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ON DIMER COVERINGS OF RECTANGLES OF FIXED WIDTH

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For fixed k let A_n denote the number of dimer coverings of a $k \times n$ rectangle. Various properties of the generating function $\sum A_n x^n$ are obtained, in particular answering questions of Klarner and Pollack and of Hock and McQuistan. An explicit expression for the molecular freedom for dimers on a saturated $k \times n$ lattice space is also obtained. The results are consequences of the explicit formula for A_n obtained by Kasteleyn and by Temperley and Fisher.

Let k be a fixed positive integer, and let $A_n = A_{n,k}$ denote the number of ways to tile a $k \times n$ rectangle with $nk/2$ dimers (or dominoes). (Of course $A_n = 0$ if nk is odd.) Form the generating function

$$F_k(x) = \sum_{n \geqq 0} A_n x^n.$$

It is well known (e.g., [5]) that $F_k(x)$ represents a rational function, say $F_k(x) = P_k(x)/Q_k(x)$ with P_k, Q_k polynomials with integer coefficients, and $Q_k(0) = 1$. We do not assume that $F_k(x)$ is reduced to lowest terms. If

$$Q_k(x) = 1 - \alpha_1 x - \cdots - \alpha_q x^q,$$

then it follows that

$$A_{n+q} = \alpha_1 A_{n+q-1} + \cdots + \alpha_q A_n \tag{1}$$

for all n sufficiently large (and for all $n \geqq 0$ if and only if $\deg P_k < \deg Q_k$; we will show below that $\deg Q_k - \deg P_k = 2$). For the basic facts concerning rational generating functions, see [8]. The largest root of the polynomial $x^q Q_k(1/x)$, when $F_k(x)$ is reduced to lowest terms, is denoted by μ_k; and the number $\lambda_k = \mu_k^{2/k}$ is called the *molecular freedom* for dimers on a saturated $k \times n$ lattice space.

Recently Klarner and Pollack [5] computed $P_k(x)$ and $Q_k(x)$ for $1 \leqq k \leqq 8$, while Hock and McQuistan [3] computed $Q_k(x)$ for $1 \leqq k \leqq 10$. They also computed numerically the values of μ_k for $1 \leqq k \leqq 8$ and $1 \leqq k \leqq 10$, respectively. Both papers raised various questions about the properties of $P_k(x)$ and $Q_k(x)$. Here we will

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answer these and other questions and will give an explicit formula for μ_k.

Our results are direct consequences of Kasteleyn's formula for A_n (also obtained by Temperley and Fisher [9] and later by Lieb [6]), which Kasteleyn shows [4, eqn. (15)] can be written in the form

$$A_n=\begin{cases}\prod_{j=1}^{[k/2]}\dfrac{c_j^{n+1}-\bar{c}_j^{n+1}}{2b_j}, & nk \text{ even},\\ 0, & nk \text{ odd}\end{cases}\tag{2}$$

where

$$c_j=\cos\frac{j\pi}{k+1}+\left(1+\cos^2\frac{j\pi}{k+1}\right)^{1/2},$$

$$\bar{c}_j=\cos\frac{j\pi}{k+1}-\left(1+\cos^2\frac{j\pi}{k+1}\right)^{1/2},$$

$$b_j=\left(1+\cos^2\frac{j\pi}{k+1}\right)^{1/2}.$$

Note that $c_j\bar{c}_j=-1$.

Write $l=[k/2]$, and let S be any subset of $\{1,\dots,l\}$ and $\bar{S}=\{1,\dots,l\}-S$. Define

$$c_S=\left(\prod_{j\in S}c_j\right)\left(\prod_{j\in\bar{S}}\bar{c}_j\right).$$

Then (2) shows that

$$A_n=\left[\prod_{j=1}^{l}(2b_j)^{-1}\right]\sum_S(-1)^{|\bar{S}|}c_S^{n+1},\tag{3}$$

provided nk is even, where S ranges over all subsets of $\{1,\dots,l\}$.

