Further Combinatorial Properties of Two Fibonacci Lattices

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In an earlier paper on differential posets, two lattices \( \text{Fib}(r) \) and \( Z(r) \) were defined for each positive integer \( r \), and were shown to have some interesting combinatorial properties. In this paper the investigation of \( \text{Fib}(r) \) and \( Z(r) \) is continued. A bijection \( \Psi : \text{Fib}(r) \to Z(r) \) is shown to preserve many properties of the lattices, though \( \Psi \) is not an isomorphism. As a consequence we give an explicit formula which generalizes the rank generating function of \( \text{Fib}(r) \) and of \( Z(r) \). Some additional properties of \( \text{Fib}(r) \) and \( Z(r) \) are developed related to the counting of chains.

1. Introduction

In [3] two lattices, denoted \( \text{Fib}(r) \) and \( Z(r) \), were defined for each positive integer \( r \) and were shown to have some interesting combinatorial properties. (\( \text{Fib}(1) \) had previously been considered in [1], where it was called the “Fibonacci lattice”.) In particular, \( \text{Fib}(r) \) and \( Z(r) \) have a unique minimal element \( 0 \), are graded, and have the same (finite) number of elements of each rank. When \( r = 1 \), the number of elements of rank \( n \) is the Fibonacci number \( F_{n+1} \) (where \( F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1} \)). There is a rank-preserving bijection \( \psi : \text{Fib}(r) \to Z(r) \), which satisfies \( e(x) = e(\psi(x)) \) for all \( x \in \text{Fib}(r) \), where \( e(x) \) denotes the number of maximal chains in the interval \( [0, x] \) (see [3, Prop. 5.7]).

In this paper we show that in fact the intervals \( [0, x] \) and \( [0, \psi(x)] \) have the same number of chains (or multichains) of any specified length. These numbers are relatively easy to compute for \( \text{Fib}(r) \), so we have ‘transferred’ this result to \( Z(r) \). As a consequence, we show that for any fixed \( n \geq 1 \),

\[
\sum_{x_1, x_2, \ldots, x_n} q^{e(x)} = \prod_{i=1}^{n} (1 - rq - ((i-1)r + 1)q^2)^{-1},
\]

where the sum ranges over all \( n \)-element multichains in \( \text{Fib}(r) \) or in \( Z(r) \), and where \( \rho \) denotes rank. Our results can also be interpreted in terms of the zeta polynomial [2, Ch. 3.11] of certain subposets of \( \text{Fib}(r) \) and \( Z(r) \). The proof in [3] that \( [0, x] \) and \( [0, \psi(x)] \) have the same number of maximal chains does not extend to chains of smaller lengths, so we use a new method of proof here.

We will use the notation

\[ N = \{0, 1, 2, \ldots\}, \quad P = \{1, 2, 3, \ldots\}. \]

2. Multichains in \( \text{Fib}(r) \) and \( Z(r) \)

We first define the lattices \( \text{Fib}(r) \) and \( Z(r) \). Let \( A(r) = \{1, 1_2, \ldots, 1_r, 2\} \) be an alphabet with \( r \) types of 1's and with one 2. (When \( r = 1 \) we simply let \( A(1) = \{1, 2\} \).) Then \( \text{Fib}(r) \) and \( Z(r) \) have the same set of elements, namely the set \( A(r)^* \) of all finite words with letters in \( A(r) \) (including the empty word \( \emptyset \)). The cover relations (and hence by transitivity the entire partial order) of \( \text{Fib}(r) \) and \( Z(r) \) are defined as follows. We say that \( v \) covers \( u \) in \( \text{Fib}(r) \) if \( u \) is obtained from \( v \) by changing a single 2 to a 1, for some \( i \), or by deleting the last letter in \( v \) if it is a 1. For instance, the word \( v = 221_21_21_1 \) in \( \text{Fib}(2) \) covers the words \( 121_21_21_1, 121_21_21_1, 121_21_21_1, 221_21_21_1, 221_21_21_1, 221_21_21_1, 221_21_21_1, 221_21_21_1, \) and \( 221_21_21_2 \). We say that \( v \) covers \( u \) in \( Z(r) \) if \( u \) can be obtained
from \( v \) by changing a single 2 to a 1, for some \( i \), provided that all letters preceding this 2 are also 2's, or by deleting the first letter which is not a 2 (if it occurs). Thus in \( Z(2) \) the word \( v = 221221211 \) covers the words \( 122122111, 12121221211, 212122121, 212122121 \) and \( 22121211 \). (Note that \( v \) covers 7 words in \( \text{Fib}(r) \) and 5 in \( Z(r) \).)

