Smith Normal Form and Combinatorics

Richard P. Stanley
Smith normal form

$A$: $n \times n$ matrix over commutative ring $R$ (with 1)

Suppose there exist $P, Q \in \text{GL}(n, R)$ such that

$$PAQ := B = \text{diag}(d_1, d_1d_2, \ldots, d_1d_2 \cdots d_n),$$

where $d_i \in R$. We then call $B$ a Smith normal form (SNF) of $A$. 
Smith normal form

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where $d_i \in R$. We then call $B$ a Smith normal form (SNF) of $A$.

**NOTE.** (1) Can extend to $m \times n$.

(2) unit $\cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n$.

Thus SNF is a refinement of $\det$. 
Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a unit in $R$. 
Row and column operations

Can put a matrix into SNF by the following operations.

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- Add a multiple of a column to another column.
- Multiply a row or column by a unit in $R$.

Over a field, SNF is row reduced echelon form (with all unit entries equal to 1).
Existence of SNF

**PIR**: principal ideal ring, e.g., \( \mathbb{Z}, K[x], \mathbb{Z}/m\mathbb{Z} \).

If \( R \) is a PIR then \( A \) has a unique SNF up to units.
Existence of SNF

**PIR**: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.

If $R$ is a PIR then $A$ has a unique SNF up to units.

Otherwise $A$ “typically” does not have a SNF but may have one in special cases.
Not known in general for which rings $R$ does every matrix over $R$ have an SNF.
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Necessary condition: $R$ is a **Bézout ring**, i.e., every finitely generated ideal is principal.

**Example.** ring of entire functions and ring of all algebraic integers (not PIR’s)
Not known in general for which rings $R$ does every matrix over $R$ have an SNF.

Necessary condition: $R$ is a **Bézout ring**, i.e., every finitely generated ideal is principal.

**Example.** ring of entire functions and ring of all algebraic integers (not PIR’s)

**Open:** every matrix over a Bézout domain has an SNF.
$R$: a PID

$A$: an $n \times n$ matrix over $R$ with rows $v_1, \ldots, v_n \in R^n$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
**Algebraic interpretation of SNF**

\( R \): a PID

\( A \): an \( n \times n \) matrix over \( R \) with rows

\[ v_1, \ldots, v_n \in R^n \]

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( A \)

**Theorem.**

\[ R^n / (v_1, \ldots, v_n) \cong (R/e_1 R) \oplus \cdots \oplus (R/e_n R). \]
Algebraic interpretation of SNF

\( \mathbb{R} \): a PID

\( \mathbb{A} \): an \( n \times n \) matrix over \( \mathbb{R} \) with rows \( v_1, \ldots, v_n \in \mathbb{R}^n \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( \mathbb{A} \)

\textbf{Theorem.}

\[ \mathbb{R}^n / (v_1, \ldots, v_n) \cong (\mathbb{R}/e_1\mathbb{R}) \oplus \cdots \oplus (\mathbb{R}/e_n\mathbb{R}) . \]

\( \mathbb{R}^n / (v_1, \ldots, v_n) \): \textbf{(Kasteleyn) cokernel} of \( \mathbb{A} \)
An explicit formula for SNF

\( \mathbb{R} \): a PID

\( A \): an \( n \times n \) matrix over \( \mathbb{R} \) with \( \det(A) \neq 0 \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( A \)
An explicit formula for SNF

$\mathcal{R}$: a PID

$A$: an $n \times n$ matrix over $\mathcal{R}$ with $\det(A) \neq 0$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$

**Theorem.** $e_1 e_2 \cdots e_i$ is the gcd of all $i \times i$ minors of $A$.

**minor**: determinant of a square submatrix.

**Special case**: $e_1$ is the gcd of all entries of $A$. 
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix}
  3 & -1 & -1 \\
  -1 & 3 & -1 \\
  -1 & -1 & 3
\end{bmatrix}$$

Matrix-tree theorem $\implies \det(A) = 16$, the number of spanning trees of $K_4$. 
Reduced Laplacian matrix of $K_4$:

$$
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\end{bmatrix}
$$

Matrix-tree theorem $\implies \text{det}(A) = 16$, the number of spanning trees of $K_4$.