Lemma. *We have*

$$\prod_{j=1}^{l}b_j^2=d_k2^{-k},$$

where $d_0=1$, $d_1=2$, $d_k=2d_{k-1}+d_{k-2}$. *Explicitly,*

$$d_k=\frac{(1+\sqrt{2})^{k+1}-(1-\sqrt{2})^{k+1}}{2\sqrt{2}}.\tag{4}$$

Proof. When k is even, equation (4) is the case $u=1$ of a formula of Kasteleyn [4, eqn. (14)], and when k is odd, Kasteleyn's proof is also valid. The recurrence $d_k=2d_{k-1}+d_{k-2}$ follows from (4) since $1\pm\sqrt{2}$ are roots of the polynomial x^2-2x-1.

One can also view this lemma (as well as [4, eqn. (14)]) as a standard result on the Chebyshev polynomial

$$U_k(x)=2^k\prod_{j=1}^{k}\left(x-\cos\frac{j\pi}{k+1}\right),$$

after observing that

$$\mathrm{i}^k 2^k \prod_{j=1}^{l} b_j^2 = U_k(\mathrm{i}), \quad \text{where } \mathrm{i}^2 = -1.$$

Theorem. (a) *The polynomial* $Q_k(x)$ *can be taken to be*

$$Q_k(x) = \begin{cases} \prod_S (1 - c_S x), & k \text{ even} \\ \prod_S (1 - c_S^2 x^2), & k \text{ odd,} \end{cases} \tag{5}$$

where S *ranges over all subsets of* $\{1, \ldots, l\}$. *Hence* A_n *satisfies a linear recurrence* (1) (*which by* (d) *below will be valid for all* $n \geqq 0$) *of degree* $q_k = \deg Q_k = 2^{[(k+1)/2]}$. *Moreover, all the roots of* $Q_k(x)$ *are real and nonzero, and exactly half the roots are positive.*

(b) *The largest reciprocal root of* $Q_k(x)$ *is*

$$\mu_k = \prod_{j=1}^{l} c_j, \tag{6}$$

which occurs with multiplicity one and which is not a reciprocal root of $P_k(x) = Q_k(x) F_k(x)$. *The molecular freedom* $\lambda_k = \mu_k^{2/k}$ *satisfies*

$$\lambda = \lim_{k \to \infty} \lambda_k = \mathrm{e}^{2G/\pi} = 1.79162\cdots, \tag{7}$$

where $G = \sum_{s \geq 0} (-1)^s (2s+1)^{-1}$ *is Catalan's constant.*

(c) *Asymptotically we have*

$$A_n \sim a_k \mu_k^{n+1}, \quad \text{as } n \to \infty \text{ with } nk \text{ even,} \tag{8}$$

where

$$a_k = 2^{\delta} d_k, \tag{9}$$

where d_k *is given by* (4) *and where* $\delta = 0$ *if* k *is even and* $\delta = \frac{1}{2}$ *if* k *is odd. Moreover,*

$$\lim_{k \to \infty} a_k^{2/k} = 1/(\sqrt{2} + 1) = \sqrt{2} - 1.$$

(d) $P_k(x)$ *has degree* $p_k = 2^{[(k+1)/2]} - 2 = q_k - 2$. *Hence* A_n *satisfies* (1) *for* all $n \geq 0$.

(e) *If* $k > 1$, *then* $P_k(x) = -x^{p_k} P_k(1/x)$. *If* k *is odd or divisible by* 4, *then* $Q_k(x) = x^{q_k} Q_k(1/x)$. *If* $k \equiv 2 \pmod 4$, *then* $Q_k(x) = -x^{q_k} Q_k(1/x)$. *If* k *is odd, then* $P_k(x) = P_k(-x)$ *and* $Q_k(-x) = Q_k(x)$. (The statements about $Q_k(x)$ are equivalent to property (d) of the roots observed by Hock and McQuistan [3, p. 104] for $k \leqq 10$.)

(f) *For* k *odd write*

$$\begin{aligned} Q_k(x) &= \beta_0 - \beta_1 x^2 + \beta_2 x^4 - \cdots + \beta_r x^{2r} \\ &= \gamma_0 - \binom{r}{1} \gamma_1 x^2 + \binom{r}{2} \gamma_2 x^4 - \cdots + \gamma_r x^{2r}, \end{aligned}$$

where $r=2^l$. *Then the numbers* γ_i *are positive and log-concave (i.e.,* $\gamma_i^2 \geqq \gamma_{i-1}\gamma_{i+1}$*). Thus they are also unimodal (i.e., increase monotonically to a maximum, and then decrease monotonically). (This implies that the* β_i*'s are also positive, log-concave, and unimodal.)*

