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# The Conjugate Trace and Trace of a Plane Partition

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The *conjugate trace* and *trace* of a plane partition are defined, and the generating function for the number of plane partitions $\pi$ of $n$ with $\leqslant r$ rows and largest part $\leqslant m$, with conjugate trace $t$ (or trace $t$, when $m = \infty$), is found. Various properties of this generating function are studied. One consequence of these properties is a formula which can be regarded as a $q$-analog of a well-known result arising in the representation theory of the symmetric group.

## 1. INTRODUCTION

A *plane partition* $\pi$ of $n$ is an array of non-negative integers,

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \tag{1}$$

for which $\sum_{i,j} n_{ij} = n$ and the rows and columns are in non-increasing order:

$$n_{ij} \geqslant n_{(i+1)j}, \qquad n_{ij} \geqslant n_{i(j+1)}, \qquad \text{for all} \quad i, j \geqslant 1.$$

The non-zero entries $n_{ij} > 0$ are called the *parts* of $\pi$. If there are $\lambda_i$ parts in the $i$-th row of $\pi$, so that, for some $r$,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > \lambda_{r+1} = 0,$$

then we call the partition $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ of the integer $p = \lambda_1 + \cdots + \lambda_r$ the *shape* of $\pi$, denoted by $\lambda$. We also say that $\pi$ has

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*r rows* and *p parts*. Similarly if $\lambda_i'$ is the number of parts in the $i$-th column of $\pi$, then, for some $c$,

$$\lambda_1' \geqslant \lambda_2' \geqslant \cdots \geqslant \lambda_c' > \lambda_{c+1}' = 0.$$

The partition $\lambda_1' \geqslant \lambda_2' \geqslant \cdots \geqslant \lambda_c'$ of $p$ is the *conjugate partition* to $\lambda$ [3, Ch. 19.2], and we say that $\pi$ has *c columns*.

If $n_{ij} > 0$, then the integer $h_{ij} = \lambda_i + \lambda_j' - i - j + 1$ is called the *hook length* of $n_{ij}$. Thus $h_{ij}$ is the number of non-zero entries directly to the right or directly below $n_{ij}$, counting $n_{ij}$ itself once. We also write $h_1, h_2, \ldots, h_p$ for the hook lengths of the parts of $\pi$ (in some arbitrary order). If $n_{ij} > 0$, the integer $c_{ij} = j - i$ is called the *content* of $n_{ij}$ (this terminology is derived from [7, Ch. 4.3]), and we write $c_1, c_2, \ldots, c_p$ for the contents of the parts of $\pi$ (in some order). Note that the hook lengths and contents of the parts of $\pi$ depend only on the shape $\lambda$ of $\pi$, and not on the parts $n_{ij}$ themselves. We sometimes refer to the $h_i$'s and $c_i$'s as the hook lengths and contents of $\lambda$.

If the non-zero entries of $\pi$ are strictly decreasing in each column, we say that $\pi$ is *column-strict*.

Throughout this paper we employ the notation:

$$\begin{aligned} (\mathbf{k}) &= 1 - x^k, \\ (\mathbf{k})! &= (\mathbf{1})(\mathbf{2}) \cdots (\mathbf{k}), \\ \binom{\mathbf{k}}{\mathbf{j}} &= \frac{(\mathbf{k})!}{(\mathbf{j})!(\mathbf{k}-\mathbf{j})!}. \end{aligned}$$

This latter expression is called a *Gaussian coefficient* (or *generalized binomial coefficient*). It reduces to an ordinary binomial coefficient $\binom{k}{j}$ when $x = 1$.

Let $\gamma_n$ be the number of column-strict plane partitions of $n$ of shape $\lambda$ and largest part $\leqslant m$. Define the generating function

$$F_m(\lambda) = \sum_{n=0}^{\infty} \gamma_n x^n,$$

and define $F(\lambda) = \lim_{m\to\infty} F_m(\lambda)$. Thus the coefficient of $x^n$ in $F(\lambda)$ is equal to the number of column-strict partitions of $n$ of shape $\lambda$. It is proved in [8, Th. 15.3] that

$$\begin{aligned} F_m(\lambda) &= x^a \frac{(\mathbf{m}+\mathbf{c}_1)(\mathbf{m}+\mathbf{c}_2)\cdots(\mathbf{m}+\mathbf{c}_p)}{(\mathbf{h}_1)(\mathbf{h}_2)\cdots(\mathbf{h}_p)}, \\ F(\lambda) &= x^a/(\mathbf{h}_1)(\mathbf{h}_2)\cdots(\mathbf{h}_p), \end{aligned} \tag{2}$$

where

$$a = \sum_{i=1}^{c} \binom{\lambda_i' + 1}{2} = \sum_{i=1}^{r} i\lambda_i . \tag{3}$$

