: number of n-cycles in $\mathfrak{S}_n$ is $(n-1)!$.

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Theorem. *Let* $\boldsymbol{b(n)}$ *be the number of* ***alternating*** n*-cycles in* $\mathfrak{S}_n$*. Then if* n *is odd,*

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}.$$

Special case

Corollary. *Let p be an odd prime. Then*

$$b(p) = \frac{1}{p}\left(E_p - (-1)^{(p-1)/2}\right).$$

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Combinatorial proof?

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Recall: the expected number of fixed points of $w \in \mathfrak{S}_n$ is 1.

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$\boldsymbol{J(n)}$ $=$ expected number of fixed points of **reverse alternating** $w \in \mathfrak{S}_n$

Theorem. $J(2n) = \dfrac{1}{E_{2n}}\left(E_{2n} - (-1)^n\right)$.

Similar (but more complicated) for n odd or for alternating permutations.

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Combinatorial proof?

Descent sets

Let $\boldsymbol{w} = a_1a_2\cdots a_n \in \mathfrak{S}_n$. **Descent set** of w:

$$\boldsymbol{D(w)} = \{i \,:\, a_i > a_{i+1}\} \subseteq \{1,\dots,n-1\}$$

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$$\begin{aligned} D(4157623) &= \{1,4,5\} \\ D(4152736) &= \{1,3,5\} \text{ (alternating)} \\ D(4736152) &= \{2,4,6\} \text{ (reverse alternating)} \end{aligned}$$

$\beta_n(S)$

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w	$D(w)$
123	$\emptyset$
213	$\{1\}$
312	$\{1\}$
132	$\{2\}$
231	$\{2\}$
321	$\{1,2\}$

$$\beta_3(\emptyset) = 1, \quad \beta_3(1) = 2$$
$$\beta_3(2) = 2, \quad \beta_3(1,2) = 1$$

Fix n. Let $S \subseteq \{1, \cdots, n-1\}$. Let $\boldsymbol{u_S} = t_1 \cdots t_{n-1}$, where

$$\boldsymbol{t_i} = \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases}$$

u_S

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Example. $n = 8,\ S = \{2, 5, 6\} \subseteq \{1, \ldots, 7\}$

$$\boldsymbol{u_S} = ababba$$

A noncommutative gen. function

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Thus

$$\Psi_3(a,b) = aa + 2ab + 2ba + bb$$

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$$= (a+b)^2 + (ab+ba)$$

The cd-index

Theorem. *There exists a noncommutative polynomial $\Phi_n(c,d)$, called the **cd-index** of $\mathfrak{S}_n$, with **nonnegative** integer coefficients, such that*

$$\Psi_n(a,b) = \Phi_n(a+b, ab+ba).$$

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Example. Recall

$$\Psi_3(a,b) = aa + 2ab + 2ba + b^2 = (a+b)^2 + (ab+ba).$$

Therefore

$$\Phi_3(c,d) = c^2 + d.$$

Small values of $\Phi_n(c, d)$

$$
\begin{aligned}
\Phi_1 &= 1 \\
\Phi_2 &= c \\
\Phi_3 &= c^2 + d \\
\Phi_4 &= c^3 + 2cd + 2dc \\
\Phi_5 &= c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \\
\Phi_6 &= c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 \\
&\quad +12cd^2 + 10dcd + 12d^2c.
\end{aligned}
$$

Expansion of cd-monomials

Let $\mathfrak{m} = \mathfrak{m}(c, d)$ be a cd-monomial, e.g., c^2d^2cd.
Expand $\mathfrak{m}(a + b, ab + ba)$:

$$(a + b)^2(ab + ba)^2(a + b)(ab + ba) = a^9 + ba^8 + \cdots .$$

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$$(a + b)^2(ab + ba)^2(a + b)(ab + ba) = a^9 + ba^8 + \cdots .$$

Key observation: $\mathfrak{m}(a + b, ab + ba)$ is a sum of distinct monomials, always including $ababab\cdots$ and $bababa\cdots$.

Alternating permutations

The coefficient of $ababab\cdots$ and of $babab a\cdots$ in $\Psi_n(a,b)$ is

$$\beta_n(2,4,6,\dots) = \beta_n(1,3,5,\dots) = E_n.$$

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$$\beta_n(2,4,6,\dots) = \beta_n(1,3,5,\dots) = E_n.$$

Hence:

Theorem. (a) $\Phi_n(1,1) = E_n$ *(can interpret coefficients combinatorially).*

(b) *For all* $S \subseteq \{1,\dots,n-1\}$,

$$\beta_n(S) \le \beta_n(1,3,5,\dots) = E_n.$$

An example

$$\Phi_5 = \mathbf{1}c^4 + \mathbf{3}c^2d + \mathbf{5}cdc + \mathbf{3}dc^2 + \mathbf{4}d^2$$

$$1 + 3 + 5 + 3 + 4 = 16 = E_5$$

Comments

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NOTE. Not hard to show that

$$\beta_n(S) < E_n$$

unless $S = \{1, 3, 5, \dots\}$ or $S = \{2, 4, 6, \dots\}$.


THE END