The character generator of SU(n)[a)]

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A simple combinatorial method for writing the character generator of SU(n) is described.

1. INTRODUCTION

Generating functions have proved to be a useful tool in the representation theory of continuous and discrete groups.[1] In the case of a compact semisimple Lie group G, the character generator is the starting point for obtaining many other generating functions of interest. The character generator for irreducible representations of a connected simply-connected semisimple Lie group G is defined by

$$X_A(\alpha) = \sum \chi_\lambda(\alpha) A_1^{r_1} \cdots A_l^{r_l},$$

where l is the rank of G, the summation extends over all nonnegative integers $r_1,\dots,r_l$, and $\chi_\lambda(\alpha)$ is the character of the finite irreducible representation of G with highest weight $\lambda = r_1\lambda_1 + \cdots + r_l\lambda_l$. Here $\lambda_1,\dots,\lambda_l$ are the fundamental weights of G. Thus the coefficient of $A_1^{r_1}\cdots A_l^{r_l} \times \alpha_1^{\mu_1}\cdots\alpha_l^{\mu_l}$ (which we abbreviate as $A^r\alpha^\mu$) in $X_A(\alpha)$ is the multiplicity of the weight $\mu = (\mu_1,\dots,\mu_l)$ (written with respect to some basis for the weight space). It follows easily from Weyl's character formula that $X_A(\alpha)$ is a rational function of A and α. For many applications it is desirable to write $X_A(\alpha)$ as a sum of terms of the form

$$A^s\alpha^v \Big/ \prod_{i=1}^{d} (1 - A_j\alpha_{l_1}\alpha_{l_2}\cdots\alpha_{l_h}), \tag{1}$$

where j, h, and $l_1, l_2,\dots,l_h$ depend on i, and where d is the same for all terms and is necessarily equal to $\frac{1}{2}(\dim G + \operatorname{rank} G)$. The method[1] used for computing $X_A(\alpha)$ does not directly yield a sum of terms of the form (1), and it is unknown in general whether $X_A(\alpha)$ can always be written in this form. We will describe a different method for computing $X_A(\alpha)$ when $G = \mathrm{SU}(n)$, which automatically expresses $X_A(\alpha)$ as a sum of terms (1). Each term can be read off by inspection from a certain type of tableau, and we state a formula for the total number of terms. Our derivation will be purely combinatorial, based on the well-known description of the characters of SU(n) in terms of Young tableaux.

2. BASIC CONCEPTS AND FUNDAMENTAL THEOREMS

We now introduce the necessary combinatorial concepts and terminology. Fix integers $m_1 > m_2 > \cdots > m_k > 0$, and set $\mathbf{m} = (m_1,\dots,m_k)$. Let $\mathbf{r} = (r_1,\dots,r_k)$ be a k–tuple of nonnegative integers, and let $Y_\mathbf{r}$ be the Young diagram with r_i columns of length i. Thus $Y_\mathbf{r}$ is a left-justified array of squares, with $r_i + r_{i+1} + \cdots + r_k$ squares in row i. Let ρ be an array obtained by inserting positive integers into the squares of $Y_\mathbf{r}$ subject to the rules: (i) Every row is non-increasing, (ii) every column is strictly decreasing, and (iii) no entry in row i exceeds m_i. For instance, if $\mathbf{m} = (5,4,2)$ and $\mathbf{r} = (4,2,3)$, then a typical ρ looks like

$$\begin{matrix} 5 & 5 & 4 & 4 & 4 & 3 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 2 \\ 2 & 1 & 1. \end{matrix}$$

