

Unimodality and Lie Superalgebras

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It is well-known how the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ can be used to prove that certain sequences of integers are unimodal and that certain posets have the Sperner property. Here an analogous theory is developed for the Lie superalgebra $\mathfrak{osp}(1, 2)$. We obtain new classes of unimodal sequences (described in terms of cycle index polynomials) and a new class of posets (the "superanalogue" of the lattice $L(m, n)$ of Young diagrams contained in an $m \times n$ rectangle) which have the Sperner property.

1. Introduction

Let m and n be integers with $m \leq n$. A sequence a_m, a_{m+1}, \dots, a_n of real numbers is *symmetric* [about $\frac{1}{2}(m+n)$] if $a_{m+i} = a_{n-i}$ for $0 \leq i \leq n-m$, and *unimodal* if $a_m \leq a_{m+1} \leq \dots \leq a_j \geq a_{j+1} \geq \dots \geq a_n$ for some j . We also call the Laurent polynomial $a_m q^m + a_{m+1} q^{m+1} + \dots + a_n q^n$ symmetric or unimodal if its coefficients a_m, a_{m+1}, \dots, a_n have the corresponding property. It is well known how the representation theory of the Lie algebra $\mathfrak{sl}(2) = \mathfrak{sl}(2, \mathbb{C})$ can be used to prove certain sequences are symmetric and unimodal. This goes back to Dynkin [5, p. 332] and is further discussed, for example, in [1], [16], [18]. In particular, every finite-dimensional complex semisimple Lie algebra \mathfrak{G} contains a copy of $\mathfrak{sl}(2)$, known as a "principal three-dimensional subalgebra," which leads to a wide variety of unimodal sequences (explicitly described in [16]).

Here we derive an analogous theory for Lie superalgebras. The analogue of $\mathfrak{sl}(2)$ is the orthosymplectic superalgebra $\mathfrak{osp}(1, 2)$ [also denoted by $B(0, 1)$ or $\mathfrak{osp}(2, 4)$]. It is no longer true that every finite-dimensional complex semisimple Lie superalgebra contains a principal $\mathfrak{osp}(1, 2)$. Indeed, the only general class of

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superalgebras which do contain a principal $\mathfrak{osp}(1,2)$ are those denoted by $\mathfrak{gl}(n+1/n)$. [Strictly speaking, $\mathfrak{gl}(n+1/n)$ is not semisimple, and we should be dealing instead with $\mathfrak{sl}(n+1/n)$, also denoted by $A(n, n-1)$ or $\mathfrak{spl}(n+1, n)$. It is more convenient to work with $\mathfrak{gl}(n+1/n)$, and this superalgebra is close enough to being semisimple to cause no difficulties.]

The Lie algebra $\mathfrak{sl}(2)$ can also be used to prove that certain partially ordered sets have some desirable extremal properties, in particular the Sperner property. This use of $\mathfrak{sl}(2)$ had its origins in [17] and was first explicitly formulated in [14]. In Section 8 we give a "superanalogue" in which $\mathfrak{sl}(2)$ is replaced by $\mathfrak{osp}(1,2)$.

2. Review of $\mathfrak{sl}(2)$

First we review the relevant background concerning $\mathfrak{sl}(2)$, so that the analogy with $\mathfrak{osp}(1,2)$ will be clear. The Lie algebra $\mathfrak{sl}(2)$ is spanned by the three matrices

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

with the bracket operation $[A, B] = AB - BA$. Let $\mathfrak{gl}(n) = \mathfrak{gl}(n, \mathbb{C})$ denote the Lie algebra of all $n \times n$ complex matrices, and let

$$\phi: \mathfrak{sl}(2) \rightarrow \mathfrak{gl}(N)$$

be a representation (= Lie-algebra homomorphism) of $\mathfrak{sl}(2)$. Then ϕh is similar to a diagonal matrix with integer eigenvalues, say $\text{diag}(j_1, \dots, j_N)$, $j_i \in \mathbb{Z}$. We then define the *character* $\text{ch} \phi$ of ϕ to be the polynomial

$$\text{ch} \phi = q^{j_1} + \dots + q^{j_N}.$$

(In the precise definition of $\text{ch} \phi$ as given e.g. in [6, §22.5], the symbol q is regarded as a certain element in the group algebra of the dual to the weight lattice of $\mathfrak{sl}(2)$, but for our purposes we may merely regard q as an indeterminate.)

The representation ϕ can be written as a direct sum of irreducible representations, and $\text{ch} \phi$ can be uniquely written as a nonnegative integral linear combination of irreducible characters. The Lie algebra $\mathfrak{sl}(2)$ has one irreducible representation ϕ_{n-1} (up to equivalence) of every dimension $n \geq 1$. [The image of $\mathfrak{sl}(2)$ under ϕ_{n-1} is a "principal three-dimensional subalgebra" of $\mathfrak{gl}(n)$.] The character of ϕ_{n-1} is given by

$$\text{ch} \phi_n = q^{-n} + q^{-n+2} + q^{-n+4} + \dots + q^n.$$

For these basic facts about $\mathfrak{sl}(2)$, see e.g. [6, §7].

It follows that when we write $\text{ch } \phi$ as a linear combination of irreducibles, we obtain, for certain nonnegative integers m_i ,

$$\begin{aligned} \text{ch } \phi &= m_0 \text{ch } \phi_0 + m_1 \text{ch } \phi_1 + \dots \\ &= m_0 + m_1(q^{-1} + q) + m_2(q^{-2} + 1 + q^2) + \dots \\ &= \sum_i b_i q^i, \end{aligned}$$

where $b_i = b_{-i}$ and $b_i - b_{i+2} = m_i \geq 0$ for $i \geq 0$. Hence we obtain:

THEOREM 2.1. *If $\text{ch } \phi = \sum b_i q^i$, then the two sequences $\dots, b_{-4}, b_{-2}, b_0, b_2, b_4, \dots$ and $\dots, b_{-3}, b_{-1}, b_1, b_3, \dots$ are symmetric (about 0) and unimodal.*

