A CHROMATIC-LIKE POLYNOMIAL FOR ORDERED SETS

Richard P. Stanley
Harvard University

This paper surveys some results appearing in a section of the author's doctoral dissertation [4, Ch. IV, Section 5]. For further details, generalizations, and applications, see [4].

Let \( P \) be a finite (partially) ordered set with \( p > 0 \) elements and longest chain of length \( \ell \) (or cardinality \( \ell + 1 \)). A chain (totally ordered set) with \( p \) elements is denoted \( C_p \).

Definition (i) A map \( \sigma : \overline{P} \rightarrow C_n \) is said to be order-preserving if \( X \leq Y \Rightarrow \sigma(X) \leq \sigma(Y) \). Define \( \Omega(n) \) to be the number of order-preserving maps \( \sigma : \overline{P} \rightarrow C_n \).

(ii) A map \( \tau : \overline{P} \rightarrow C_n \) is said to be strictly order-preserving if \( X < Y \Rightarrow \tau(X) < \tau(Y) \). Define \( \Omega(n) \) to be the number of strict order-preserving maps \( \tau : \overline{P} \rightarrow C_n \).

(iii) \( e_n \) denotes the number of surjective order-preserving maps \( \sigma : \overline{P} \rightarrow C_n \).

(iv) \( \overline{e}_n \) denotes the number of surjective strict order-preserving maps \( \tau : \overline{P} \rightarrow C_n \).

For instance, if \( P = C_p \), then \( \Omega(n) = \binom{n+p-1}{p} \) and \( \overline{\Omega}(n) = \binom{n}{p} \); while if \( P \) is a disjoint union of \( p \) points, then \( \Omega(n) = \overline{\Omega}(n) = n^p \). For any \( P \), the number \( e_P \) is equal to the number of ways of extending \( P \) to a total order and is an important numerical invariant of \( P \). It is not hard to see that \( \Omega(n) \) is equal to the number of semi-ideals in the
direct product \( P \times C_{n-1} \) (see [1] for definitions). In particular, \( \Omega(1) = 1 \) and \( \Omega(2) \) is the number of semi-ideals of \( P \).

**Theorem 1.** \( \Omega(n) \) and \( \bar{\Omega}(n) \) are polynomials in \( n \) of degree \( p \) and leading coefficient \( e^p_p \) given by

\[
\Omega(n) = \sum_{s=1}^{p} e_s \binom{n}{s}
\]

\[
\bar{\Omega}(n) = \sum_{s=1}^{p} \bar{e}_s \binom{n}{s}
\]

**Proof.** For each of the \( \binom{n}{s} \) subsets \( S \) of \( C_n \) of size \( s \), there are \( e_s \) (resp. \( \bar{e}_s \)) order-preserving (resp. strict order-preserving) maps of \( P \) onto \( S \), and the theorem follows. \( \square \)

In the language of the calculus of finite differences,

\[
e_s = \Delta^s \Omega(0)
\]

\[
\bar{e}_s = \Delta^s \bar{\Omega}(0)
\]

The polynomial \( \bar{\Omega}(n) \) is an ordered set analog of the chromatic polynomial of a graph. \( \bar{\Omega}(n) \) counts the number of ways of "coloring" \( P \) with the colors \( 1, 2, \ldots, n \) such that no two comparable elements of \( P \) have the same color, and such that this coloring is "compatible" with the ordering of \( P \). One point at which the analogy breaks down is that the coefficients of \( \bar{\Omega}(n) \) need not alternate in sign, the smallest such \( P \) having five elements.