We call ρ a *column-strict plane partition*[2] of *type* $(\mathbf{m},\mathbf{r})$. Introduce new variables $X_1, X_2,\dots$, and set

$$M(\rho) = X_1^{a_1}X_2^{a_2}\cdots,$$

where a_i parts of ρ are equal to i. Thus, for the above example, $M(\rho) = X_1^5X_2^4X_3^3X_4^3X_5^2$. In general, $a_i = 0$ if $i > m_1$, and $\Sigma\, a_i = \Sigma\, ir_i$. Given $\mathbf{m} = (m_1,\dots,m_k)$, define the generating function

$$F_\mathbf{m}(A,X) = \sum_\rho A^\mathbf{r} M(\rho), \tag{2}$$

where the sum is over all column-strict plane partitions ρ of type $(\mathbf{m},\mathbf{r})$ for some $\mathbf{r} = (r_1,\dots,r_k)$. We will give a method for computing $F_\mathbf{m}(A,X)$ as a sum of terms of the form

$$A^sX^v \Big/ \prod_{i=1}^{m} (1 - A_jX_{l_1}\cdots X_{l_j}), \tag{3}$$

where j and $l_1,\dots,l_j$ depend on i, and where $m = m_1 + \cdots + m_k$. From this it will be easy to obtain the character generator for SU(n).

We now define the type of tableaux necessary to describe the terms (3) of $F_\mathbf{m}(A,X)$. A *shifted Young diagram* $Z_\mathbf{m}$ of shape $\mathbf{m} = (m_1,\dots,m_k)$ consists of an array of $m = m_1 + \cdots + m_k$ squares, with m_i squares in row i, and with row $i+1$ indented one space to the right from row i. A *standard shifted Young tableau* (SSYT) of shape $\mathbf{m}$ is obtained by inserting the integers $1,2,\dots,m$ into the squares of $Z_\mathbf{m}$ without repetition such that every row and column is increasing.[3] For instance, an example of an SSYT of shape (7,4,3,2) is given by

$$\begin{matrix} 1 & 2 & 3 & 5 & 9 & 14 & 16 \\ & 4 & 6 & 7 & 10 & & \\ & & 8 & 11 & 13 & & \\ & & & 12 & 15. & & \end{matrix} \tag{4}$$

If π is an SSYT, define the *sub*-SSYT $\pi^{(i)}$ to be the SSYT obtained from π by deleting all entries $> i$. For instance, if π is given by (4), then $\pi^{(16)} = \pi$, $\pi^{(3)} = 123$, and

$$\pi^{(8)} = \begin{matrix} 1 & 2 & 3 & 5 \\ & 4 & 6 & 7 \\ & & 8. & \end{matrix}$$

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If π is an SSYT of shape $\mathbf{m} = (m_1,...,m_k)$, define a monomial $\Gamma(\pi) = A_k X_{m_1} X_{m_2} \cdots X_{m_k}$. For instance, if π is given by (4), then $\Gamma(\pi) = A_4 X_2 X_3 X_4 X_7$ and $\Gamma(\pi^{(8)}) = A_3 X_1 X_3 X_4$. We now state the fundamental theorem which explains how a formula for $F_{\mathbf{m}}(A,X)$ can be read off from the set of all SSYT of shape $\mathbf{m}$.

Theorem: (i) We have

$$F_{\mathbf{m}}(A,X) = \sum_{\pi} \prod_{i\in K_\pi} \Gamma(\pi^{(i)}) \Big/ \prod_{i=1}^{m} [1 - \Gamma(\pi^{(i)})], \tag{5}$$

where π ranges over all SSYT of shape $\mathbf{m}$, and K_π is the set of those i for which $i+1$ appears in π in a row above i.

(ii) To obtain the character generator for SU(n) in the form (1), with respect to the basis $\lambda_1,...,\lambda_{n-1}$ of fundamental weights, take $\mathbf{m} = (n,n-1,...,2)$ in (5) and set $X_i = \alpha_{i-1}^{-1}\alpha_i$ for $1 \leqslant i \leqslant n$ (where we set $\alpha_0 = \alpha_n = 1$). (If one prefers the characters with respect to a different basis for the weight space, replace each α_i by an appropriate $\alpha_1^{u_{1i}} \cdots \alpha_{n-1}^{u_{n-1,i}}$.) More generally, if $\lambda_1,...,\lambda_{n-1}$ are the fundamental weights of SU(n) in their usual order, then to get the generating function for those characters of SU(n) corresponding to a highest weight $r_1\lambda_1 + \cdots + r_k\lambda_k$ for some fixed $k \leqslant n-1$, take $\mathbf{m} = (n,n-1,...,n-k+1)$ and $X_i = \alpha_{i-1}^{-1}\alpha_i$, $1 \leqslant i \leqslant n$.