3. Schur functions

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition of length $l(\lambda) := \#\{i | \lambda_i \neq 0\} \leq n$, i.e., $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, $\lambda_i \in \mathbf{Z}$. Write $|\lambda| = \lambda_1 + \dots + \lambda_n$. Then λ indexes a certain irreducible representation $\psi_\lambda: \mathfrak{gl}(n) \rightarrow \mathfrak{gl}(N)$, whose description is essentially due to Schur [in the context of the Lie group $\text{GL}(n)$]. We will not bother to define ψ_λ here, but will merely state the properties of interest to us. If $A \in \mathfrak{gl}(n)$ has eigenvalues $\alpha_1, \dots, \alpha_n$, then $\psi_\lambda(A)$ has all its eigenvalues of the form $a_1 \alpha_1 + \dots + a_n \alpha_n$, where a_1, \dots, a_n are nonnegative integers independent of $\alpha_1, \dots, \alpha_n$, and $\sum a_i = |\lambda|$. Define the *character* $\text{ch } \psi_\lambda$ of ψ_λ to be the polynomial

$$\text{ch } \psi_\lambda = \sum x_1^{a_1} \dots x_n^{a_n},$$

the sum being over all eigenvalues $\sum a_i \alpha_i$ of A . Then $\text{ch } \psi_\lambda$ is a symmetric function of x_1, \dots, x_n denoted $s_\lambda(x) = s_\lambda(x_1, \dots, x_n)$ and called a *Schur function*. The basic properties of Schur functions are discussed in [12] and [15].

Consider now the composite representation

$$\mathfrak{sl}(2) \xrightarrow{\phi_{n-1}} \mathfrak{gl}(n) \xrightarrow{\psi_\lambda} \mathfrak{gl}(N)$$

of $\mathfrak{sl}(2)$. Now $\phi_{n-1}(h)$ has eigenvalues $-n+1, -n+3, \dots, n-1$, so $\psi_\lambda \phi_{n-1}(h)$ has eigenvalues $(-n+1)a_1 + (-n+3)a_2 + \dots + (n-1)a_n$. Hence

$$\begin{aligned} \text{ch}(\psi_\lambda \phi_{n-1}) &= \sum q^{-(n-1)a_1 - (n-3)a_2 + \dots + (n-1)a_n} \\ &= q^{-(n+1)|\lambda|} \sum (q^2)^{a_1 + 2a_2 + \dots + na_n} \\ &= q^{-(n+1)|\lambda|} s_\lambda(q^2, q^4, \dots, q^{2n}). \end{aligned} \quad (1)$$

We deduce from Theorem 2.1:

THEOREM 3.1. For any partition λ of length $\leq n$, the polynomial $s_\lambda(q, q^2, \dots, q^n)$ is symmetric and unimodal.

Note: If $l(\lambda) > n$, then $s_\lambda(x_1, \dots, x_n) = 0$, so the condition $l(\lambda) \leq n$ of the previous theorem is irrelevant.

A simple explicit formula for $s_\lambda(q, q^2, \dots, q^n)$ appears in [12, Example 1, p. 27] or [15, Theorem 15.3]. Theorem 3.1 is implicit in [5, p. 332], is more explicit in [16, Example 2], and is also given in [12, Example 4, p. 67]. The coefficient of q^i in $s_\lambda(q, q^2, \dots, q^n)$ has a combinatorial interpretation—it is the number of column-strict plane partitions (as defined in [12, Example 13, p. 48] and [15, §1]) of shape λ , largest part $\leq n$, and sum of parts equal to i .

Let us also mention that when $\lambda = (m)$, the partition with a single part equal to m , then

$$s_m(q, \dots, q^n) = q^m \binom{n+m-1}{m},$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1-q^a)(1-q^{a-1}) \cdots (1-q^{a-b+1})}{(1-q^b)(1-q^{b-1}) \cdots (1-q)}$$

denotes a *q-binomial coefficient*.

There is a generalization of Theorem 3.1 pointed out to me by A. Kerber, and proved by him in a different way than that given below. Let $p_\lambda(x)$ denote the power-sum symmetric function [12, pp. 15–16], defined by

$$p_m(x) = \sum_i x_i^m, \quad p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_l}(x),$$

where $l = l(\lambda)$. If $w \in S_m$, the symmetric group on m letters, then $\rho(w)$ denotes the partition whose parts are equal to the cycle lengths of w . The irreducible (ordinary) characters χ^λ of S_m are indexed by partitions λ of m . We then have the famous formula of Frobenius (see [12, Chapter I.7])

$$s_\lambda(x) = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) p_{\rho(w)}(x). \tag{2}$$

Thus

$$s_\lambda(q, q^2, \dots, q^n) = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) \prod_i (q^i + q^{2i} + \cdots + q^{ni})^{c_i(w)}, \tag{3}$$

where w has $c_i(w)$ cycles of length i .

Now if H is a subgroup of S_n and χ a character of H , then define [7, 5.1.27] the *generalized cycle index* of H with respect to χ by

$$\text{Cyc}(H, \chi) = \frac{1}{|H|} \sum_{w \in H} \chi(w^{-1}) \prod_i x_i^{c_i(w)}.$$

If $\chi=1$ is the trivial character, we write $\text{Cyc}(H)$ for $\text{Cyc}(H, 1)$. Equation (3) becomes

$$s_\lambda(q, q^2, \dots, q^n) = \text{Cyc}(S_m, \chi^\lambda)(x_i \rightarrow q^i + q^{2i} + \dots + q^{ni}), \quad (4)$$

where the notation indicates that we substitute $q^i + q^{2i} + \dots + q^{ni}$ for x_i in $\text{Cyc}(S_m, \chi^\lambda)$.

We may generalize (4) as follows. Let

$$f(q) = a_0 + a_1 q + \dots + a_r q^r$$

be any polynomial with symmetric, unimodal, nonnegative integer coefficients. Let $n = f(1)$, and let $\phi: \mathfrak{sl}(2) \rightarrow \mathfrak{gl}(n)$ be the representation with character $\text{ch } \phi = q^{-r} f(q^2)$. Let $\psi_\lambda: \mathfrak{gl}(n) \rightarrow \mathfrak{gl}(N)$ be as above, where $l(\lambda) \leq n$ and $|\lambda| = m$. Then we obtain in the same way as (1) and (4) that

$$\text{ch}(\psi_\lambda \phi) = q^{-mr} s_\lambda \left(\underbrace{1, 1, \dots, 1}_{a_0}, \underbrace{q^2, q^2, \dots, q^2}_{a_1}, \dots, \underbrace{q^{2r}, \dots, q^{2r}}_{a_r} \right),$$

so from (2) we get that

$$\text{Cyc}(S_m, \chi^\lambda)(x_i \rightarrow f(q^i)) \quad (5)$$

is symmetric (about $\frac{1}{2}mr$) and unimodal. The substitution $x_i \rightarrow f(q^i)$ is sometimes called the *Polya composition* with $f(q)$, denoted

$$\text{Cyc}(S_m, \chi^\lambda)[f(q)].$$

Since the center of symmetry of the polynomial (5) is at $\frac{1}{2}mr$ (independent of λ), it follows that any nonnegative linear combination of polynomials (5), where m and f are fixed, will also be symmetric (about $\frac{1}{2}mr$) and unimodal. Hence we may replace χ^λ in (5) with any ordinary character χ of S_m . Thus we have proved:

THEOREM 3.2. *Let χ be an ordinary character of S_m , and let $f(q)$ be a polynomial with nonnegative integral unimodal coefficients, satisfying $q^r f(1/q) = f(q)$. Let*

$$g(q) = \text{Cyc}(S_m, \chi)[f(q)]. \quad (6)$$

Then $g(q)$ has nonnegative integral unimodal coefficients, and $q^{mr} g(1/q) = g(q)$.