(g) *Define*

$$T_k(x) = \begin{cases} \prod_{|S| \text{ even}} (1-c_S x), & k \text{ even}, \\ \prod_{|S| \text{ even}} (1-c_S^2 x^2), & k \text{ odd}, \end{cases}$$

$$\bar{T}_k(x) = \begin{cases} \prod_{|S| \text{ odd}} (1-c_S x), & k \text{ even}, \\ \prod_{|S| \text{ odd}} (1-c_S^2 x^2), & k \text{ odd}, \end{cases}$$

so that $Q_k(x) = T_k(x)\bar{T}_k(x)$. *Then the coefficients of* $T_k(x)$ *and* $\bar{T}_k(x)$ *lie in the field* $\mathbb{Q}(d_k^{1/2})$, *where* d_k *is given by the Lemma, and if* $d_k^{1/2} \notin \mathbb{Q}$, *then the coefficients of any monomial* x^j *in* $T_k(x)$ *and* $\bar{T}_k(x)$ *are conjugate in* $\mathbb{Q}(d_k^{1/2})$. *If* $d_k^{1/2} \in \mathbb{Q}$, *then* $T_k(x)$ *and* $\bar{T}_k(x)$ *have rational coefficients (so* $Q_k(x)$ *is reducible over* $\mathbb{Q}$*).* (J. Lagarias has shown me a proof that d_k is a square if and only if $k=0$ or $k=6$). *When* $k=6$ *we have*

$$T_6(x) = (1-x)(1-6x+5x^2-x^3),$$

$$\bar{T}_6(x) = (1+x)(1+5x+6x^2+x^3).$$

(The fact that ± 1 *are roots of* $Q_6(x)$ *is equivalent to the surprising identity* $c_1 = c_2 c_3$ *for* $k=6$.) *Moreover, when* k *is even,*

$$P_k(x) = d_k^{-1/2}(T_k(x)\bar{T}_k'(x) - T_k'(x)\bar{T}_k(x)).$$

Proof. (a) From (2) it follows that $F_k(x) = A_k(x)/B_k(x)$, where $B_k(x) = \prod_S (1-c_S x)$ and where $A_k(x)$ is a polynomial. Hence to prove (5) it suffices to show that the coefficients of $C_k(x)$ are integers where

$$C_k(x) = \begin{cases} B_k(x), & k \text{ even}, \\ B_k(x)B_k(-x), & k \text{ odd}. \end{cases}$$

Equivalently, if σ is an automorphism of the splitting field of the field $L = \mathbb{Q}(c_S \mid S \subseteq \{1, \ldots, l\})$ (actually, L is Galois extension of $\mathbb{Q}$, but this is irrelevant), and if t is a root of $C_k(x)$ of multiplicity m, then σt is also a root of $C_k(x)$ of multiplicity m. (Probably all roots of $C_k(x)$ have multiplicity one; see the conjecture below.)

Set $D = \prod_{j=1}^{l} (2b_j)^{-1}$. By the Lemma D^2 is a rational number, so $\sigma D = \pm D$. Applying σ to (3) yields (since A_n is rational)

$$A_n = \sigma A_n = \pm D \sum_S (-1)^{|\bar{S}|}(\sigma c_S)^{n+1}, \quad nk \text{ even}. \tag{10}$$

Suppose $t = c_S$, so that m is equal to the number of T for which $c_S = c_T$. Since $c_j > 0$

and $\bar{c}_j < 0$, it follows that $c_S > 0$ if and only if $|\bar{S}|$ is even, and hence $(-1)^{|\bar{S}|} = (-1)^{|\bar{T}|}$ whenever $c_S = c_T$. Thus the coefficient of t^n in (3) when all equal expressions c_X^{n+1} are combined is equal to $(-1)^{|\bar{S}|}Dtm$.

Now all functions $f(n) = \sum_r a_r \gamma_r^n$, where the γ_r's are distinct nonzero complex numbers and the a_r's nonzero complex numbers, are different. It follows from (3) and (10) that when k is even the coefficient of $(\sigma t)^n$ in (3) when all equal expressions c_X^{n+1} are combined is equal to $\pm(-1)^{|\bar{S}|}D(\sigma t)m$. Hence exactly m values of T satisfy $\sigma t = c_T$, so that σt is a root of $C_k(x)$ of multiplicity m as desired.