(iii) The number $g^{\mathbf{m}}$ of terms in the sum (5) (equivalently, the number of SSYT of shape $\mathbf{m}$) is given by

$$g^{\mathbf{m}} = \frac{m!}{m_1! \cdots m_k!} \prod_{1<i<j<k} \frac{m_i - m_j}{m_i + m_j},$$

where $\mathbf{m} = (m_1,...,m_k)$. In particular,

$$g^{(n,n-1,...,2)} = \begin{cases} \dfrac{\binom{n+1}{2}!\,2!\,4!\cdots(n-2)!}{(n+1)!\,(n+3)!\cdots(2n-1)!}, & n \text{ even} \\[2ex] \dfrac{\binom{n+1}{2}!\,2!\,4!\cdots(n-1)!}{n!\,(n+2)!\cdots(2n-1)!}, & n \text{ odd}. \end{cases}$$

3. PROOF OF FUNDAMENTAL THEOREM

(i) The right-hand side of (5) may be rewritten as

$$\sum_{\pi} \sum_{b_1,...,b_m} \Gamma(\pi^{(1)})^{b_1} \cdots \Gamma(\pi^{(m)})^{b_m}, \tag{6}$$

where $b_1,...,b_m$ ranges over all sequences of nonnegative integers such that $b_i > 0$ if $i\in K_\pi$. To each term $\Gamma(\pi^{(1)})^{b_1} \cdots \Gamma(\pi^{(m)})^{b_m}$ of (6), associate a column-strict plane partition ρ by defining ρ to have b_i columns with entries $l_1 > \cdots > l_j$, where $\pi^{(i)}$ has shape $(l_1,...,l_j)$. If ρ is of type $(\mathbf{m},\mathbf{r})$ then $\Gamma(\pi^{(1)})^{b_1} \cdots \Gamma(\pi_{(m)})^{b_m}$ is just the monomial $A^{\mathbf{r}}M(\rho)$ appearing in (2). Hence to prove (i), we need to show that the map $(\pi,\mathbf{b}) \to \rho$ defined above between (a) ordered pairs $(\pi,\mathbf{b})$ where π is a SSYT of shape $\mathbf{m}$ and $\mathbf{b}$ is a sequence of nonnegative integers $b_1,...,b_m$ such that $b_i > 0$ if $i\in K_\pi$, and (b) column-strict plane partitions ρ of type $(\mathbf{m},\mathbf{r})$ for some $\mathbf{r}$, is a one-to-one correspondence.

Given $(\pi,\mathbf{b})$ define $a_i = b_i + b_{i+1} + \cdots + b_m$. Thus $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$, and $a_i > a_{i+1}$ if $i\in K_\pi$. Clearly we can recover $\mathbf{b}$ from $\mathbf{a} = (a_1,...,a_m)$ by $b_i = a_i - a_{i+1}$. Now let σ be the array obtained by replacing i in π by a_i. Then σ is a *shifted plane partition*[3] of shape $\mathbf{m}$, i.e., an array obtained by inserting nonnegative integers into the squares of $Z_{\mathbf{m}}$ so that every row and column is nonincreasing.

We can recover ρ from σ by defining the ith column of ρ to be the shape of the shifted plane partition consisting of all entries of σ which are $\geqslant i$. Hence we need to show that the map $(\pi,\mathbf{a}) \to \sigma$ just defined between (a) ordered pairs $(\pi,\mathbf{a})$ where π is a SSYT of shape $\mathbf{m}$ and $\mathbf{a}$ is a sequence $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$ of integers such that $a_i > a_{i+1}$ if $i\in K_\pi$, and (b) shifted plane partitions σ of shape $\mathbf{m}$, is a one-to-one correspondence. This will follow from a general result about partially ordered sets which we now describe.