In particular, it is easily seen (e.g., by Frobenius reciprocity) that for any subgroup G of S_m and any ordinary character χ of G we have

$$\text{Cyc}(G, \chi) = \text{Cyc}(S_m, \text{ind}_G^{S_m} \chi),$$

where $\text{ind}_G^{S_m} \chi$ denotes the induction of χ to S_m . There follows the result of Kerber:

COROLLARY 3.3. *Let G be a subgroup of S_m , χ an ordinary character of G , and $f(q)$ as in Theorem 3.2. Then*

$$\text{Cyc}(G, \chi)[f(q)]$$

satisfies the conclusions to Theorem 3.2.

Let us note that if $f(q)$ satisfies all the conditions of Theorem 3.2 except integrality, then $g(q)$ [as defined by (6)] need not have unimodal coefficients. For instance,

$$\text{Cyc}(S_2)\left[\frac{1}{2} + \frac{1}{2}q\right] = \frac{1}{8}(3 + 2q + 3q^2).$$

4. The superalgebra $\text{osp}(1, 2)$

A Lie superalgebra is a vector space \mathfrak{G} (which we will always take over \mathbb{C}), together with two subspaces \mathfrak{G}_0 and \mathfrak{G}_1 for which $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$, and a binary operation $[A, B]$ satisfying certain axioms. Rather than give the precise definition here, we will be content with defining examples of concern to us. Our basic reference is [8] and the useful summary [9]. All the results stated below without proof can essentially be found in these references.

Let V be a complex vector space of dimension $m + n$, and let V_0 and V_1 be subspaces satisfying $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. The Lie superalgebra $\text{gl}(m/n)$ is defined as follows. As a vector space it is given by $\text{End } V$, the set of linear transformations $A: V \rightarrow V$. For $i = 0, 1$, define

$$\text{End}_i V = \{A \in \text{End } V : AV_j \subseteq V_{i+j}\},$$

where the subscript $i + j$ is taken modulo 2. Thus

$$\text{End } V = \text{End}_0 V \oplus \text{End}_1 V.$$

Define a binary operation $[A, B]$ on $\text{End } V$ by

$$[A, B] = AB - (-1)^{ij}BA,$$

where $A \in \text{End}_i V$, $B \in \text{End}_j V$, and extending to all of $\text{End } V$ by bilinearity

Choose an ordered basis for V whose first m elements form a basis for V_0 and last n for V_1 . Then $\text{End } V$ can be identified with the space of all $(m+n) \times (m+n)$ complex matrices

$$A = \begin{matrix} & \overbrace{\quad}^m & \overbrace{\quad}^n & \\ m \left\{ \begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right. & & & \\ & n \left\{ \begin{array}{c|c} A_3 & A_4 \end{array} \right. & & \end{matrix},$$

where $\text{End}_0 V$ consists of those matrices with $A_2 = 0$ and $A_3 = 0$, and $\text{End}_1 V$ of those with $A_1 = 0$ and $A_4 = 0$.

Any subspace \mathfrak{G} of $\mathfrak{gl}(m/n)$ satisfying $\mathfrak{G} = [\mathfrak{G} \cap \mathfrak{gl}(m/n)_0] \oplus [\mathfrak{G} \cap \mathfrak{gl}(m/n)_1]$ and closed under the operation $[A, B]$ is itself a Lie superalgebra, with $\mathfrak{G}_i = \mathfrak{G} \cap \mathfrak{gl}(m/n)_i$. In particular, define the *orthosymplectic* Lie superalgebra $\text{osp}(1,2) \subset \mathfrak{gl}(1/2)$ [sometimes denoted $\text{osp}(2,4)$ or $B(0,1)$] to be the set of all 3×3 complex matrices of the form

$$\left[\begin{array}{c|cc} 0 & \alpha & \beta \\ \hline \beta & \gamma & \delta \\ -\alpha & \varepsilon & -\gamma \end{array} \right]$$

Thus $\dim \text{osp}(1,2) = 5$.

A (finite-dimensional) *representation* of a Lie superalgebra $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ is a linear transformation $\phi: \mathfrak{G} \rightarrow \mathfrak{gl}(m/n) = \text{End}_0 V \oplus \text{End}_1 V$, such that $\phi \mathfrak{G}_i \subseteq \text{End}_i V$ and $\phi[A, B] = [\phi A, \phi B]$. We write $A \cdot v$ for $(\phi A)(v)$ and think of $A \in \mathfrak{G}$ as acting on V . Two representations

$$\phi: \mathfrak{G} \rightarrow \text{End}_0 V \oplus \text{End}_1 V,$$

$$\psi: \mathfrak{G} \rightarrow \text{End}_0 W \oplus \text{End}_1 W$$

are *equivalent* if there is an isomorphism $\sigma: V \rightarrow W$ and a bijection $\pi: \{0,1\} \rightarrow \{0,1\}$ such that $\sigma V_i = W_{\pi i}$ ($i=0,1$) and $\sigma(\phi A) = (\psi A)\sigma$ for all $A \in \mathfrak{G}$. A representation $\phi: \mathfrak{G} \rightarrow \text{End}_0 V \oplus \text{End}_1 V$ is *irreducible* if V has no proper \mathfrak{G} -invariant subspace $W = W_0 \oplus W_1$ where $W_i = W \cap V_i$.

THEOREM 4.1.

(a) Every representation $\phi: \text{osp}(1,2) \rightarrow \text{End } V$ is completely reducible, i.e., we can write V as a direct sum of irreducible invariant subspaces.

(b) For each integer $n \geq 0$ there is an irreducible representation $\phi_n: \text{osp}(1,2) \rightarrow \mathfrak{gl}(n+1/n)$, and this accounts for all inequivalent (finite-dimensional) irreducible representations of $\text{osp}(1,2)$.