It is easily seen that \( \text{Fib}(r) \) and \( Z(r) \) are graded posets with \( \emptyset = \phi \) (the empty word), and rank function given by

\[
\rho(a_1 a_2 \cdots a_k) = a_1 + a_2 + \cdots + a_k,
\]

where \( a_i \in A(r) \), and where we add the \( a_i \)'s as integers (ignoring subscripts on the 1's).

It is also easily seen \([1, 3\), after Def. 5.6\] that \( \text{Fib}(1) \) is a distributive lattice, while \( \text{Fib}(r) \) for any \( r \) is upper-semimodular. More strongly, if \( x \in \text{Fib}(r) \) and \( x^* \) is the join of all elements covering \( x \), then the interval \([x, x^*]\) is the product of a boolean algebra with the modular lattice of rank two and cardinality \( r + 2 \). In particular, \( \text{Fib}(2) \) is 'join-distributive'. (In \([3\) it was erroneously claimed that \( \text{Fib}(r) \) is join-distributive for any \( r \).)

We will need the following result from \([3, \text{Prop. 5.4}]:\)

**Proposition 2.1.** \( Z(r) \) is a modular lattice for which every complemented interval has length \( \leq 2 \).

Given \( x \in A(r)^* \) and \( n \in \mathbb{P} \), let \( M_n(x) = M_n(x, r) \) (respectively, \( N_n(x) = N_n(x, r) \)) denote the number of multichains \( \emptyset = x_0 \leq x_1 \leq \cdots \leq x_n = x \) in \( \text{Fib}(r) \) (respectively, \( Z(r) \)) of length \( n \) with top \( x \). It is clear from the definitions of \( \text{Fib}(r) \) and \( Z(r) \) that if \( x \) and \( x' \) are two words in \( A(r)^* \) differing only in the subscripts on the 1's, then there are automorphisms of \( \text{Fib}(r) \) and of \( Z(r) \) which send \( x \) to \( x' \). Hence \( M_n(x) = M_n(x') \) and \( N_n(x) = N_n(x') \). For this reason we often suppress the subscripts on the 1's in \( x \) when writing \( M_n(x) \) or \( N_n(x) \) for particular \( x \). For instance, \( M_n(211y) \) denotes \( M_n(21,1,y) \) for any \( i, j \in \{1, \ldots, r\} \) and \( y \in A(r)^* \).

In the terminology of \([2, \text{Ch. 3.11}] \), \( M_n(x) \) and \( N_n(x) \) are (as functions of \( n \)) the *zeta polynomials* of the interval \([0, x]\) of \( \text{Fib}(r) \) and \( Z(r) \), respectively.

**Lemma 2.2.** Let \( u \in A(r)^* \). Then

\[
M_n(1u) = \sum_{i=1}^{n} M_i(u), \quad M_n(2u) = \sum_{i=1}^{n} ((i-1)r + 1)M_i(u). \tag{1,2}
\]

**Proof.** Let \( 1 \leq i \leq n \) and \( 1 \leq j \leq r \). Given a multichain \( \emptyset = u_0 \leq u_1 \leq \cdots \leq u_i = u \) in \( \text{Fib}(r) \), associate with it the multichain \( \emptyset = x_0 = x_1 = \cdots = x_{n-i} < 1, u_1 \leq \cdots \leq 1, u_i = 1, u \) in \( \text{Fib}(r) \). This sets up a bijection which proves (1).