Let P be any finite partially ordered set (poset) with m elements, and let $\omega: P \to \{1,2,...,m\}$ be a fixed order-preserving bijection (so $x < y$ in P implies $\omega(x) < \omega(y)$). Let $\mathscr{L}(P)$ be the set of *all* order-preserving bijections $\pi: P \to \{1,2,...,m\}$. If $\pi \in \mathscr{L}(P)$, let S_π denote the set of all integer sequences $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$ such that $a_i > a_{i+1}$ if $\omega\pi^{-1}(i) > \omega\pi^{-1}(i+1)$. Finally, let $\mathscr{A}(P)$ consist of all order-reversing maps $\sigma: P \to \{0,1,2,...\}$ [i.e., $x < y$ in P implies $\sigma(x) \geqslant \sigma(y)$]. According to Ref. 4 or Theorem 6.2 of Ref. 5, we have:

Lemma: Define a map $\Phi(\pi,\mathbf{a}) = \sigma$ between ordered pairs $(\pi,\mathbf{a})$ where $\pi \in \mathscr{L}(P)$ and $\mathbf{a} \in S_\pi$, and the set $\mathscr{A}(P)$, by the rule $\sigma(x) = a_{\pi^{-1}(x)}$. Then Φ is a one-to-one correspondence.

We may regard the shifted Young diagram $Z_{\mathbf{m}}$ as a poset, with the elements (squares) increasing as we read left-to-right or top-to-bottom. Choose $\omega: Z_{\mathbf{m}} \to \{1,2,...m\}$ to increase by unit amounts along each row. E.g., for $\mathbf{m} = (5,3,1)$, ω is given by

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & \\ & & 9. & & \end{matrix}$$

It is clear that a map $\sigma \in \mathscr{A}(Z_{\mathbf{m}})$ is nothing more than a shifted plane partition of shape $\mathbf{m}$, and that an order-preserving bijection $\pi \in \mathscr{L}(Z_{\mathbf{m}})$ is just an SSYT. It follows from the lemma and our choice of ω that we have exactly the one-to-one correspondence $(\pi,\mathbf{a}) \to \sigma$ needed to complete the proof of (i).

(ii) This follows immediately from (i) and the well-known description of the irreducible representations of SU(n) in terms of Young tableaux.

(iii) The number $g^{\mathbf{m}}$ of SSYT of shape $\mathbf{m}$ has been calculated implicitly by Schur,[7] and more explicitly in Refs. 3 and 8.

4. EXAMPLES

We will use the Fundamental Theorem to compute the character generators of SU(3) and SU(4). These two cases are at least implicit in Ref. 6.

For the case of SU(3), there are two SSYT π of shape (3,2). For each of these π, we need to compute (by inspection) the shape $(l_1,...,l_j)$ of each of the five sub-SSYT $\pi^{(1)},...,\pi^{(5)}$ and hence obtain the monomial $\Gamma(\pi^{(i)}) = A_j X_{l_1} \cdots X_{l_j}$. We also compute by inspection the set K_π of i in π such that $i+1$ appears in a higher row than i. Then π will contribute a term $\Pi_{i\in K_\pi}\Gamma(\pi^{(i)})/\Pi_{i=1}^{m}[1 - \Gamma(\pi^{(i)})]$ to $F_{\mathbf{m}}(A,X)$. Substituting $X_1 = \alpha_1$, $X_2 = \alpha_1^{-1}\alpha_2$, $X_3 = \alpha_2^{-1}$ yields the character generator $X_A(\alpha)$. The table below gives the relevant information for each SSYT π.

