A Chromatic Symmetric Function Conjecture

Richard P. Stanley

M.I.T.
Basic notation

\( G \): simple graph with \( d \) vertices

\( V \): vertex set of \( G \)

\( E \): edge set of \( G \)

Coloring of \( G \):

any \( \kappa : V \rightarrow \mathbb{P} = \{1, 2, \ldots\} \)

Proper coloring:

\( uv \in E \Rightarrow \kappa(u) \neq \kappa(v) \)
The chromatic symmetric function

\[ X_G = X_G(x_1, x_2, \ldots) = \sum_{\text{proper } \kappa : V \to \mathbb{P}} x^\kappa, \]

the **chromatic symmetric function** of \( G \), where

\[ x^\kappa = \prod_{v \in V} x_{\kappa(v)} = x_1^{\#_{\kappa^{-1}}(1)} x_2^{\#_{\kappa^{-1}}(2)} \ldots. \]
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\[ X_G(1^n) := X_G(1, 1, \ldots, 1) = \chi_G(n), \]

the chromatic polynomial of \( G \).
Example of a monomial

\[ x^\kappa = x_1^2 x_2 x_3^2 x_5 \]
Simple examples

\[ X_{\text{point}} = x_1 + x_2 + x_3 + \cdots = e_1. \]

More generally, let

\[ e_k = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}, \]

the \( k \)th \textbf{elementary symmetric function}. Then

\[ X_{K_n} = n! e_n \]
\[ X_{G+H} = X_G \cdot X_H. \]
Acyclic orientations

**Acyclic orientation**: an orientation $\mathcal{O}$ of the edges of $G$ that contains no directed cycle.
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**Theorem** (RS, 1973). Let $a(G)$ denote the number of acyclic orientations of $G$. Then

$$a(G) = (-1)^d \chi_G(-1).$$

Easy to prove by induction, by deletion-contraction, bijectively, geometrically, etc.
Fundamental theorem of symmetric functions. Every symmetric function can be uniquely written as a polynomial in the $e_i$’s, or equivalently as a linear combination of $e_\lambda$’s.

Write $\lambda \vdash d$ if $\lambda$ is a partition of $d$, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = d.$$
A refinement of $a(G)$

Note that if $\lambda \vdash d$, then $e_\lambda(1^n) = \prod \binom{n}{\lambda_i}$, so

$$e_\lambda(1^n)|_{n=-1} = \prod \left(\frac{-1}{\lambda_i}\right) = (-1)^d.$$  

Hence if $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$, then

$$a(G) = \sum_{\lambda \vdash d} c_\lambda.$$
Sinks

**Sink** of an acyclic orientation (or digraph): vertex for which no edges point out (including an isolated vertex).

\[ a_k(G) \]: number of acyclic orientations of \( G \) with \( k \) sinks

\( \ell(\lambda) \): length (number of parts) of \( \lambda \)
Theorem. Let $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. Then

$$\sum_{\lambda \vdash d} c_\lambda = a_k(G').$$
The sink theorem

Theorem. Let $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. Then

$$\sum_{\lambda \vdash d, \ell(\lambda) = k} c_\lambda = a_k(G').$$

Proof based on quasisymmetric functions.
Theorem. Let $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. Then

$$\sum_{\lambda \vdash d, \ell(\lambda) = k} c_\lambda = a_k(G).$$

Proof based on quasisymmetric functions.

Open: Is there a simpler proof?
Example. Let \( G \) be the claw \( K_{13} \).

Then

\[
X_G = 4e_4 + 5e_31 - 2e_22 + e_211.
\]

Thus \( a_1(G) = 1, \ a_2(G) = 5 - 2 = 3, \ a_3(G) = 1, \ a(G) = 5. \)
Example. Let $G$ be the claw $K_{13}$.

Then

$$X_G = 4e_4 + 5e_3 - 2e_2 + e_1.$$ 

Thus $a_1(G) = 1$, $a_2(G) = 5 - 2 = 3$, $a_3(G) = 1$, $a(G) = 5$.