Again given \( \emptyset = u_0 \leq u_1 \leq \cdots \leq u_i = u \) in \( \text{Fib}(r) \), define the following \((i-1)r + 1\) multichains of length \( n \) from \( \emptyset \) to \( 2u \) in \( \text{Fib}(r) \):

\[
\emptyset = x_0 = x_1 = \cdots = x_{n-i} < 1, u_1 \leq \cdots \leq 1, u_i = 1, u, \quad 1 \leq k \leq r, \quad 2 \leq s \leq i,
\]

\[
\emptyset = x_0 = x_1 = \cdots = x_{n-i} < 2, u_1 \leq \cdots \leq 2, u_i = 2, u.
\]

Every multichain of length \( n \) from \( \emptyset \) to \( 2u \) occurs exactly once in this way, so (2) follows.

**Lemma 2.3.** For any \( i \geq 0 \) and any \( u \in A(1)^* \), we have

\[
N_n(2^1u) - N_{n-1}(2^1u) = r \sum_{j=1}^{i} N_n(2^{j-1}2^j1u) + N_n(2^iu) - iN_n(2^{i-1}1u), \tag{3}
\]

\[
N_n(2^i) - N_{n-1}(2^i) = r \sum_{j=1}^{i} N_n(2^{j-1}2^j) - (ir-1)N_n(2^{i-1}). \tag{4}
\]

(Set \( N_n(2^{-1}u) = 0 \) and \( N_n(2^{-1}) = 0 \) in the case \( i = 0 \).)
PROOF. Let \( P \) be any locally finite poset for which every principal order ideal \( A_x := \{ y \in P : y \leq x \} \) is finite. Let \( L_n(x) \) be the number of multichains \( x_1 \leq x_2 \leq \cdots \leq x_n = x \) in \( P \). Clearly,

\[
L_n(x) = \sum_{y \leq x} L_{n-1}(y).
\]

Hence, letting \( \mu \) denote the Möbius function of \( P \) we have, by the Möbius inversion formula [2, Prop. 3.7.1],

\[
L_{n-1}(x) = \sum_{y \leq x} L_n(y) \mu(y, x).
\]

Since \( \mu(x, x) = 1 \) there follows

\[
L_n(x) - L_{n-1}(x) = -\sum_{y < x} L_n(y) \mu(y, x). \tag{5}
\]

Now, given \( x \in Z(r) \), let \( x_* \) be the meet of elements which \( x \) covers. (Since \( Z(r) \) is a lattice by Proposition 2.1, it follows that \( x_* \) exists.) By a well known property of Möbius functions (e.g. [2, Cor. 3.9.5]), we have \( \mu(y, x) = 0 \) unless \( x_* \leq y \leq x \). But by Proposition 2.1, the interval \([x_*, x]\) has length at most 2 (since a finite modular lattice is complemented if and only if 0 is a meet of coatoms).

If \([x_*, x]\) has length 0, then \( x = 0 \) and the lemma is clearly valid (put \( i = 0 \) in (4) to obtain \( 0 = 0 \)).

If \([x_*, x]\) has length 1, then \([x_*, x] = [u, 1] \) or \([x_*, x] = [1, 2]\); the latter case only for \( r = 1 \) (so \( j = 1 \)). Then \( \mu(x_*, x) = -1 \), and equations (3) (with \( i = 0 \)) and (4) (with \( i = r = 1 \)) coincide with (5).