Note: Unlike the situation for the Lie algebra $\mathfrak{gl}(n)$, not every finite-dimensional representation of $\mathfrak{gl}(m/n)$ is completely reducible.

Let $h = \text{diag}(0, 1, -1) \in \text{osp}(1,2)$. If $\phi: \text{osp}(1,2) \rightarrow \mathfrak{gl}(m/n)$ is a representation, then ϕh is similar to a diagonal matrix with integer eigenvalues a_1, \dots, a_{m+n} . We

then define the *character* of ϕ by

$$\text{ch } \phi = q^{a_1} + \cdots + q^{a_{m+n}}.$$

THEOREM 4.2. Let $\phi_n: \text{osp}(1, 2) \rightarrow \text{gl}(n+1/n)$ be the irreducible representation of Theorem 4.1(b). Then

$$\text{ch } \phi_n = q^{-n} + q^{-n+1} + \cdots + q^n.$$

In fact, writing $\text{gl}(n+1/n) = \text{End}(V_0 \oplus V_1)$ where $\dim V_0 = n+1$ and $\dim V_1 = n$, then $\phi_n h$ restricted to V_0 has eigenvalues $-n, -n+2, \dots, n$, while $\phi_n h$ restricted to V_1 has eigenvalues $-n+1, -n+3, \dots, n-1$.

COROLLARY 4.3. Let $\phi: \text{osp}(1, 2) \rightarrow \text{gl}(m/n)$ be any representation. Then the Laurent polynomial

$$\text{ch } \phi = \sum_{i=-N}^N b_i q^i.$$

is unimodal and symmetric about 0 (i.e., $b_i = b_{-i}$).

Proof: By Theorems 4.1 and 4.2, $\text{ch } \phi$ is a nonnegative integer linear combination of the Laurent polynomials $\text{ch } \phi_n = q^{-n} + q^{-n+1} + \cdots + q^n$, and the proof follows. \square

5. Super-Schur functions

We now turn to the superalgebra analogue of Schur functions. Let $\Gamma(m, n)$ be the set of all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \leq n$ if $i \geq m+1$. Thus the Young diagram of λ lies inside a hook of arm height m and leg width n . Then λ indexes a certain irreducible representation $\psi_\lambda: \text{gl}(m/n) \rightarrow \text{gl}(M/N)$, as described, e.g., in [3].

Suppose $A \in \text{End}_0 V$, where $\text{gl}(m/n) = \text{End } V$. Let $\alpha_1, \dots, \alpha_m$ be the eigenvalues of A restricted to V_0 , and β_1, \dots, β_n the eigenvalues of A restricted to V_1 . Then the eigenvalues of $\psi_\lambda(A)$ have the form $a_1 \alpha_1 + \cdots + a_m \alpha_m + b_1 \beta_1 + \cdots + b_n \beta_n$, where the a_i 's and b_j 's are nonnegative integers independent of the α_i 's and β_j 's. Moreover, $\sum a_i + \sum b_j = |\lambda|$. Define the *character* $\text{ch } \psi_\lambda$ of ψ_λ to be the polynomial

$$\text{ch } \psi_\lambda = \sum x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}, \quad (7)$$

the sum being over all eigenvalues $\sum a_i \alpha_i + \sum b_j \beta_j$ of $\psi_\lambda(A)$.

While the evaluation of the characters $\text{ch } \psi_\lambda$ is included in the general theory of Kač, a combinatorial description appears in [2], [4], and most explicitly in [3]. We will use the notation $s_\lambda(x/y)$ for these characters and call them *super-Schur functions*; Berele and Regev denote them by $\text{HS}_\lambda(x; y)$ and call them "hook Schur functions." They are polynomials which are symmetric in the x_i 's and y_j 's separately, and have the additional "cancellation property"

$$s_\lambda(x/y)|_{x_i = -y_i} = s_\lambda(x/y)|_{x_i = y_i = 0}. \quad (8)$$

Such polynomials are essentially the "bisymmetric functions" of [13, §5]. [More precisely, $s_\lambda(x/-y)$ is a bisymmetric function.]

The super-Schur functions are given by the formula [4, (9); 3, Definition 6.3]

$$s_\lambda(x/y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda/\mu'}(y). \quad (9)$$

Here $s_{\lambda/\mu'}$ denotes a skew Schur function [12, Chapter I.5] and $'$ denotes the conjugate partition. In the terminology of D. E. Littlewood [10; 11, Chapter 6.4], $s_\lambda(x/-y)$ is a Schur function of the series

$$\frac{\prod_{i=1}^m (1-x_i)}{\prod_{j=1}^n (1-y_j)}.$$

In the notation of λ -rings [12, pp. 26-27], the polynomial $s_\lambda(x/-y)$ corresponds to the operation $S^\lambda(X-Y)$, where $X = x_1 + \dots + x_m$ and $Y = y_1 + \dots + y_n$. If ω_y denotes the automorphism of the ring of symmetric functions in the variable $y = (y_1, y_2, \dots)$ as described in [12, pp. 14-17] (regard ω_y as commuting with the x_i 's), then

$$s_\lambda(x/y) = \omega_y s_\lambda(x, y), \quad (10)$$

where $s_\lambda(x, y)$ denotes the Schur function s_λ in the two sets of variables x and y . Since ω_y is an algebra automorphism preserving the standard scalar product [12, Chapter I.4] on symmetric functions, almost every formula involving Schur functions has a "superanalogue" obtained by applying ω_y .

The formula (9), together with the well-known combinatorial definition of skew Schur functions [12, p. 42; 15, §12], gives a combinatorial definition of $s_\lambda(x/y)$. Namely, set $x_1 < \dots < x_m < y_1 < \dots < y_n$, and fill the Young diagram of shape λ with x_i 's and y_j 's such that:

- The entries weakly increase in every row and column.
- The x_i 's strictly increase in columns.
- The y_j 's strictly increase in rows.

Let $p(T)$ be the product of the entries of the resulting tableau T . Then

$$s_\lambda(x/y) = \sum_T p(T), \quad (11)$$

summed over all tableaux of shape λ satisfying (a)-(c).