1. $\pi = \begin{matrix} 1 & 2 & 3 \\ & 4 & 5 \end{matrix} \quad K_\pi = \phi$

i	1	2	3	4	5
$\pi^{(i)}$	1	1 2	1 2 3	1 2 3 4	1 2 3 4 5
$\Gamma(\pi^{(i)})$	A_1X_1	A_1X_2	A_1X_3	$A_2X_1X_3$	$A_2X_2X_3$

2. $\pi = \begin{matrix} 1 & 2 & 4 \\ & 3 & 5 \end{matrix} \quad K_\pi = \{3\}$

i	1	2	3	4	5
$\pi^{(i)}$	1	1 2	1 2 3	1 2 4 3	1 2 4 3 5
$\Gamma(\pi^{(i)})$	A_1X_1	A_1X_2	$A_2X_1X_2$	$A_2X_1X_3$	$A_2X_2X_3$

Hence

$$F_{(3,2)}(A,X) = \frac{1}{(1-A_1X_1)(1-A_1X_2)(1-A_1X_3)(1-A_2X_1X_3)(1-A_2X_2X_3)} + \frac{A_2X_1X_2}{(1-A_1X_1)(1-A_1X_2)(1-A_2X_1X_2)(1-A_2X_1X_3)(1-A_2X_2X_3)}.$$

Thus the character generator for SU(3) is given by:

$$X_A(\alpha) = \frac{1}{(1-\alpha_1A_1)(1-\alpha_1{}^{-1}\alpha_2A_1)(1-\alpha_2{}^{-1}A_1)(1-\alpha_1\alpha_2{}^{-1}A_2)(1-\alpha_1{}^{-1}A_2)} + \frac{\alpha_2A_2}{(1-\alpha_1A_1)(1-\alpha_1{}^{-1}\alpha_2A_1)(1-\alpha_2A_2)(1-\alpha_1\alpha_2{}^{-1}A_2)(1-\alpha_1{}^{-1}A_2)}.$$

For the case of SU(4), there are 12 SSYT of shape (4, 3, 2). For each one we list the set K_π and the shapes $(l_1,\ldots,l_j)$ of each $\pi^{(i)}$, so $\Gamma(\pi^{(i)}) = A_jX_{l_1}\cdots X_{l_j}$.

(1) $\pi = \begin{matrix} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & 9 \end{matrix} \quad K_\pi = \phi$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots l_j$	1	2	3	4	4,1	4,2	4,3	4,3,1	4,3,2

(2) $\pi = \begin{matrix} 1 & 2 & 3 & 4 \\ & 5 & 6 & 8 \\ & & 7 & 9 \end{matrix} \quad K_\pi = \{7\}$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	4	4,1	4,2	4,2,1	4,3,1	4,3,2

(3) $\pi = \begin{matrix} 1 & 2 & 3 & 5 \\ & 4 & 6 & 7 \\ & & 8 & 9 \end{matrix} \quad K_\pi = \{4\}$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	3,1	4,1	4,2	4,3	4,3,1	4,3,2

(4) $$\pi = \begin{array}{cccc} 1 & 2 & 3 & 5 \\ & 4 & 6 & 8 \\ & & 7 & 9 \end{array} \quad K_\pi = \{4,7\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	3,1	4,1	4,2	4,2,1	4,3,1	4,3,2

(5) $$\pi = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ & 4 & 5 & 7 \\ & & 8 & 9 \end{array} \quad K_\pi = \{5\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	3,1	3,2	4,2	4,3	4,3,1	4,3,2

(6) $$\pi = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ & 4 & 5 & 8 \\ & & 7 & 9 \end{array} \quad K_\pi = \{5,7\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	3,1	3,2	4,2	4,2,1	4,3,1	4,3,2

(7) $$\pi = \begin{array}{cccc} 1 & 2 & 3 & 7 \\ & 4 & 5 & 8 \\ & & 6 & 9 \end{array} \quad K_\pi = \{6\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	3	3,1	3,2	3,2,1	4,2,1	4,3,1	4,3,2

(8) $$\pi = \begin{array}{cccc} 1 & 2 & 4 & 5 \\ & 3 & 6 & 7 \\ & & 8 & 9 \end{array} \quad K_\pi = \{3\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	2,1	3,1	4,1	4,2	4,3	4,3,1	4,3,2