We now come to the crucial lemma (whose proof will not be given here) in analyzing the polynomials \( \Omega(n) \) and \( \bar{\Omega}(n) \). Let \( \omega \) be any surjective order-preserving map \( P \rightarrow \{1, 2, \ldots, p\} \), i.e., \( \omega \) is an extension of \( P \) to a total order. We denote the elements of \( P \) by \( X_1, \ldots, X_p \), where \( \omega(X_i) = i \). List all permutations \( i_1, i_2, \ldots, i_p \) of \( 1, 2, \ldots, p \) with the
property that if \( X < Y \) in \( P \), then \( \omega(X) \) appears before \( \omega(Y) \) in \( i_1, i_2, \ldots, i_p \). There are \( p \) such permutations. Put a "\( \leq \)" between two consecutive terms \( i_j \) and \( i_{j+1} \) if \( i_j \leq i_{j+1} \); otherwise put a "\( < \)" sign. Denote the array thus obtained by \( \bar{\lambda} \). Denote by \( \overline{\lambda} \) the array obtained from \( \lambda \) by changing all "\( < \)" signs to "\( \leq \)" signs and "\( \leq \)" signs to "\( < \)" signs. We say a map \( \sigma: P \to C_n \) is compatible with a permutation \( i_1, \ldots, i_p \) appearing in \( \lambda \) (or \( \overline{\lambda} \)) if \( \sigma(X_i) \leq \sigma(X_{i+1}) \leq \cdots \leq \sigma(X_{i+p}) \) and \( \sigma(X_{i+j}) < \sigma(X_{i+j+1}) \) whenever a "\( < \)" sign appears in \( \overline{\lambda} \).

\[ \begin{array}{l}
\textbf{Example:} \text{ Let } P \text{ and } \omega \text{ be given by } 1 \rightarrow 2 \rightarrow 3 \rightarrow 4. \text{ Then } \lambda \text{ and } \overline{\lambda} \text{ are given by }\\
1 \leq 2 \leq 3 \leq 4 \quad 1 < 2 < 3 < 4 \\
2 < 1 \leq 3 \leq 4 \quad 2 < 1 < 3 < 4 \\
1 \leq 2 \leq 4 < 3 \quad 1 < 2 < 4 \leq 3 \\
2 < 1 \leq 4 < 3 \quad 2 < 1 < 4 \leq 3 \\
2 \leq 4 < 1 \leq 3 \quad 2 < 4 \leq 1 < 3 \\
\lambda \quad \overline{\lambda}
\end{array} \]

\textbf{Lemma (i)} Every order-preserving map \( \sigma: P \to C_n \) is compatible with exactly one permutation in \( \lambda \).

\( (ii) \text{ Every strict order-preserving map } \tau: P \to C_n \text{ is compatible with exactly one permutation in } \overline{\lambda}. \quad \square \)

Thus we obtain alternative expressions for \( \Omega(n) \) and \( \overline{\Omega}(n) \) by summing the contributions coming from each permutation in \( \lambda \) and \( \overline{\lambda} \).

If exactly \( s \) "\( < \)" signs appear in a given permutation, then this permutation is easily seen to contribute a term \( (n+p-1-s) \) to \( \Omega(n) \) or \( \overline{\Omega}(n) \). Thus by the lemma, we obtain
Theorem 2: Let \( w_s \) (resp. \( \overline{w}_s \)) be the number of permutations in \( \Sigma \) (resp. \( \overline{\Sigma} \)) with exactly \( s \) "<" signs. Then

\[
\Omega(n) = \sum_{s=0}^{p-1} w_s \left( \binom{p+n-1-s}{p} \right)
\]

\[
\overline{\Omega}(n) = \sum_{s=0}^{p-1} \overline{w}_s \left( \binom{p+n-1-s}{p} \right).
\]

But clearly \( \overline{w}_s = w_{p-1-s} \). Substituting into Theorem 2 and comparing the resulting expression for \( \overline{\Omega}(n) \) with the expression for \( \Omega(n) \), we obtain the following fundamental result.

Theorem 3: \( \overline{\Omega}(n) = (-1)^p \Omega(-n) \).