What about SNF?
An example (continued)

\[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & -1 \\
8 & -4 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & 0 \\
8 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
0 & 4 & 0 \\
4 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]
Reduced Laplacian matrix of $K_n$

$L_0(K_n) = nI_{n-1} - J_{n-1}$

$\det L_0(K_n) = n^{n-2}$
Reduced Laplacian matrix of $K_n$

$$L_0(K_n) = nI_{n-1} - J_{n-1}$$
$$\det L_0(K_n) = n^{n-2}$$

**Trick:** $2 \times 2$ submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i|e_{i+1}$, we get the SNF $\text{diag}(1, n, n, \ldots, n)$. 
Laplacian matrices of general graphs

SNF of the Laplacian matrix of a graph: very interesting

connections with sandpile models, chip firing, abelian avalanches, etc.
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no time for further details 😞
SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.
Is the question interesting?

\( \text{Mat}_k(n) \): all \( n \times n \ \mathbb{Z} \)-matrices with entries in \([-k, k]\) (uniform distribution)

\( p_k(n, d) \): probability that if \( M \in \text{Mat}_k(n) \) and \( \text{SNF}(M) = (e_1, \ldots, e_n) \), then \( e_1 = d \).
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\textbf{Recall:} \( e_1 = \gcd \) of \( 1 \times 1 \) minors (entries) of \( M \)
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Recall: \( e_1 = \gcd \) of \( 1 \times 1 \) minors (entries) of \( M \)

Theorem. \( \lim_{k \to \infty} p_k(n, d) = \frac{1}{d n^2 \zeta(n^2)} \)
Specifying some $e_i$

with Yinghui Wang
Specifying some $e_i$

with Yinghui Wang (王颖慧)
Two general results.

Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{P}$, $\alpha_i | \alpha_{i+1}$.

$\mu_k(n)$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_i = \alpha_i$ for $1 \leq \alpha_i \leq n - 1$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$

Then $\mu(n)$ exists, and $0 < \mu(n) < 1$. 
Let $\alpha_n \in \mathbb{P}$.

$\nu_k(n)$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

Then

$$\lim_{k \to \infty} \nu_k(n) = 0.$$
Sample result

\( \mu_k(n) \): probability that the SNF of a random \( A \in \text{Mat}_k(n) \) satisfies \( e_1 = 2, \ e_2 = 6. \)

\[
\mu(n) = \lim_{k \to \infty} \mu_k(n).
\]
\[ \mu(n) = 2^{-n^2} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i} \right) \]

\[ \cdot \frac{3}{2} \cdot 3^{-(n-1)^2} \left( 1 - 3^{(n-1)^2} \right) \left( 1 - 3^{-n} \right)^2 \]

\[ \cdot \prod_{p>3} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i} \right). \]
\( \kappa(n) \): probability that an \( n \times n \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \)
\( \kappa(n) \): probability that an \( n \times n \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \)

Theorem. \( \kappa(n) = \frac{\prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right)}{\zeta(2) \zeta(3) \cdots} \)
Cyclic cokernel

$$\kappa(n):$$ probability that an $$n \times n \mathbb{Z}$$-matrix has SNF $$\text{diag}(e_1, e_2, \ldots, e_n)$$ with $$e_1 = e_2 = \cdots = e_{n-1} = 1$$

**Theorem.** 
$$\kappa(n) = \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right) \frac{\zeta(2) \zeta(3) \cdots}{\zeta(6) \prod_{j \geq 4} \zeta(j)}$$

**Corollary.** 
$$\lim_{n \to \infty} \kappa(n) = \frac{1}{\zeta(6) \prod_{j \geq 4} \zeta(j)} \approx 0.846936 \cdots.$$
Small number of generators

\( g \): number of generators of cokernel (number of entries of SNF \( \neq 1 \)) as \( n \to \infty \)

previous slide: \( \text{Prob}(g = 1) = 0.846936 \cdots \)
Small number of generators

