Order **6**: 15–18, 1989.

A Recurrence for Linear Extensions

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Communicated by I. Rival

(Received: 30 April 1989; accepted: 1 May 1989)

Abstract. The number $e(P)$ of linear extensions of a finite poset P is expressed in terms of $e(Q)$ for certain smaller posets Q. The proof is based on M. Schützengerger's concept of promotions of linear extensions.

AMS subject classification (1980). 06A10.

Key words. Linear extension, promotion, comparability graph.

Let P be a finite poset, and let $e(P)$ denote the number of linear extensions of P [6, p. 110]. If A denotes the set of minimal elements of P, then it is easily seen that

$$e(P) = \sum_{x \in A} e(P - x). \tag{1}$$

In this paper, we give a generalization of (1) which is not so apparent. Our proof will be based on the concept of promotions of linear extensions due to M. Schützenberger [4].

Suppose $C: x_0 < x_1 < \cdots < x_m$ is a saturated chain in P (so x_{i+1} covers x_i for $0 \leqslant i < m$). Define a new poset P_C as follows: Replace in P the elements of C by new elements $x_{01}, x_{12}, \ldots, x_{m-1,m}$ subject to the relations (and those implied by transitivity)

* Partially supported by NSF Grant #DMS-8700995.
** Partially supported by NSF Grant #DMS-8401376.

$$x_{01} < x_{12} < \cdots < x_{m-1,m},$$

$$y < x_{i,i+1}, \quad \text{if } y \in P - C \text{ and } y < x_{i+1} \text{ in } P,$$

$$y > x_{i,i+1}, \quad \text{if } y \in P - C \text{ and } y > x_i \text{ in } P.$$

(Think of $x_{i,i+1}$ as the 'merge' of x_i and x_{i+1}.) When $m = 0$, so C consists of the single point $x = x_0$, then the definition of P_C becomes $P_C = P - x$.

THEOREM. *Let P be a finite poset. Let $\mathscr{C}$ be a set of saturated chains of P such that every maximal chain of P contains exactly one element of $\mathscr{C}$. Then*

$$e(P) = \sum_{C \in \mathscr{C}} e(P_C).$$

Proof. Let $\mathscr{L}(Q)$ denote the set of all linear extensions of the poset Q. We regard a linear extension σ of a (finite) poset Q as an order-preserving bijection of Q onto some chain K. Two linear extensions $\sigma : Q \to K$ and $\sigma' : Q \to K'$ will be considered identical if $f\sigma = \sigma'$, where f is the unique order-preserving bijection $K \to K'$. We will now construct a bijection

$$\phi : \mathscr{L}(P) \to \bigcup_{C \in \mathscr{C}} \mathscr{L}(P_C),$$

from which the proof of the theorem clearly follows.

Let $\sigma : P \to \{1, 2, \ldots, n\}$ be a linear extension of P. Following Schützenberger [4], we define a certain maximal chain $M : y_1 < y_2 < \cdots < y_r$ of P as follows. Let y_1 be the (minimal) element of P satisfying $\sigma(y_1) = 1$. Once $y_1, \ldots, y_t$ are defined, stop if y_i is a maximal element of P. Otherwise let y_{i+1} be that element of P which covers y_i and which has the smallest value $\sigma(z)$ among all elements z covering y_i. This inductively defines a maximal chain $M = M(\sigma)$. By assumption, M contains a unique chain $C : y_s = x_0 < x_1 < \cdots < x_m$ which is an element of $\mathscr{C}$(so $x_i = y_{s+i}$). Now define a function $\phi\sigma : P_C \to \{2, 3, \ldots, n\}$ as follows.

$$\phi\sigma(x) = \begin{cases} \sigma(x), & \text{if } x \notin M \text{ or if } x \in M \text{ but } x > x_m, \\ \sigma(x_{i+1}), & \text{if } x = x_{i,i+1}, \\ \sigma(y_{i+1}), & \text{if } x = y_i < x_0. \end{cases}$$

We claim that the map $\phi : \mathscr{L}(P) \to \cup \mathscr{L}(P_C)$ is a bijection. First we check that $\phi\sigma \in \mathscr{L}(P_C)$. Clearly, $\phi\sigma$ is a bijection, so we need to show that if y covers x in P_C then $\phi\sigma(x) < \phi\sigma(y)$. Let C' denote the chain $y_1 < y_2 < \cdots < y_{s-1}$ of P and of P_C, and let C'' denote the chain $x_{01} < x_{12} < \cdots < x_{m-1,m}$ of P_C. We have to check a number of cases, depending on whether x and y belong to C', C'', or $P - (C' \cup C'')$. These cases are all straightforward; we do four of them as a sample.