Example: To compute $s_{21}(x_1, x_2/y_1)$, we have

$x_1 x_1$	$x_1 x_2$	$x_1 x_1$	$x_1 x_2$	$x_2 x_2$
x_2	x_2	y_1	y_1	y_1
	$x_1 y_1$	$x_1 y_1$	$x_2 y_1$	
	x_2	y_1	y_1	

$$\text{so } s_{21}(x_1, x_2/y_1) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 y_1 + 2x_1 x_2 y_1 + x_2^2 y_1 + x_1 y_1^2 + x_2 y_1^2.$$

It is perhaps of interest to note that we may choose *any* ordering of the x_i 's and y_j 's in the combinatorial description of $s_\lambda(x/y)$. More precisely:

THEOREM 5.1. Fix an arbitrary linear ordering of the set $\{x_1, \dots, x_m, y_1, \dots, y_n\}$. Fill the Young diagram of shape λ with x_i 's and y_j 's such that:

- (i) The entries weakly increase in every row and column.
- (ii) Any x_i appears at most once in each column.
- (iii) Any y_j appears at most once in each row.

Then $s_\lambda(x/y) = \sum_T p(T)$, summed over all tableaux of shape λ satisfying (i)–(iii), where $p(T)$ is the product of the entries of T .

Proof: Let h_i and e_i denote the complete homogeneous and elementary symmetric functions, respectively, as defined in [12, Chapter I.2]. Thus $h_i = s_i$ and $e_i = s_{1^i}$. By (11) and the Littlewood-Richardson rule [12, Chapter I.9] (or by applying the automorphism ω_y to the scalar product $\langle s_\lambda, h_\mu \rangle$), the coefficient of $x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}$ in $s_\lambda(x/y)$ is equal to the coefficient of s_λ when the product $h_{a_1} \cdots h_{a_m} e_{b_1} \cdots e_{b_n}$ is expanded as a linear combination of Schur functions. But the factors of this product can be written in any order, and by the Littlewood-Richardson rule this yields the desired result. \square

There is an alternative combinatorial interpretation of $s_\lambda(x/y)$ which makes the cancellation property (8) obvious. We merely state the result without proof; it is not difficult to deduce it from Theorem 5.1 by letting $m = n = \infty$ and choosing the ordering $x_1 < y_1 < x_2 < y_2 < \cdots$.

THEOREM 5.2. Fill in the Young diagram of shape λ with positive integers such that:

- (i) The entries weakly increase in every row and column.
- (ii) The entries strictly increase along any diagonal running from the upper left to lower right. (Equivalently, no 2×2 square has all its entries equal.)

Let T be the resulting tableau. Let m_i denote the number of entries of T equal to i , r_i the number of rows of T which contain an i , and c_i the number of columns of T which contain an i . Define

$$q(T) = \prod_{i \geq 1} x_i^{m_i - r_i} y_i^{m_i - c_i} (x_i + y_i)^{r_i + c_i - m_i}.$$

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Then

$$s_\lambda(x/y) = \sum_T q(T),$$

where T ranges over all tableaux satisfying (i) and (ii).

6. Application to unimodality

Consider the composite representation

$$\text{osp}(1, 2) \xrightarrow{\phi_n} \text{gl}(n+1/n) \xrightarrow{\psi_\lambda} \text{gl}(M/N)$$

of $\mathfrak{osp}(1,2)$. Let $h = \text{diag}(0, 1, -1) \in \mathfrak{osp}(1,2)$ as in Section 4. Write $\mathfrak{gl}(n+1/n) = \text{End}(V_0 \oplus V_1)$. By Theorem 4.2, the eigenvalues of $\phi_n(h)$ restricted to V_0 are $-n, -n+2, \dots, n$, while those restricted to V_1 are $-n+1, -n+3, \dots, n-1$. Thus if

$$s_\lambda(x/y) = \sum x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} y_1^{b_1} \cdots y_n^{b_n},$$

then by the definition (7) of $s_\lambda(x/y)$ there follows

$$\begin{aligned} \text{ch } \psi_\lambda \phi_n &= \sum q^{-na_1 + (-n+2)a_2 + \cdots + na_{n-1} + (-n+1)b_1 + (-n+3)b_2 + \cdots + (n-1)b_n} \\ &= q^{-n|\lambda|} \sum q^{2a_2 + 4a_3 + \cdots + 2na_{n+1} + b_1 + 3b_2 + \cdots + (2n-1)b_n} \\ &= q^{-n|\lambda|} s_\lambda(1, q^2, \dots, q^{2n}/q, q^3, \dots, q^{2n-1}). \end{aligned}$$

Hence from Corollary 4.3, we conclude:

THEOREM 6.1. *Let $\lambda \in \Gamma(n+1, n)$. Then the polynomial $s_\lambda(1, q^2, \dots, q^{2n}/q, q^3, \dots, q^{2n-1})$ is symmetric about $n|\lambda|$ and is unimodal.*

Remark: By (9) we have

$$\begin{aligned} s_\lambda(1, q^2, \dots, q^{2n}/q, q^3, \dots, q^{2n-1}) &= \sum_{\mu \subset \lambda} s_\mu(1, q^2, \dots, q^{2n}) s_{\lambda/\mu}(q, q^3, \dots, q^{2n-1}) \\ &= \sum_{\mu \subset \lambda} f_\mu(q), \text{ say.} \end{aligned}$$

It is easily seen that $f_\mu(q) = q^{2n|\lambda|} f_\mu(1/q)$ and that $f_\mu(q)$ is even or odd depending on whether $|\lambda/\mu| = |\lambda| - |\mu|$ is even or odd. Moreover, it follows from Theorem 3.1 and the nonnegativity of the integers $c_{\mu\nu}^\lambda$ in the expansion $s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu$ [12, Chapter 1.9] that the coefficients of the even-degree (respectively, odd-degree) terms of $f_\mu(q)$ for $|\lambda/\mu|$ even (respectively, odd) are unimodal. Hence it is a consequence of the representation theory of $\mathfrak{sl}(2)$ and $\mathfrak{gl}(n)$ alone that the two polynomials

$$\sum_{\substack{\mu \subset \lambda \\ |\lambda/\mu| \text{ even}}} f_\mu(q) \quad \text{and} \quad \sum_{\substack{\mu \subset \lambda \\ |\lambda/\mu| \text{ odd}}} f_\mu(q), \quad (12)$$

respectively, have unimodal coefficients of their even-degree (respectively, odd-degree) terms; and both are symmetric about the same point (viz., $n|\lambda|$). The superalgebra structure has the effect of "unifying" (12) into a single polynomial with unimodal coefficients. This is analogous to (but far less profound than) the way in which superalgebra theory unifies particles with half-integer spin (fermions) and integer spin (bosons).