Finally, assume that \([x_*, x]\) has length 2. If \( x \) covers \( k \) elements \( y \), then \( \mu(y, x) = -1 \), and \( \mu(x_*, x) = k - 1 \). Now if \( x = 2^i 1_k u \) (with \( i \geq 1 \)) then \( x \) covers the \( i + 1 \) elements \( y = 2^{-1}1_m 2^{-i}1_k u \) (1 \( \leq j \leq i \) and \( 1 \leq m \leq r \)) or \( y = 2^i u \); and \( x_* = 2^{-1}1_k u \). If \( x = 2^i \) (with \( i > 0 \), and with \( i > 1 \) if \( r = 1 \)) then \( x \) covers the \( ir \) elements \( y = 2^{-1}1_k 2^{r-i} \) (1 \( \leq j \leq i \), 1 \( \leq k \leq r \)) and \( x_* = 2^{-1} \). Thus equations (3) and (4) again coincide with (5), and the proof is complete. \( \square \)

We come to the main result of this section.

**THEOREM 2.4.** For all \( w \in A(1)^* \) and \( n \geq 1 \), we have \( M_* (w, r) = N_* (w, r) \). (Recall that in the notation \( M_* (w, r) \) and \( N_* (w, r) \), \( w \) stands for any word \( w \in A(r)^* \) obtained from \( w \) by replacing each 1 with some 1, for 1 \( \leq i \leq r \).)

**PROOF.** Given a function \( F : P \rightarrow Z \), define new functions \( \sigma F \) and \( \tau F \) by

\[
\sigma F(n) = \sum_{i=1}^{n} F(i), \quad \tau F(n) = ((n-1)r + 1) F(n).
\]

If \( w = w_1 w_2 \cdots w_k \in A(1)^* \), define the operator \( F_w \) on functions \( F : P \rightarrow Z \) by replacing each 1 in \( w \) with \( \sigma \) and each 2 with \( \sigma \tau \). For instance, \( F_{22221} = \sigma \sigma \tau \sigma \sigma \sigma \). Let \( I : P \rightarrow Z \) be defined by \( I(n) = 1 \) for all \( n \). Then it follows from Lemma 2.2 and the initial condition \( M_* (\phi) = 1 \) that

\[
M_* (w) = \Gamma_w I(n). \tag{6}
\]

Hence (since clearly \( N_* (\phi) = 1 \)) it suffices to show that the right-hand side of (6) satisfies the same recurrence, given by Lemma 2.3, that \( N_* (w) \) satisfies.

We claim that the operators \( \sigma \) and \( \tau \) satisfy the relation

\[
r \sigma^2 = \tau \sigma - \sigma \tau + r \sigma; \tag{7}
\]
for we have
\[ r\sigma^2 F(n) = r \sum_{i=1}^{n} (n-1+i)F(i), \quad \tau\sigma F(n) = ((n-1)r+1) \sum_{i=1}^{n} F(i), \]
\[ \sigma\tau F(n) = \sum_{i=1}^{n} ((i-1)r+1)F(i), \quad r\sigma F(n) = \sum_{i=1}^{n} F(i), \]
from which (7) is immediate.

Now suppose that \( w = 2^i1_u \in A(1)^* \). In order to show that \( I_w(n) \) satisfies the same recurrence (3) as does \( N_n(w) \), it suffices to show that for any \( F : \mathbb{P} \to \mathbb{Z} \),
\[
(\sigma\tau)^i\sigma F(n) - (\sigma\tau)^i\sigma F(n-1) = r \sum_{j=1}^{i} (\sigma\tau)^{i-j-1} \sigma(\sigma\tau)^{j-i} \sigma F(n) + (\sigma\tau)^i F(n) - ir(\sigma\tau)^{i-1} \sigma F(n).
\] (8)

We have
\[
r \sum_{j=1}^{i} (\sigma\tau)^{i-j-1} \sigma(\sigma\tau)^{j-i} \sigma = \sum_{j=1}^{i} (\sigma\tau)^{i-j-1} (\tau\sigma - \sigma\tau + r\sigma)(\tau\sigma)^{j-i}, \quad \text{by (7)}
\]
\[= \sum_{j=1}^{i} [(\sigma\tau)^{i-j-1}(\tau\sigma)^{j-i+1} - (\sigma\tau)^{i-j}(\tau\sigma)^{j-i} + \tau(\sigma\tau)^{j-i}],
\]
\[= (\tau\sigma)^i - (\sigma\tau)^i + ir(\sigma\tau)^{i-1}. \] (9)