(a) $x, y \in P - (C' \cup C'')$. Then $\phi\sigma(x) = \sigma(x) < \sigma(y) = \phi\sigma(y)$, since $\sigma \in \mathscr{L}(P)$.

(b) $x = x_{i,i+1} \in C''$ and $y \in P - C''$ (so also $y \in P - C'$ since $x < y$). Then by definition of P_C we have $y > x_i$ in P. Since by definition of C, $\sigma(z)$ is smallest among all z covering x_i when $z = x_{i+1}$, we have $\phi\sigma(x) = \sigma(x_{i+1}) < \sigma(y) = \phi\sigma(y)$.

(c) $x \in P - (C' \cup C'')$ and $y = x_{i,i+1} \in C''$. Then $x < x_{i+1}$ in P and $\phi\sigma(x) = \sigma(x) < \sigma(x_{i+1}) = \phi\sigma(y)$.

(d) $x = y_i \in C'$ and $y = x_{j,j+1} \in C''$. Then, since $y_{i+1} < x_{i+1}$ in P, we have $\phi\sigma(x) = \sigma(y_{i+1}) < \sigma(x_{i+1}) = \phi\sigma(y)$.

In a similar fashion the remaining cases are handled, so $\phi\sigma \in \mathscr{L}(P_C)$ as claimed.

To complete the proof that ϕ is a bijection, we define its inverse ψ. Let $\tau: P_C \to \{2, 3, \ldots, n\}$ be a linear extension of P_C. Define $\psi\tau: P \to \{1, 2, \ldots, n\}$ as follows. First define a saturated chain $N: z_r < z_{r-1} < \cdots < z_0 = x_0$ in P as follows. We have set $z_0 = x_0$. Once z_i is defined, let z_{i+1} be the element z which z_i covers and which has the largest value $\tau(z)$ among all elements z covered by z_i. (Note that all z covered by z_i lie in $P - C$, since $z < x_0$, so $\tau(z)$ is defined.) Continue until reaching a minimal element z_r of P. We now define

$$\psi\tau(x) = \begin{cases} \tau(x), & \text{if } x \notin C \text{ and } x \notin N, \\ \tau(x_{i,i+1}), & \text{if } x = x_i \in C \text{ and } 1 \leqslant i \leqslant m, \\ \tau(z_{i+1}), & \text{if } x = z_i \text{ and } 0 \leqslant i \leqslant r-1, \\ 1, & \text{if } x = z_r. \end{cases}$$

It is routine to check that $\phi\psi$ and $\psi\phi$ are identity maps, so the proof is complete. □

A special case of the previous theorem which includes (1) is the following.

COROLLARY. *Let P be a finite poset, and let A be an antichain of P which intersects every maximal chain. Then*

$$e(P) = \sum_{x \in A} e(P - x). \qquad \square$$

Note. The above theorem had earlier been obtained by D. Sturtevant (unpublished) in the special case that $\mathscr{C}$ is the set of maximal chains of P. Sturtevant observed that this result leads to an easy proof that $e(P)$ depends only on the comparability graph of P. It is somewhat easier to use the previous corollary instead of Sturtevant's result to prove this property of $e(P)$. To do so, let $\mathrm{Com}(P)$ denote the comparability graph of P.

PROPOSITION. *If P and Q are finite posets with* $\mathrm{Com}(P) \cong \mathrm{Com}(Q)$ *(as undirected graphs), then $e(P) = e(Q)$.*

Proof. Induction on $\#P$, the result being clear for $\#P = 0$. Assume $\#P > 0$. A subset A of the vertices of $\mathrm{Com}(P)$ is an antichain of P which intersects every maximal chain if and only if A is independent (i.e., no two vertices of A are connected by an edge) and intersects every maximal clique. Pick such an A. Note that $\mathrm{Com}(P - x) = \mathrm{Com}(P) - x$, where $\mathrm{Com}(P) - x$ denotes the graph $\mathrm{Com}(P)$ with vertex x and all incident edges removed. By induction $e(P - x)$ depends only on $\mathrm{Com}(P - x)$, so we can denote it $e(\mathrm{Com}(P - x))$. Then by the corollary,

$$e(P) = \sum_{x \in A} e(\mathrm{Com}(P - x)).$$

Since the right-hand side depends only on $\mathrm{Com}(P)$, the same is true of the left. □

For other proofs of the previous proposition, see [1], [2], [3], [5, Cor. 4.5], [6, Exercise 3.60].

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