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$\text{Prob}(g \leq 2) = 0.99462688 \cdots$
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**Theorem.** \( \text{Prob}(g \leq \ell) = 1 - (3.46275 \cdots)2^{-(\ell + 1)^2}(1 + O(2^{-\ell})) \)
Small number of generators

$g$: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

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\begin{align*}
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\end{align*}

Theorem. $\text{Prob}(g \leq \ell) = 1 - (3.46275 \cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$
\[ 3.46275 \ldots = \frac{1}{\prod_{j \geq 1} \left( 1 - \frac{1}{2^j} \right)} \]
Example of SNF computation

$\lambda$: a partition $(\lambda_1, \lambda_2, \ldots)$, identified with its Young diagram

(3,1)
Example of SNF computation

λ: a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\[
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \\
\end{array}
\]

(3,1)

λ*: λ extended by a border strip along its entire boundary
**Example of SNF computation**

\( \lambda \): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\[
(3,1)
\]

\( \lambda^* \): \( \lambda \) extended by a border strip along its entire boundary

\[
(3,1)^* = (4,4,2)
\]
Initialization

Insert 1 into each square of $\lambda^*/\lambda$.

\[
(3,1)^* = (4,4,2)
\]
Let $t \in \lambda$. Let $M_t$ be the largest square of $\lambda^*$ with $t$ as the upper left-hand corner.
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Let \( t \in \lambda \). Let \( M_t \) be the largest square of \( \lambda^* \) with \( t \) as the upper left-hand corner.
Determinantal algorithm

Suppose all squares to the southeast of \( t \) have been filled. Insert into \( t \) the number \( n_t \) so that \( \det M_t = 1 \).
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\[
\begin{array}{ccc}
1 & 1 & 2 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 
Determinantal algorithm

Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{array}{ccc}
3 & 5 & 2 & 1 \\
3 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]
Determinantal algorithm

Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

$$
\begin{array}{ccc|c|c}
9 & 5 & 2 & \multicolumn{1}{c|}{} & 1 \\
3 & 2 & 1 & 1 & 1 \\
\multicolumn{3}{c|}{} & 1 & 1 \\
\end{array}
$$
Uniqueness

Easy to see: the numbers $n_t$ are well-defined and unique.
Uniqueness

Easy to see: the numbers $n_t$ are well-defined and unique.

Why? Expand $\det M_t$ by the first row. The coefficient of $n_t$ is 1 by induction.
If \( t \in \lambda \), let \( \lambda(t) \) consist of all squares of \( \lambda \) to the southeast of \( t \).
If $t \in \lambda$, let $\lambda(t)$ consist of all squares of $\lambda$ to the southeast of $t$.

\begin{equation*}
\lambda = (4,4,3)
\end{equation*}
If $t \in \lambda$, let $\lambda(t)$ consist of all squares of $\lambda$ to the southeast of $t$. 

\[
\lambda = (4,4,3)
\]

\[
\lambda(t) = (3,2)
\]
\( u_\lambda = \#\{\mu : \mu \subseteq \lambda\} \)
$u_\lambda = \#\{\mu : \mu \subseteq \lambda\}$

Example. $u_{(2,1)} = 5$:

\[\begin{array}{cccc}
\cdot & \cdot & \cdot & \phi \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & & \\
\end{array}\]
\( u_\lambda = \# \{ \mu : \mu \subseteq \lambda \} \)

Example. \( u_{(2,1)} = 5 \):

\[
\begin{array}{cccc}
\emptyset & & & \\
\emptyset & & & \\
\end{array}
\]

There is a determinantal formula for \( u_\lambda \), due essentially to MacMahon and later Kreweras (not needed here).
Berlekamp (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.

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**Theorem.** $n_t = u \lambda(t)$
Berlekamp (1963) first asked for \( n_t \) (mod 2) in connection with a coding theory problem.

Carlitz-Roselle-Scoville (1971): combinatorial interpretation of \( n_t \) (over \( \mathbb{Z} \)).