When k is odd (3) is valid only for n even. The above argument applied to $A'_n = A_{2n}$ shows that c_S^2 and σc_S^2 are roots of $C_k(\sqrt{x})$ of the same multiplicity, so that $\pm c_S$ and $\sigma(\pm c_S)$ are roots of $R_k(x)$ of the same multiplicity, completing the proof of (5).

Clearly the numbers c_j are real and, as already observed, satisfy $c_j > 0$, $\bar{c}_j < 0$. From this we immediately have that the roots (or reciprocal roots) of $Q_k(x)$ are real and nonzero, and that exactly half of the roots are positive. A different proof that the denominator of $F_k(x)$, when reduced to lowest terms, has real roots appears in [5, p. 47].

(b) Clearly $c_j > |\bar{c}_j| > 0$, so the largest c_S is uniquely obtained by letting $S = \{1, \ldots, l\}$, yielding (6). This largest reciprocal root μ_k cannot be a reciprocal root of $P_k(x)$ since the term μ_k^n appears in (3) with nonzero coefficient, so that μ_k must be a reciprocal root of the *least* denominator of $F_k(x)$. A different proof that the largest reciprocal root of the least denominator of $F_k(x)$ has multiplicity one appears in [1, p. 284].

One can compute $\lim_{k\to\infty} \lambda_k$ directly from (6) by expressing $\lim_{k\to\infty} \log \mu_k^{2/k}$ in terms of a Riemann integral in a standard way, yielding

$$\log \lambda = \frac{2}{\pi} \int_0^{\pi/2} \log(\cos x + (1+\cos^2 x)^{1/2})\, \mathrm{d}x.$$

The above integral is essentially evaluated, e.g., in [4, p. 1216], and is equal to Catalan's constant G. Hence $\lambda = \mathrm{e}^{2G/\pi}$.

Alternatively, Kasteleyn [4] and Temperley and Fisher [9] showed that

$$\lim_{\substack{k,n\to\infty \\ kn \text{ even}}} A_{n,k}^{2/nk} = \mathrm{e}^{2G/\pi}.$$

But (always assuming kn is even)

$$\lim_{k,n\to\infty} A_{n,k}^{2/nk} = \lim_{k\to\infty} \left(\lim_{n\to\infty} A_n^{1/n} \right)^{2/k} = \lim_{k\to\infty} \mu_k^{2/k}$$

and again (6) follows. This computation of λ_k is mentioned in [6, eqn. (7)].

(c) From (3) and (6), the coefficient a_k of μ_k^{n+1} in A_n is given by (9), so (8) follows.

From (9) and the explicit expression (4) for d_k it is clear that $\lim a_k^{2/k} = (1+\sqrt{2})^{-1}$. (It is also possible to prove this result without explicitly evaluating d_k, by expressing $\lim (2/k) \log (b_1 \cdots b_l)$ as a Riemann integral.)

(d) It follows from the form (3) of A_n and basic facts about rational generating functions [8, Theorem 4.1] that $p_k < q_k$. Then by [7, Proposition 5.2], we have that $q_k - p_k$ is equal to the largest integer m for which $A_{-1} = A_{-2} = \cdots = A_{-m+1} = 0$, where A_{-n} is defined by substituting $-n$ for n in (2) or (3). Clearly by (2) we have $A_{-1} = 0$. On the other hand, since $c_j \bar{c}_j = -1$, it follows that $A_{-2} = \pm A_0 = \pm 1$, and the proof follows.

(e) Since $c_j \bar{c}_j = -1$, we have $c_S \bar{c}_S = (-1)^l$. Hence if k is odd then the reciprocal roots $\pm c_S$ of $Q_k(x)$ come in groups of four of the form $c_S, -c_S, c_{\bar{S}} = \pm c_S^{-1}, -c_{\bar{S}} = \mp c_S^{-1}$. This implies $Q_k(x) = x^{q_k} Q_k(1/x)$ and $Q_k(x) = Q_k(-x)$. If k is divisible by 4, then the reciprocal roots come in pairs c_S and $c_{\bar{S}} = c_S^{-1}$, which implies $Q_k(x) = x^{q_k} Q_k(1/x)$. If $k \equiv 2 \pmod 4$, then the reciprocal roots come in pairs c_S and $c_{\bar{S}} = -c_S^{-1}$, which implies $Q_k(x) = -x^{q_k} Q_k(1/x)$.