But for any \( G : \mathbb{P} \to \mathbb{Z} \) we have
\[ \sigma G(n) - \sigma G(n-1) = G(n). \]

Thus
\*
(\sigma\tau)^i\sigma F(n) - (\sigma\tau)^i\sigma F(n-1) = \sigma(\tau\sigma)^i F(n) - \sigma(\tau\sigma)^i F(n-1)
\*
= (\tau\sigma)^i F(n).
\] (10)

Hence (8) follows from (9) and (10), as desired.

There remains the case \( w = 2^i \). We need to show that for any \( F : \mathbb{P} \to \mathbb{Z} \),
\[ (\sigma\tau)^i F(n) - (\sigma\tau)^i F(n-1) = r \sum_{j=1}^{i} (\sigma\tau)^{i-j} \sigma(\sigma\tau)^{j-i} F(n) - (ir-1)(\sigma\tau)^{i-1} F(n). \]
The proof is analogous to that of (8) and will be omitted. \( \square \)

**Corollary 2.5.** For all \( w \in A(r)^* \), the intervals \([\phi, w]\) in \( \text{Fib}(r) \) and \( \text{Z}(r) \) have the same number of elements.

**Proof.** Put \( n = 2 \) in Theorem 2.4. \( \square \)

It would be interesting to find a simple bijective proof of Corollary 2.5. The intervals \([\phi, w]\) in \( \text{Fib}(r) \) and \( \text{Z}(r) \) do not in general have the same rank-generating function (e.g. \( w = 1, 2^i \)).

We have the following generalization of Corollary 2.5:

**Corollary 2.6.** For any \( w \in A(r)^* \) and any \( j \in \mathbb{P} \), the intervals \([\phi, w]\) in \( \text{Fib}(r) \) and \( \text{Z}(r) \) have the same number of \( j \)-element chains.

**Proof.** For any finite poset \( P \), let \( L_n(P) \) be the number of multichains \( x_1 \leq x_2 \leq \cdots \leq x_{n-1} \) of length \( n \) in \( P \), and let \( c_j \) be the number of \( j \)-element chains. Then (see [2, Prop. 3.11.1])
\[ L_n(P) = \sum_{j=1}^{n} \binom{n-2}{j-1}. \] (11)
From this it follows easily that the numbers \( L_n(P) \) uniquely determine the \( c_i \)'s. The proof now follows from Theorem 2.4. \( \square \)

3. A Generalized Rank-generating Function

The rank-generating function of a poset \( P \) with rank function \( \rho: P \to \mathbb{N} \) (defined by \( \rho(x) = \) length of longest chain of \( P \) with top element \( x \)) is given [2, p. 99] by

\[
F(P, q) = \sum_{x \in P} q^{\rho(x)}.
\]

For \( \text{Fib}(r) \) and \( Z(r) \) we have (see [3, Th. 5.3 and Prop. 5.7])

\[
F(\text{Fib}(r), q) = F(Z(r), q) = (1 - rq - q^2)^{-1}, \quad (12)
\]

Now, given \( P \) as above and \( n \in \mathbb{P} \), define

\[
F_n(P, q) = \sum_{\pi_1, \ldots, \pi_n} q^{\rho(\pi_n)},
\]

summed over all \( n \)-element multichains in \( P \). The main result of this section is the following:

**Theorem 3.1.** Let \( n \in \mathbb{P} \). Then

\[
F_n(\text{Fib}(r), q) = F_n(Z(r), q) = \prod_{i=0}^{n} (1 - r(q - ((i - 1)r + 1)q^2))^{-1}.
\]