Example: Suppose λ consists of a single row of length l . Then

$$\begin{aligned} s_i(1, q^2, \dots, q^{2n}/q, q^3, \dots, q^{2n-1}) &= \sum_{i=0}^l s_i(1, q^2, \dots, q^{2n}) s_{l-i}(q, q^3, \dots, q^{2n-1}) \\ &= \sum_{i=0}^l q^{(l-i)^2} \begin{bmatrix} n+i \\ i \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l-i \end{bmatrix}_{q^2}, \end{aligned}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}_{q^2}$ denotes the q -binomial coefficient in the variable q^2 . If we denote the above expression by $P_{ln}(q)$, then

$$\sum_{i \geq 0} P_{ln}(q) t^i = \frac{(1+qt)(1+q^3t) \cdots (1+q^{2n-1}t)}{(1-t)(1-q^2t) \cdots (1-q^{2n}t)}.$$

Even though $P_{ln}(q)$ has a simple, elementary definition, we don't know how to prove its coefficients are unimodal without using superalgebras. If instead we took λ to consist of a single column of length l , then we would obtain the unimodality of the polynomials $P'_{ln}(q)$ defined by

$$\sum_{i \geq 0} P'_{ln}(q) t^i = \frac{(1+t)(1+q^2t) \cdots (1+q^{2n}t)}{(1-qt)(1-q^3t) \cdots (1-q^{2n-1}t)}.$$

Example: Suppose $\lambda = (\lambda_1, \lambda_2, \dots)$ where $\lambda_{n+1} \geq n$. The representation ψ_λ of $\mathfrak{gl}(n+1/n)$ is then called *typical*, and by [3, Theorem 6.20] or [9, §2.4] we have

$$s_\lambda(x_1, \dots, x_{n+1}/y_1, \dots, y_n) = s_\alpha(x) s_{\beta'}(y) \prod_{i=1}^{n+1} \prod_{j=1}^n (x_i + y_j),$$

where $\alpha = (\lambda_1 - n, \lambda_2 - n, \dots, \lambda_{n+1} - n)$ and $\beta' = (\lambda_{n+2}, \lambda_{n+3}, \dots)$. In particular, if $\lambda = (n, n, \dots, n)$ ($n+1$ times), then

$$s_\lambda(x_1, \dots, x_{n+1}/y_1, \dots, y_n) = \prod (x_i + y_j),$$

a result essentially due to Littlewood [10, Theorem XVIII; 11, Theorem XVIII, p. 115]. Thus by Theorem 6.1 the polynomial

$$\prod_{i=0}^n \prod_{j=1}^n (q^{2i} + q^{2j-1}) = q^{i^n(n+1)(4n-1)} \prod_{i=1}^n (1 + q^{2i-1})^{2(n-i+1)}$$

is unimodal.

7. A cycle-index generalization

Theorem 6.1 can be generalized in the same way as Theorem 3.1 was extended to Theorem 3.2.

THEOREM 7.1. *Let χ be an ordinary character of the symmetric group S_m , and let $f(q)$ be a polynomial with nonnegative integral unimodal coefficients, satisfying $q^{2r}f(1/q) = f(q)$ for some integer $r \geq 0$. Let*

$$g(q) = \text{Cyc}(S_m, \chi)(x_i \rightarrow f((-1)^{i-1}q^i)).$$

Then $g(q)$ has nonnegative integral unimodal coefficients, and $q^{2mr}g(1/q) = g(q)$.

Proof: Let $\phi: \text{osp}(1,2) \rightarrow \text{gl}(k/n)$ have character $q^{-r}f(q)$. (If c denotes the middle coefficient of $f(q)$, then $k = \frac{1}{2}[f(1)+c]$ and $n = \frac{1}{2}[f(1)-c]$.) Let $\chi = \sum c_\lambda \chi^\lambda$ be the decomposition of χ into irreducibles. Let $\psi: \text{gl}(k/n) \rightarrow \text{gl}(M/N)$ be given by $\psi = \sum c_\lambda \psi_\lambda$, so $\text{ch } \psi = \sum c_\lambda s_\lambda(x/y)$. Suppose $f(q) = a_0 + a_1q + \dots + a_{2r}q^{2r}$. Then

$\text{ch } \psi \phi$

$$= \sum c_\lambda s_\lambda \left(\underbrace{1, \dots, 1}_{a_0}, \underbrace{q^2, \dots, q^2}_{a_2}, \dots, \underbrace{q^{2r}, \dots, q^{2r}}_{a_{2r}} / \underbrace{q, \dots, q}_{a_1}, \dots, \underbrace{q^{2r-1}, \dots, q^{2r-1}}_{a_{2r-1}} \right). \quad (13)$$

Now it follows from (2), (10), and the fact that $\omega_y p_i(y) = (-1)^{i-1} p_i(y)$ [12, p. 16] that

$$s_\lambda(x/y) = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) \prod_i [p_i(x) - (-1)^i p_i(y)]^{c_i(w)},$$

where $|\lambda| = m$ and w has $c_i(w)$ cycles of length i . Hence

$$\begin{aligned} \text{ch } \psi \phi &= \frac{1}{m!} \sum_\lambda c_\lambda \sum_{w \in S_m} \chi^\lambda(w) \prod_i [a_0 + a_1 q^{2i} + \dots + a_{2r} q^{2ri} \\ &\quad - (-1)^i (a_1 q^i + \dots + a_{2r-1} q^{(2r-1)i})]^{c_i(w)} \\ &= \frac{1}{m!} \sum_{w \in S_m} \chi(w) \prod_i f((-1)^{i-1} q^i)^{c_i(w)} \\ &= \text{Cyc}(S_m, \chi)(x_i \rightarrow f((-1)^{i-1} q^i)). \end{aligned}$$

Thus by Theorem 6.1, $g(q)$ has nonnegative integral unimodal coefficients. Each term in the sum on the right-hand side of (13) is easily seen to be symmetric about mr , so the same is true of $\text{ch } \psi\phi$. \square

COROLLARY 7.2. *Let G be a subgroup of S_m , χ an ordinary character of G , and $f(q)$ as in Theorem 7.1. Then the polynomial*

$$\text{Cyc}(G, \chi)(x_i \rightarrow f((-1)^{i-1} q^i))$$

has nonnegative integral unimodal coefficients and is symmetric about mr .

Proof: Exactly the same as the deduction of Corollary 3.3 from Theorem 3.2. \square

As in Section 3, we note that if $f(q)$ is not required to have integral coefficients, then $g(q)$ need not be unimodal. For instance, let $f(q) = \frac{1}{2} + q + \frac{1}{2}q^2$. Then

$$\text{Cyc}(S_2)(x_i \rightarrow f((-1)^{i-1} q^i)) = \frac{1}{8}(3 + 4q + 2q^2 + 4q^3 + 3q^4).$$

8. The Sperner property

The Lie algebra $\mathfrak{sl}(2)$ has been used [14] to show that certain posets have the Sperner property. We will briefly review these results here and indicate their analogues for $\mathfrak{osp}(1, 2)$. We follow [14] in notation and terminology.