The numbers \( w_s \) are natural generalizations of the Eulerian numbers [3, pp214-215]. When \( P \) is a disjoint union of \( p \) points, then \( w_s \) is equal to the number of permutations of \( 1, 2, \ldots, p \) with exactly \( s \) decreases between consecutive terms. This is the combinatorial definition of the Eulerian numbers \( A_{p,s+1} \). We also have the generating functions

\[
\sum_{n=0}^{\infty} \Omega(n) x^n = \left( \sum_{s=0}^{p-1} w_s x^{s+1} \right) / (1-x)^{p+1}
\]

\[
\sum_{n=0}^{\infty} \overline{\Omega}(n) x^n = \left( \sum_{s=0}^{p-1} \overline{w}_s x^{s+1} \right) / (1-x)^{p+1}
\]

Theorem 3 allows the determination of all integer zeros of \( \Omega(n) \). We state a slightly stronger result.
Corollary 1. We have $\Omega(0) = \Omega(-1) = \ldots = \Omega(-\ell) = 0$, while for $n > 0,$
$$(-1)^n \Omega(-\ell - n) \geq \Omega(n) > 0.$$ 

One can ask when equality holds in the inequality at the end of Corollary 1. A complete answer is provided by the following two theorems. They are proved by constructing in an obvious way a strict order-preserving map $\tau : P \to C_{n+\ell}$ corresponding to a given order-preserving map $\sigma : P \to C_n$, and analyzing when this correspondence is bijective.

Theorem 4. $\Omega(-\ell - 1) = (-1)^n$ if and only if every element of $P$ is contained in a chain of length $\ell$. $\square$

Theorem 5. The following three conditions are equivalent.

(i) $\Omega(-\ell - n) = (-1)^n \Omega(n)$ for some integer $n > 1$.

(ii) $\Omega(-\ell - n) = (-1)^n \Omega(n)$ for all $n$.

(iii) Every maximal chain of $P$ has length $\ell$.

It is not difficult to find ordered sets satisfying the conditions of Theorem 4 but not of Theorem 5. There are exactly such non-isomorphic ordered sets with six elements and none smaller. Theorem 5 leads to some interesting identities which appear to be difficult to prove by purely combinatorial reasoning.

Corollary 2. If every maximal chain of $P$ has length $\ell$, then

(i) $2e_{p-1} = (p+\ell-1)e_p$

(ii) $2\bar{e}_{p-1} = (p-\ell-1)e_p$

(iii) The coefficient of $n^{p-1}$ in $\Omega(n)$ is $\frac{\ell e_p}{2(p-1)!}$

(iv) $\sum_{s=1}^{\ell} \frac{p}{s} e_s = 2\ell \sum_{s=1}^{\ell} e_s$. 

Proof. By Theorem 5, we have

\[ \Omega(n) = \sum_{s=1}^{p} c_s \binom{n}{s} = (-1)^p \sum_{s=1}^{p} c_s \binom{-t-n}{s} \]

Equating coefficients of \( n^{p-1} \) gives (i) while (ii) is proved similarly using \( \bar{\Omega}(n) \). (iii) is then an immediate consequence of (i). We omit the proof of (iv) which involves a somewhat more complicated manipulation. \( \square \)

As a consequence of formula (i) or (ii) of the previous corollary, we get a curious though not very significant result. I have been unable to find a direct combinatorial proof of this fact.

Corollary 3. If every maximal chain of \( P \) has length \( \ell \), then either \( p-\ell \) is odd or \( e_p \) is even.

The preceding corollary motivates the following conjecture: Let \( P \) be any finite ordered set. If the length of every maximal chain of \( P \) has the same parity as \( p \), then \( e_p \) is even.

In conclusion we mention that various methods are available for explicitly determining \( \Omega(n) \) for special classes of ordered sets \( P \). For instance, one of the more interesting such classes consists of those \( P \) which are the direct product of two chains, say \( P = C_r \times C_s \). It can then be shown that

\[ \Omega(n) = \frac{\binom{r+n-1}{r} \binom{r+n}{r} \cdots \binom{r+s+n-3}{r}}{\binom{r}{r} \binom{r+1}{r} \cdots \binom{r+s-1}{r}} \]

This formula is closely related to MacMahon's solution of the "generalized ballot problem". [2, Section 103].
References


3. John Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc. (New York, 1958)