(9) $$\pi = \begin{array}{cccc} 1 & 2 & 4 & 5 \\ & 3 & 6 & 8 \\ & & 7 & 9 \end{array} \quad K_\pi = \{3,7\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	2,1	3,1	4,1	4,2	4,2,1	4,3,1	4,3,2

$$(10)\quad \pi = \begin{array}{cccc} 1 & 2 & 4 & 6 \\ & 3 & 5 & 7 \\ & & 8 & 9 \end{array} \quad K_\pi = \{3,5\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	2,1	3,1	3,2	4,2	4,3	4,3,1	4,3,2

$$(11)\quad \pi = \begin{array}{cccc} 1 & 2 & 4 & 6 \\ & 3 & 5 & 8 \\ & & 7 & 9 \end{array} \quad K_\pi = \{3,5,7\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	2,1	3,1	3,2	4,2	4,2,1	4,3,1	4,3,2

$$(12)\quad \pi = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ & 3 & 5 & 8 \\ & & 6 & 9 \end{array} \quad K_\pi = \{3,6\}$$

i	1	2	3	4	5	6	7	8	9
$l_1,\ldots,l_j$	1	2	2,1	3,1	3,2	3,2,1	4,2,1	4,3,1	4,3,2

Thus we obtain the following expression for the character generator $X_A(\alpha)$ of SU(4):

$$\begin{aligned}
&(1-\alpha_1A_1)(1-\alpha_1^{-1}\alpha_2A_1)(1-\alpha_1\alpha_2^{-1}A_3)(1-\alpha_1^{-1}A_3)X_A(\alpha)\\
&= \frac{1}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_3^{-1}A_1)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2^{-1}A_2)}\\
&\quad + \frac{\alpha_2\alpha_3^{-1}A_3}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_3^{-1}A_1)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2\alpha_3^{-1}A_3)}\\
&\quad + \frac{\alpha_1\alpha_2^{-1}\alpha_3A_2}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2^{-1}A_2)}\\
&\quad + \frac{\alpha_1A_2A_3}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2\alpha_3^{-1}A_3)}\\
&\quad + \frac{\alpha_1^{-1}\alpha_3A_2}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2^{-1}A_2)}\\
&\quad + \frac{\alpha_1^{-1}\alpha_2A_2A_3}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2\alpha_3^{-1}A_3)}\\
&\quad + \frac{\alpha_3A_3}{(1-\alpha_2^{-1}\alpha_3A_1)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_3A_2)(1-\alpha_3A_3)(1-\alpha_2\alpha_3^{-1}A_3)}\\
&\quad + \frac{\alpha_2A_2}{(1-\alpha_2A_2)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2^{-1}A_2)}\\
&\quad + \frac{\alpha_2^2\alpha_3^{-1}A_2A_3}{(1-\alpha_2A_2)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1\alpha_3^{-1}A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2\alpha_3^{-1}A_3)}\\
&\quad + \frac{\alpha_1^{-1}\alpha_2\alpha_3A_2^2}{(1-\alpha_2A_2)(1-\alpha_1\alpha_2^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_3A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2^{-1}A_2)}
\end{aligned}$$

$$+ \frac{\alpha_1^{-1}\alpha_2^2 A_2^2 A_3}{(1-\alpha_2 A_2)(1-\alpha_1\alpha_2^{-1}\alpha_3 A_2)(1-\alpha_1^{-1}\alpha_3 A_2)(1-\alpha_1^{-1}\alpha_2\alpha_3^{-1}A_2)(1-\alpha_2\alpha_3^{-1}A_3)}$$
$$+ \frac{\alpha_2\alpha_3 A_2 A_3}{(1-\alpha_2 A_2)(1-\alpha_1\alpha_2^{-1}\alpha_3 A_2)(1-\alpha_1^{-1}\alpha_3 A_2)(1-\alpha_3 A_3)(1-\alpha_2\alpha_3^{-1}A_3)}.$$

There seems little point in writing down the character generator of SU(5), which by part (iii) of the theorem has 286 terms. Even more impractical is the character generator of SU(6), with 33592 terms.