A *ranked poset* P of length r is a partially ordered P together with a partition $P = \bigcup_{i=0}^r P_i$ into $r+1$ nonvoid ranks P_i , $0 \leq i \leq r$, such that elements in P_i cover only elements in P_{i-1} . Assuming P is connected; then the ranking of P , if it exists, is unique. We will assume P is finite, and we set $p_i = |P_i|$. A ranked poset P is *strongly Sperner* if for every $k \geq 1$ no union of k antichains of P contains more elements than does the union of the k largest ranks of P . In particular ($k=1$), a strongly Sperner poset is *Sperner*, i.e., no antichain of P contains more than $\max p_i$ elements. A ranked poset of length r is *rank-symmetric* if $p_i = p_{r-i}$, for $0 \leq i \leq r$. It is *rank-unimodal* if $p_0 \leq p_1 \leq \dots \leq p_k \geq p_{k+1} \geq \dots \geq p_r$ for some $0 \leq k \leq r$. Finally, a ranked poset is *Peck* if it is rank-symmetric, rank-unimodal, and strongly Sperner.

If $P = \bigcup_{i=0}^r P_i$ is any ranked poset, define a graded complex vector space

$$\tilde{P} = \tilde{P}_0 \oplus \tilde{P}_1 \oplus \dots \oplus \tilde{P}_r,$$

where \tilde{P}_i is the complex vector space with basis P_i . A linear operator X on \tilde{P} is a *lowering operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i-1}$, and a *raising operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i+1}$. A raising operator X is an *order-raising operator* if for all $a \in P_i$ we have

$$Xa = \sum_b \theta(a, b)b,$$

where $\theta(a, b) = 0$ unless b covers a in P . For any ranked poset P of length r , define a linear operator H on \bar{P} by

$$Ha = (2i - r)a, \quad a \in P_i.$$

We now say that a ranked poset P carries a representation of $\mathfrak{sl}(2)$ if there exists a lowering operator Y and an order-raising operator X on \bar{P} such that $XY - YX = H$. This is equivalent to the statement that if x, h, y are the matrices spanning $\mathfrak{sl}(2)$ defined in Section 2, then the linear transformation $\phi: \mathfrak{sl}(2) \rightarrow \mathfrak{gl}(\bar{P})$ defined by $\phi(x) = X, \phi(h) = H, \phi(y) = Y$ is a homomorphism of Lie algebras.

We now state (as a single theorem) the results of [17, Lemma 1.1] and [14, Theorem 1].

THEOREM 8.1. *Let P be a ranked poset of length r . The following three conditions are equivalent:*

- (i) P is Peck.
- (ii) There exists an order-raising operator X on \bar{P} such that

$$X^{r-2i}|_{\bar{P}_i}: \bar{P}_i \rightarrow \bar{P}_{i-1}$$

is an isomorphism of vector spaces for every $0 \leq i < r/2$.

- (iii) P carries a representation of $\mathfrak{sl}(2)$.

Moreover, an order-raising operator X satisfies (ii) if and only if P carries a representation of $\mathfrak{sl}(2)$ whose order-raising operator is X .

We wish to give a "super-analogue" of Theorem 8.1. Suppose P is ranked of even length $2r$. Write $\bar{P} = \bar{P}^0 \oplus \bar{P}^1$, where $\bar{P}^0 = \bar{P}_0 \oplus \bar{P}_2 \oplus \cdots \oplus \bar{P}_{2r}$ and $\bar{P}^1 = \bar{P}_1 \oplus \bar{P}_3 \oplus \cdots \oplus \bar{P}_{2r-1}$. Define the following three elements of $\mathfrak{osp}(1, 2)$:

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \quad (14)$$

Then $\mathfrak{osp}(1, 2)$ has a vector space basis consisting of h, x, y , and

$$x^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad y^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The superalgebra structure on $\mathfrak{osp}(1, 2)$ is defined by the relations

$$[h, x] = hx - xh = x,$$

$$[h, y] = hy - yh = -y,$$

$$[x, y] = xy + yx = h,$$

$$[x, x] = 2x^2, \quad [y, y] = 2y^2.$$

DEFINITION 8.2. Let P be a ranked poset of length $2r$. Define a linear operator H on \tilde{P} by

$$Ha = (i-r)a, \quad a \in P_i. \quad (15)$$

Then P is said to carry a representation of $\mathfrak{osp}(1,2)$ if there exists a lowering operator Y and an order-raising operator X such that $XY + YX = H$. Equivalently, the linear transformation

$$\phi: \mathfrak{osp}(1,2) \rightarrow \mathfrak{gl}(\tilde{P}^0/\tilde{P}^1)$$

defined by $\phi(h) = H$, $\phi(x) = X$, $\phi(y) = Y$, $\phi(x^2) = X^2$, $\phi(y^2) = Y^2$ is a homomorphism of Lie superalgebras.

THEOREM 8.3. Let P be a ranked poset of length $2r$. Then P carries a representation of $\mathfrak{osp}(1,2)$ if and only if P is Peck.

Proof: Assume P carries a representation of $\mathfrak{osp}(1,2)$ with order-raising operator X . While a proof that P is Peck could be given along the lines of [14, Theorem 1], it is easier to appeal to Theorem 8.1. One easily checks that h, x^2, y^2 span a sub-superalgebra of $\mathfrak{osp}(1,2)$ isomorphic to the Lie algebra $\mathfrak{sl}(2)$. Hence the posets

$$P^0 = P_0 \oplus P_2 \oplus \cdots \oplus P_{2r}$$

and

$$P^1 = P_1 \oplus P_3 \oplus \cdots \oplus P_{2r-1}$$

each carry representations of $\mathfrak{sl}(2)$ with order-raising operator X^2 . By Theorem 8.1, the linear transformations

$$(X^2)^{r-2i} \Big|_{\tilde{P}_{2i}}: \tilde{P}_{2i} \rightarrow \tilde{P}_{2(r-i)}, \quad 0 \leq i \leq \frac{r}{2},$$

$$(X^2)^{r-1-2i} \Big|_{\tilde{P}_{2i+1}}: \tilde{P}_{2i+1} \rightarrow \tilde{P}_{2(r-i)-1}, \quad 0 \leq i < \frac{r-1}{2},$$

are isomorphisms. But then for any $0 \leq i < r$, the linear transformations

$$X^{2r-2i} \Big|_{\tilde{P}_i}: \tilde{P}_i \rightarrow \tilde{P}_{2r-i}$$

are isomorphisms. By Theorem 8.1, P is Peck.