**Proof.** It follows from Theorem 2.4 that \( F_n(\text{Fib}(r), q) = F_n(Z(r), q) \). We prove Theorem 3.1 for \( \text{Fib}(r) \) by induction on \( n \). The case \( n = 1 \) is given by (12). Now assume the result for \( n - 1 \). Write

\[
F_n(\text{Fib}(r), q) = \sum_{i=0} f_i(q) q^i.
\]

We claim that

\[
f_n(t) - f_{n-1}(t) = rf_n(t - 1) + ((n - 1)r + 1)f_n(t - 2), \quad (14)
\]

for \( n > 0 \). (When \( n = 0 \), (14) is valid for \( t \geq 3 \).)

Now, using the notation of the previous section, we have

\[
f_n(t) = \sum_{\rho(u) = t} M_n(u),
\]

summed over all words \( u \in A(r)^* \) of rank \( t \).

For each \( u \in A(r)^* \) of rank \( t - 1 \) there are \( r \) words \( u = 1\mu \) (provided that \( t \geq 1 \)); while for each \( u \in A(r)^* \) of rank \( t - 2 \) there is one word \( u = 2\mu \) of rank \( t \) (provided that \( t \geq 2 \)). Hence

\[
f_n(t) = r \sum_{\rho(u) = t - 1} M_n(1\mu) + \sum_{\rho(u) = t - 2} M_n(2\mu).
\]

By (1) and (2) there follows

\[
f_n(t) = r \sum_{\rho(u) = t - 1} M_n(u) + \sum_{\rho(u) = t - 2} \sum_{i=1}^{n} ((i - 1)r + 1)M_n(u),
\]

so (since \( n > 0 \))

\[
f_n(t) - f_{n-1}(t) = r \sum_{\rho(u) = t - 1} M_n(u) + \sum_{\rho(u) = t - 2} \sum_{i=1}^{n} ((i - 1)r + 1)M_n(u)
\]

\[
= rf_n(t - 1) + ((n - 1)r + 1)f_n(t - 2),
\]

proving (14).
Now multiply (14) by $x^t$ and sum on $t \geq 0$. This results in (writing $F_t(q)$ for $F_t(Fib(r), q)$)

$$F_n(q) - F_{n-1}(q) = rqF_n(q) + ((n-1)r + 1)q^2F_n(q),$$

for $n > 0$, whence

$$F_n(q) = F_{n-1}(q)/(1 - rq - ((n-1)r + 1)q^2).$$

The proof follows by induction. □

Given a graded poset $P$ and $t \in \mathbb{N}$, let

$$P_{[0,t]} = \{x \in P : 0 \leq \rho(x) \leq t\}. \quad (15)$$

In the terminology of [2, Ch. 3.12], $P_{[0,t]}$ is a rank-selected subposet of $P$. Thus, in the notation of (13), $f_n(t)$ is the number of $n$-element multichains in $Fib(r)_{[0,t]}$ or $Z(r)_{[0,t]}$, so $f_{n-1}(t)$ (as a function of $n$) is the zeta polynomial of $Fib(r)_{[0,t]}$ or $Z(r)_{[0,t]}$. By (11), $f_n(t)$ (or $f_{n-1}(t)$) is a polynomial of degree $t$ and leading coefficient $m_t/t!$, where $m_t$ is the number of maximal chains in $Fib(r)_{[0,t]}$ or $Z(r)_{[0,t]}$. By [3, Prop. 3.1], we have

$$\sum_{t \geq 0} m_t x^t/t! = \exp(rt + \frac{1}{2}rt^2).$$

Equivalently,

$$m_t = \sum_{\pi} r \tau^{c(\pi)}, \quad (16)$$

where $\pi$ ranges over all involutions in the symmetric group $\mathbb{S}$, and where $c(\pi)$ denotes the number of cycles of $\pi$.