5. CONCLUSIONS

The generating function $F_{\mathbf{m}}(A,X)$ has some additional properties of interest. If $m = (n,n-1,...,2)$ then write $F_{\mathbf{m}}(A,X) = F_n(A,X)$. If we set each $A_1 = 1$ in $F_n(A,X)$, then it follows, e.g., from Eq. (11.9;6) of Ref. 9 or Corollary 8.3 of Ref. 2 that

$$F_n(1,1,...,1;X) = (1 - X_1X_2\cdots X_n) \Big/ \prod_{i=1}^{n}(1-X_i) \prod_{1<i<j<n}(1-X_iX_j).$$

If we set each $X_i = 1$ and $A_i = A$ in $F_{\mathbf{m}}(A,X)$, then it follows from (5) that the coefficient of A^q in $F_{\mathbf{m}}(A,...,A,1,...,1)$ is a polynomial function $P_{\mathbf{m}}(q)$ of q of degree $m-1$ and leading coefficient $g^{\mathbf{m}}/(m-1)!$ When $m = (n,n-1,...,n-k+1)$, this polynomial $P_{n,k}(q)$ is given by

$$P_{n,k}(q) = \sum \dim(a_1\lambda_1 + \cdots + a_k\lambda_k), \tag{7}$$

where the sum is over all k–tuples of nonnegative integers $(a_1,...,a_k)$ such that $a_1 + \cdots + a_k = q$, and where $\dim\lambda$ denotes the dimension of the irreducible representation of SU(n) with highest weight λ. When $k = n-1$, the sum (7) can be explicitly evaluated using a result of Andrews[10] and independently Macdonald[11] (pp. 50–52). Namely,

$$P_{n,n-1}(q) = \begin{cases} \Delta^2 \prod_{i=0}^{l} \dfrac{(q+n+2i-2)_{4i+1}}{(n+2i)_{4i+1}}, & \text{if } n = 2l+1 \\ \Delta^2 \prod_{i=1}^{l} \dfrac{(q+n+2i-3)_{4i-1}}{(n+2i-1)_{4i-1}}, & \text{if } n = 2l, \end{cases}$$

where $(r)_s = r(r-1)(r-2)\cdots(r-s+1)$, and where Δ^2 is the second-difference operator, defined by $\Delta^2 Q(q) = Q(q+2) - 2Q(q+1) + Q(q)$. Alternatively, we have $P_{n,n-1}(q) = \Delta^2 \dim((q-2)\lambda_n)$, where λ_n is the highest weight of the spin representation of the Lie algebra $so(2n+1,\mathbb{C})$. A theoretical explanation of this fact can be given by considering the decomposition of $gl(n,\mathbb{C}) \subset so(2n+1,\mathbb{C})$ in the representation $(q-2)\lambda_n$. We will not enter into the details here.

We have described a method for writing $F_{\mathbf{m}}(A,X)$ as a sum of $g^{\mathbf{m}}$ terms of the form (3). One may wonder whether there is some alternative way to write $F_{\mathbf{m}}(A,X)$ as a sum of fewer terms of the form (3). If we have any such representation of $F_{\mathbf{m}}(A,X)$ then setting $A_i = A$ and $X_i = 1$ as above, we obtain

$$F_{\mathbf{m}}(A,...,A,1,...,1) = \sum_j \frac{A^{t_j}}{(1-A)^m} = \frac{\Sigma_j A^{t_j}}{(1-A)^m},$$

for certain integers $t_j \geqslant 0$. Hence the integers t_j are uniquely determined by $F_{\mathbf{m}}(A,X)$, not by the way in which $F_{\mathbf{m}}(A,X)$ is written as a sum of terms (3). In particular, the number of terms is always the same, namely, $g^{\mathbf{m}}$.

Let us mentions that the numbers $g^{\mathbf{m}}$ were shown by Schur[5] to be the degrees of the irreducible projective representations of the symmetric group S_m. We don't know if this connection between SU(n) and S_m is just a coincidence.

It is natural to ask whether our results for SU(n) can be extended to other simple Lie groups, in particular O(n) and Sp($2n$). We have been unable to write the character generator for these groups in the form (1) because of the lack of a combinatorial description of the characters which would allow the use of the lemma on posets. Though there exist combinatorial descriptions of the characters of these groups (e.g., Ref. 9, p. 240, and Ref. 12), they seem unsuitable for the implementaion of the Lemma.

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