When is $X_G$ e-positive (i.e., each $c_\lambda \geq 0$)?
Let $P$ be a finite poset. Let $3 + 1$ denote the disjoint union of a 3-element chain and 1-element chain:
\( P \) is \((3+1)\)-free if it contains no induced \( 3 + 1 \).

\[ (3+1)\text{-free} \quad \text{not} \]
**The main conjecture**

\[ \text{inc}(P) : \text{incomparability graph of } P \text{ (vertices are elements of } P; uv \text{ is an edge if neither } u \leq v \text{ nor } v \leq u) \]
The main conjecture

\( \text{inc}(P) \): incomparability graph of \( P \) (vertices are elements of \( P \); \( uv \) is an edge if neither \( u \leq v \) nor \( v \leq u \))

**Conjecture.** If \( P \) is \((3 + 1)\)-free, then \( X_{\text{inc}(P)} \) is \( e \)-positive.
Two comments

- Suggests that for incomparability graphs of \((3 + 1)\)-free posets, \(c_\lambda\) counts acyclic orientations of \(G\) with \(\ell(\lambda)\) sinks and some further property depending on \(\lambda\).

**Open:** What is this property?
Two comments

- Suggests that for incomparability graphs of \((3 + 1)\)-free posets, \(c_\lambda\) counts acyclic orientations of \(G\) with \(\ell(\lambda)\) sinks and some further property depending on \(\lambda\).

**Open:** What is this property?

- True if \(P\) is 3-free, i.e., \(X_G\) is \(e\)-positive if \(G\) is the complement of a bipartite graph. More generally, \(X_G\) is \(e\)-positive if \(G\) is the complement of a triangle-free (or \(K_3\)−free) graph.
Fix $k \geq 2$. Define

$$P_d = \sum_{i_1, \ldots, i_d} x_{i_1} \cdots x_{i_d},$$

where $i_1, \ldots, i_d$ ranges over all sequences of $d$ positive integers such that any $k$ consecutive terms are distinct.
A simple special case

Fix $k \geq 2$. Define

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where $i_1, \ldots, i_d$ ranges over all sequences of $d$ positive integers such that any $k$ consecutive terms are distinct.

**Conjecture.** $P_d$ is $e$-positive.
The case \( k = 2 \)

\[
P_d = \sum_{i_1, \ldots, i_d} x_{i_1} \cdots x_{i_d},
\]

where \( i_j \geq 1, \ i_j \neq i_{j+1} \).

**Theorem (Carlitz).**

\[
\sum P_d \cdot t^d = \frac{\sum_{i \geq 0} e_i t^i}{1 - \sum_{i \geq 1} (i - 1) e_i t^i}.
\]
The case $k = 2$

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where $i_j \geq 1$, $i_j \neq i_{j+1}$.

**Theorem (Carlitz).**

$$\sum P_d \cdot t^d = \frac{\sum_{i \geq 0} e_i t^i}{1 - \sum_{i \geq 1} (i - 1) e_i t^i}.$$

**Corollary.** $P_d$ is $e$-positive for $k = 2$. 
The case $k = 3$

Ben Joseph (2001) probably had a complicated Inclusion-Exclusion proof.
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$$
\sum P_d \cdot t^d = \frac{\text{numerator}}{1 - (2e_3t^3 + 6e_4t^4 + 24e_5t^5 + (64e_6 + 6e_{51} - e_{33})t^6 + \cdots )}.
$$
Schur functions $\{s_\lambda\}$ forms a linear basis for symmetric functions.

$e_\lambda$ is $s$-positive.

(Gasharov) $X_G$ is $s$-positive if $G$ is the incomparability graph of a $(3 + 1)$-free poset.

Conjecture (Gasharov). If $G$ is claw-free, then $X_G$ is $s$-positive. (Need not be $e$-positive).
When $G$ is a unit interval graph (special case of incomparability graphs of $(3 + 1)$-free posets), then Haiman found a close connection with Verma modules and Kazhdan-Lustzig polynomials.