We may ask what more can be said about the polynomials $f_n(t)$. By standard properties of rational generating functions [2, Cor. 4.3.1], we have

$$\sum_{n \geq 0} f_n(t)x^n = \frac{W_t(x)}{(1 - x)^{t+1}},$$

where for fixed $t$, $W_t(x)$ is a polynomial in $x$ (called the $f_n(t)$—Eulerian polynomial) of degree $\leq t$ with integer coefficients summing to $m_t$ (as defined in (16)). Since $Z(r)$ is a modular lattice (or since $Fib(r)$ is semimodular), it follows from known results (see [2, Example 3.13.5 and Exercise 3.67(b)]) that $W_t(x)$ has non-negative coefficients. Since $Fib(1)$ is a distributive lattice, the following combinatorial interpretation of the coefficients of $W_t(x)$ (when $r = 1$) follows easily from the theory of $P$-partitions [2, Ch. 4.5].

**Proposition 3.2.** Given a permutation $\pi \in \mathbb{S}$, write $\pi$ as a product of disjoint cycles where (a) each cycle is written with its smallest element first, and (b) the cycles are written in increasing order of their smallest element. Let $\bar{\pi}$ be the permutation (written as a word) in $\mathbb{S}$, which results from erasing all parentheses from the above cycle notation. (We may have $\bar{\pi} = \emptyset$ even though $\pi \neq \sigma$; contrast this with the standard representation of [2, p. 17].) Then, when $r = 1$, we have

$$W_t(x) = \sum_{\pi} x^{1+d(\pi^{-1})},$$

where $\pi$ ranges over all involutions in $\mathbb{S}$, and where $d(\pi^{-1})$ denotes the number of descents [2, pp. 21–23] of $(\pi)^{-1}$. 
For instance, when \( t = 4 \) we have the following table:

\[
\begin{array}{cccc}
\pi & x & x^{-1} & d(x^{-1}) \\
(1)(2)(3)(4) & 1234 & 1234 & 0 \\
(12)(3)(4) & 1234 & 1234 & 0 \\
(13)(2)(4) & 1324 & 1324 & 1 \\
(14)(2)(3) & 1423 & 1342 & 1 \\
(1)(23)(4) & 1234 & 1234 & 0 \\
(1)(24)(3) & 1243 & 1243 & 1 \\
(12)(34) & 1234 & 1234 & 0 \\
(13)(24) & 1324 & 1324 & 1 \\
(14)(23) & 1423 & 1342 & 1 \\
\end{array}
\]

Hence \( W_d(x) = 5x + 5x^2 \) when \( r = 1 \). Presumably a similar result holds for any \( r \), but we will not consider this here.

**Proposition 3.3.** Fix \( r \in \mathbb{P} \). Then the polynomials \( W_t(x) \) satisfy the recurrence

\[
W_t(x) = rW_{t-1}(x) + ((rt - 1)x - r + 1)W_{t-2}(x) + r(x(1 - x)W_{t-2}(x), \quad t \geq 3, \tag{17}
\]

with the initial conditions

\[
W_0(x) = 1, \quad W_1(x) = rx, \quad W_2(x) = (r - 1)x^2 + (r^2 + 1)x.
\]

**Proof.** Multiply (14) by \( x^n \) and sum on \( n \geq 0 \). Since (14) is valid for \( n \geq 0 \) when \( t \geq 3 \), we obtain for \( t \geq 3 \) that

\[
\frac{W_t(x)}{(1 - x)^{t+1}} - \frac{xW_t(x)}{(1 - x)^{t+1}} = \frac{rW_{t-1}(x)}{(1 - x)^{t+1}} + rx \frac{dW_{t-2}(x)}{dx} (1 - x)^{t-1} - \frac{(r - 1)W_{t-2}(x)}{(1 - x)^{t-1}}. \tag{18}
\]

When equation (18) is simplified, the recurrence (17) results. It is easy to compute \( W_t(x) \) for \( 0 \leq t \leq 2 \) by a direct argument, so the proof follows. \( \square \)