**Theorem.** \( n_t = u \chi(t) \)

**Proofs.**
1. Induction (row and column operations).
2. Nonintersecting lattice paths.
An example

<table>
<thead>
<tr>
<th>7</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
An example

\[
\begin{array}{cccc}
7 & 3 & 2 & 1 \\
2 & 1 & 1 & 1 \\
1 & 1 & & \\
& & & \phi
\end{array}
\]
Weight each $\mu \subseteq \lambda$ by $q^{\mid \lambda/\mu \mid}$. 
A $q$-analogue

Weight each $\mu \subseteq \lambda$ by $q^{\lambda/\mu}$.

$\lambda = 64431, \quad \mu = 42211, \quad q^{\lambda/\mu} = q^8$
\[ u_\lambda(q) = \sum_{\mu \subseteq \lambda} q^{\lambda/\mu} \]

\[ u_{(2,1)}(q) = 1 + 2q + q^2 + q^3 : \]

\[
\begin{array}{cccc}
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\end{array} \]

Smith Normal Form and Combinatorics – p. 31
Diagonal hooks

\[ d_i(\lambda) = \lambda_i + \lambda_i' - 2i + 1 \]

\[ d_1 = 9, \quad d_2 = 4, \quad d_3 = 1 \]
Theorem. \( M_t \) has an SNF over \( \mathbb{Z}[q] \). Write \( d_i = d_i(\lambda_t) \). If \( M_t \) is a \((k + 1) \times (k + 1)\) matrix then \( M_t \) has SNF

\[
\text{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \ldots, q^{d_1+d_2+\cdots+d_k}).
\]
Main result (with C. Bessenrodt)

**Theorem.** $M_t$ has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If $M_t$ is a $(k + 1) \times (k + 1)$ matrix then $M_t$ has SNF

$$\text{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \ldots, q^{d_1+d_2+\cdots+d_k}).$$

**Corollary.** $\det M_t = q^{\sum i d_i}$. 
Main result (with C. Bessenrodt)

**Theorem.** $M_t$ has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If $M_t$ is a $(k + 1) \times (k + 1)$ matrix then $M_t$ has SNF

$$\text{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \ldots, q^{d_1+d_2+\cdots+d_k}).$$

**Corollary.** $\det M_t = q^{\sum i d_i}$.

**Note.** There is a multivariate generalization.
An example

\[ \lambda = 6431, \quad d_1 = 9, \quad d_2 = 4, \quad d_3 = 1 \]
An example

\[ \lambda = 6431, \quad d_1 = 9, \quad d_2 = 4, \quad d_3 = 1 \]

\[ \text{SNF of } M_t : (1, q, q^5, q^{14}) \]
Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. 
A special case

Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. 
A special case

Let $\lambda$ be the staircase $\delta_n = (n - 1, n - 2, \ldots, 1)$.

$u_{\delta_{n-1}}(q)$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known $q$-analogue $C_n(q)$ of the Catalan number $C_n$. 
A $q$-Catalan example

$$C_3(q) = q^3 + q^2 + 2q + 1$$
A $q$-Catalan example

$C_3(q) = q^3 + q^2 + 2q + 1$

$$
\begin{vmatrix}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1
\end{vmatrix} \overset{\text{SNF}}{\sim} \text{diag}(1, q, q^6)
$$

since $d_1(3, 2, 1) = 1$, $d_2(3, 2, 1) = 5$. 
A $q$-Catalan example

$$C_3(q) = q^3 + q^2 + 2q + 1$$

$$\begin{vmatrix}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1
\end{vmatrix} \overset{\text{SNF}}{\sim} \text{diag}(1, q, q^6)$$

since $d_1(3, 2, 1) = 1$, $d_2(3, 2, 1) = 5$.

- $q$-Catalan determinant previously known
- SNF is new
\[
\sum_{n \geq 0} C_n(q)x^n = \frac{1}{1 - \frac{x}{1 - \frac{qx}{1 - \frac{q^2x}{1 - \cdots}}}}.
\]
\[ \sum_{n \geq 0} C_n(q)x^n = \frac{1}{1 - \frac{x}{1 - \frac{qx}{1 - \frac{q^2x}{1 - \cdots}}}}. \]