Now define

$$\bar{F}_k(x) = \sum_{n>0} A_{-n} x^n.$$

A result of Popoviciu (see e.g. [7, Proposition 5.2]) implies that

$$F_k(x) = -\bar{F}_k(1/x),$$

as rational functions. From (2) and the equality $c_j \bar{c}_j = -1$ it is clear that

$$A_{-n} = (-1)^{(n-1)l} A_{n-2}, \qquad A_{-1} = 0.$$

Hence

$$\bar{F}_k(x) = \begin{cases} x^2 F_k(x), & l \text{ even}, \\ x^2 F_k(-x), & l \text{ odd}. \end{cases}$$

Comparing with (11) yields

$$F_k(x) = \begin{cases} -(1/x^2) F_k(1/x), & l \text{ even}, \\ -(1/x^2) F_k(-1/x), & l \text{ odd}. \end{cases}$$

Comparing this result with what was just proved for $Q_k(x)$ (and using $q_k - p_k = 2$) yields the desired properties of $P_k(x)$.

(f) Let $Q(x) = \sum_{i=0}^{s} \delta_i \binom{s}{i} x^i$ be any polynomial with negative real roots. I. Newton showed (see e.g. [2, Theorem 51]) that $\delta_i^2 \geqq \delta_{i-1} \delta_{i+1}$. (This result is in fact valid for any polynomial with real roots.) Now consider for k odd the polynomial

$$Q_k(\sqrt{x}) = \prod_S (1 - c_S^2 x) = \sum_{i=0}^{r} \gamma_i \binom{r}{i} (-1)^i x^i.$$

Since c_S is real and nonzero, it follows that $c_S^2 > 0$ and hence each $\gamma_i > 0$. By Newton's result, $\gamma_i^2 \geq \gamma_{i-1} \gamma_{i+1}$. Since each $\gamma_i > 0$, this means $\gamma_i \geq \min\{\gamma_{i-1}, \gamma_{i+1}\}$ so that the γ_i's are unimodal. This completes the proof.

(g) We omit the proof, which is a rather routine consequence of what we already have shown.

In conclusion we mention the following conjecture.

Conjecture. The polynomial $Q_k(x)$ has distinct roots.

This conjecture is equivalent to the statement that $2^{[(k+1)/2]}$ is the *least* degree of a linear recurrence relation satisfied by A_n (or equivalently, that $P_k(x)$ and $Q_k(x)$ are relatively prime). To see this, note that c_S^n occurs in (3) with nonzero coefficient, so that c_S must be a reciprocal root of the denominator $R_k(x)$ when $F_k(x)$ is reduced to lowest terms. When k is even, this accounts for all $2^{[(k+1)/2]}$ roots of $Q_k(x)$. When k is odd, this only accounts for half the roots of $Q_k(x)$. However, in this case $A_n=0$ when n is odd. Thus if A_n satisfies (1), then it also satisfies (1) when every term $\alpha_i A_{n+q-i}$ with i odd is deleted. This means that the unique recurrence (1) of minimal degree satisfies $\alpha_{2i+1}=0$, so $R_k(x)=R_k(-x)$. Hence not only must all the numbers c_S be roots of $R_k(x)$, but also their negatives $-c_S$, and we have again accounted for all $2^{[(k+1)/2]}$ roots of $Q_k(x)$.

Let us point out that although we are unable to decide whether the roots of $Q_k(x)$ are distinct, it is evident from (3) that the least denominator of $F_k(x)$ has distinct roots (because the coefficient of each c_S^n is a constant, rather than a polynomial in n of degree ≥ 1). This answers a question raised in [5, p. 47].

A stronger assertion than the distinctness of the roots of $Q_k(x)$ is the statement that $Q_k(x)$ is irreducible over the rationals. In this regard, J. Lagarias has pointed out to me that the reducibility of $Q_6(x)$ implies the reducibility of $Q_k(x)$ when $k+1$ is divisible by 7. Moreover, Lagarias has proved that $Q_k(x)$ is irreducible whenever $k+1$ is an odd prime $\neq 7$. Hence in this case the above conjecture is valid.

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