The values of \( W_t(x) \) for \( 3 \leq t \leq 7 \) are given by

\[
\begin{align*}
W_3(x) &= r(3r - 2)x^2 + r(r^2 + 2)x, \\
W_4(x) &= (r - 1)(2r - 1)x^3 + (6r^3 - 2r^2 + 3r - 2)x^2 + (r^4 + 3r^2 + 1)x, \\
W_5(x) &= r(11r^2 - 12r + 3)x^3 + r(10r^3 + 12r - 6)x^2 + r(r^4 + 4r^2 + 3)x, \\
W_6(x) &= (r - 1)(2r - 1)(3r - 1)x^4 + (35r^4 - 22r^3 + 13r^2 - 12r + 3)x^3 \\
&\quad + (15r^3 + 5r^2 + 31r^2 - 8r^2 + 6r - 3)x^2 + (r^5 + 5r^4 + 6r^2 + 1)x, \\
W_7(x) &= 2r(5r - 2)(5r^2 - 5r + 1)x^4 + r(85r^4 - 10r^3 + 60r^2 - 60r + 12)x^3 \\
&\quad + r(21r^4 + 14r^3 + 65r^3 + 30r - 12)x^2 + r(r^6 + 6r^4 + 10r^3 + 4)x.
\end{align*}
\]

We conclude with a brief discussion of a natural generalization of the polynomials \( W_t(x) \). Let \( P \) be a graded poset and \( S \) a finite subset of \( P \). Generalizing (15), define the rank-selected poset \([2, \text{p. } 131]\)

\[
P_S = \{ z \in P : \rho(z) \in S \}.
\]

Let \( \alpha(P, S) \) denote the number of maximal chains of \( P_S \), and define

\[
\beta(P, S) = \sum_{T \subseteq S} (-1)^{|S - T|} \alpha(P, T).
\]
Equivalently,

\[ \alpha(P, S) = \sum_{T \in S} \beta(P, T). \]

For more information concerning the numbers \( \alpha(P, S) \) and \( \beta(P, S) \), see [2, Sect. 3.12–3.13]. In particular [2, Exer. 3.67], we have for \( P = \text{Fib}(r) \) and \( P = \text{Z}(r) \) that

\[ W_i(x) = \sum_S \beta(P, S)x^{\#(S - \{i\})}, \tag{19} \]

where \( S \) ranges over all subsets of \( \{1, \ldots, t\} \). Moreover, since \( \text{Fib}(r) \) is semimodular and \( \text{Z}(r) \) is modular, we have [2, Exam. 3.13.5] that \( \beta(\text{Fib}(r), S) \geq 0 \) and \( \beta(\text{Z}(r), S) \geq 0 \). However, it is false in general that \( \beta(\text{Fib}(r), S) = \beta(\text{Z}(r), S) \). For instance,

\[ \beta(\text{Fib}(1), \{2, 4\}) = 1, \quad \beta(\text{Z}(1), \{2, 4\}) = 2. \]

The techniques of [2, Sect. 3.12] lead to the following result, which together with (19) imply Proposition 3.2 by an easy argument (so that Proposition 3.4 may be regarded as a generalization of Proposition 3.2).

**Proposition 3.4.** Let \( S \) be a finite subset of \( \mathbb{P} \). Then \( \beta(\text{Fib}(1), S) \) is equal to the number of permutations \( \pi = (a_1, a_2, a_3, \ldots) \) of \( \mathbb{P} \) satisfying:

(a) \( a_i = i \) for all but finitely many \( i \);

(b) \( 2i \) and \( 2i + 1 \) appear to the right of \( 2j - 1 \) for all \( i \in \mathbb{P} \);

(c) \( D(\pi) = S \), where \( D(\pi) \) denotes the descent set of \( \pi \) [2, p. 21].

It would be interesting to find a similar result for \( \text{Fib}(r) \) when \( r \geq 2 \) and for \( \text{Z}(r) \) when \( r \geq 1 \).

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**References**


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