The converse is proved entirely analogously to the corresponding result for $\mathfrak{sl}(2)$ [14, p. 277]; the details are omitted. \square

It follows from Theorems 8.1 and 8.3 that if P carries a representation of $\text{osp}(1,2)$, then it also carries a representation of $\text{sl}(2)$. However, in certain cases there may be a "natural" way to define the $\text{osp}(1,2)$ representation while a direct construction of the $\text{sl}(2)$ representation appears intractable. The next result gives such an example; it describes a new class of Peck posets (or even of posets with the Sperner property), and may be regarded as the "super-analogue" of the fact [17, Section 4] that certain posets $L(m, n)$ are Peck.

THEOREM 8.4. *Let k and r be positive integers, and define $K(k, 2r)$ (respectively, $\bar{K}(k, 2r)$) to be the set of all Young diagrams Y contained in a $k \times 2r$ rectangle, such that no two rows of Y have the same odd (respectively, even) length (including, in the case of \bar{K} , no two rows of zero length, where Y is regarded as having exactly k rows). Partially order $K(k, 2r)$ and $\bar{K}(k, 2r)$ by inclusion of Young diagrams. Define the rank of a Young diagram Y to be its number of squares, so that $K(k, 2r)$ and $\bar{K}(k, 2r)$ become ranked posets. Then $K(k, 2r)$ and $\bar{K}(k, 2r)$ are Peck posets.*

Proof: First consider the case $K(k, 2r)$. We index the rows and columns of a matrix A in $\text{gl}(r+1/r)$ by the numbers $0, 2, 4, \dots, 2r, 1, 3, \dots, 2r-1$, in the order given. Let $E_{ij} \in \text{gl}(r+1/r)$ denote the matrix with a 1 in position (i, j) and 0's elsewhere. Define a linear transformation $\psi_r: \text{osp}(1,2) \rightarrow \text{gl}(r+1/r)$ as follows, where h, x, y are given by (14):

$$\begin{aligned}\psi_r(h) &= rE_{00} + (r-1)E_{11} + \dots - rE_{2r,2r}, \\ \psi_r(x) &= E_{01} + E_{12} + \dots + E_{2r-1,r}, \\ \psi_r(y) &= rE_{10} + (r-1)E_{32} + (r-2)E_{54} + \dots + E_{2r-1,2r-2} \\ &\quad - E_{21} - 2E_{43} - 3E_{64} - \dots - rE_{2r,2r-1}, \\ \psi_r(x^2) &= \psi_r(x)^2, \quad \psi_r(y^2) = \psi_r(y)^2.\end{aligned}\tag{16}$$

One easily checks by direct computation that ψ_r is a homomorphism of Lie superalgebras. [In fact, ψ_r is the irreducible representation ϕ_r of $\text{osp}(1,2)$, but it is irrelevant here that ψ_r is irreducible.]

The representation ψ_r defines an action of $\text{osp}(1,2)$ on a vector space $V_0 \oplus V_1$, where $\dim V_0 = r+1$, $\dim V_1 = r$. Define the k th supersymmetric power $\hat{S}^k(V_0 \oplus V_1)$ of the pair (V_0, V_1) to be the k th tensor power $T^k(V_0 \oplus V_1)$ modulo the subspace generated by all relations

$$v \otimes w - (-1)^{|v||w|} w \otimes v,\tag{17}$$

where $v \in V_i$, $w \in V_j$. [We may identify $\hat{S}^k(V_0 \oplus V_1)$ with the space

$$\coprod_{j=0}^k S^j(V_0) \otimes \Lambda^{k-j}(V_1),$$

where S^i and Λ^i denote the i th symmetric and exterior power, respectively.] The action ψ_r of $\mathfrak{osp}(1,2)$ on $V_0 \oplus V_1$ induces an action of $\mathfrak{osp}(1,2)$ on $T^k(V_0 \oplus V_1)$ by

$$A \cdot (v_0 \otimes v_1 \otimes \cdots \otimes v_{2r}) = \sum_{j=0}^{2r} v_0 \otimes v_1 \otimes \cdots \otimes A v_j \otimes \cdots \otimes v_{2r}. \quad (18)$$

The subspace defined by (17) is stable under this action, so $\mathfrak{osp}(1,2)$ acts on $\hat{S}^k(V_0 \oplus V_1)$ by the same rule (18). [We identify $v_0 \otimes \cdots \otimes v_{2r}$ with its image in $\hat{S}^k(V_0 \oplus V_1)$.]

Now let $x_{2r}, x_{2r-2}, \dots, x_0$ and $x_{2r-1}, x_{2r-3}, \dots, x_1$ be the ordered bases of V_0 and V_1 , respectively, which define the matrices $A \in \mathfrak{gl}(r+1/r)$. In other words, $x_{2r}, \dots, x_0, x_{2r-1}, \dots, x_1$ are the unit coordinate vectors for $V_0 \oplus V_1$ in the given order. A basis for $\hat{S}^k(V_0 \oplus V_1)$ consists of all tensors $x = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}$ for which $2r \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 0$ and for which no two odd i_j 's are equal. Hence we may identify x with the Young diagram Y in $K(k, 2r)$ whose rows are of length i_1, i_2, \dots, i_k . Thus we may identify the vector space $\hat{K}(k, 2r)$ with $\hat{S}^k(V_0 \oplus V_1)$.

It is then immediate from (16) and (18) that $\psi_r(x)$ is an order-raising operator, $\psi_r(y)$ is a lowering operator (in fact, an order-lowering operator), and $\psi_r(h)$ acts as in (15). Hence $\hat{K}(k, 2r)$ carries a representation of $\mathfrak{osp}(1,2)$, and the proof follows from Theorem 8.3.

The proof for $\bar{K}(k, 2r)$ is entirely analogous. We replace $\hat{S}^k(V_0 \oplus V_1)$ by the k th superexterior power $\hat{\Lambda}^k(V_0 \oplus V_1)$ defined by taking $T^k(V_0 \oplus V_1)$ modulo the subspace generated by all relations

$$v \otimes w - (-1)^{(r-1)(j-1)} w \otimes v,$$

or equivalently $\hat{\Lambda}^k(V_0 \oplus V_1) = \hat{S}^k(V_1 \oplus V_0)$. The details are omitted. \square

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