This result is also given implicitly by D. E. Littlewood [5, p. 124, Th. I]. Note the similarity of the expression for $F_m(\lambda)$ to a Gaussian coefficient. In fact, $F_m(\lambda)$ reduces to a Gaussian coefficient

$$\binom{\mathbf{m} + \mathbf{p} - \mathbf{1}}{\mathbf{p}}$$

(except for the factor $x^a$) when $\lambda$ has just one part, i.e., when the corresponding plane partition is linear. Thus $F_m(\lambda)$ can be regarded as a kind of "two-dimensional" Gaussian coefficient.

## 2. The Conjugate Trace of a Plane Partition

Let $\rho_n$ be the number of plane partitions of $n$ with $\leqslant r$ rows and largest part $\leqslant m$. It is known that

$$\sum_{n=0}^{\infty} \rho_n x^n = \prod_{i=1}^{m} \prod_{j=1}^{r} (1 - x^{i+j-1})^{-1}. \tag{4}$$

For a proof of an even stronger result, see, e.g., MacMahon [6, Sect. 495]. A combinatorial proof of (4) appears in Bender and Knuth [1, Th. 2]. A simple derivation of (4) from (2) appears in [8, Sect. 18].

In view of (4), it is natural to ask whether there is some combinatorial interpretation of the coefficient of $q^t x^n$ in the expansion of

$$\prod_{i=1}^{m} \prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}.$$

Using a result of Knuth [4, Th. 2] (cf. also Bender and Knuth [1, Th. B]), we can give an affirmative answer.

The result of Knuth may be stated as follows:

(K) There is a one-to-one correspondence between ordered pairs $(\pi_1, \pi_2)$ of column-strict plane partitions of the same shape and matrices $(a_{ij})$ of non-negative integers. In this correspondence,

(i) $k$ appears in $\pi_1$ exactly $\sum_i a_{ik}$ times, and

(ii) $k$ appears in $\pi_2$ exactly $\sum_i a_{ki}$ times.

It should be remarked that an implicit form of (K) was given previously by Littlewood [5, p. 103, first formula of Th. V]. Similarly, Theorem 4 of Knuth [4] (Th. A of [1]) appears in [5] as formula (11.9; 6), while Knuth's "dual correspondence" [4, Sect. 5] (Th. C of [1]) is equivalent to [5, p. 103, second formula of Th. V]. The five additional formulas obtained by Littlewood [5 (11.9; 1)–(11.9; 5)] also have interesting applications to plane partitions though we shall not discuss them here.

2.1. DEFINITION. Let $\pi$ be a plane partition. The *conjugate trace of* $\pi$ is defined to be the number of parts $n_{ij}$ of $\pi$ satisfying $n_{ij} \geqslant i$.

We write $T_{rmt}^*(n)$ for the number of plane partitions of $n$ with $\leqslant r$ rows and largest part $\leqslant m$, and with conjugate trace $t$. Also define

$$T_{rt}^*(n) = \lim_{m\to\infty} T_{rmt}^*(n),$$

$$T_t^*(n) = \lim_{r\to\infty} T_{rt}^*(n).$$

Thus, e.g., $T_t{}^*(n)$ is equal to the number of plane partitions of $n$ with conjugate trace $t$.

2.2. THEOREM. *We have*

$$\sum_{n=0}^{\infty} \sum_{t=0}^{\infty} T_{rmt}^*(n)\, q^t x^n = \prod_{i=1}^{m} \prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}.$$

*Proof.* Frobenius [2] (cf. also Sudler [9] and Littlewood [5, p. 60]) has constructed a one-to-one correspondence between linear partitions $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ of $p$ and pairs of strict partitions $\mu$ and $\nu$,

$$\begin{aligned} \lambda_1 &= \mu_1 > \cdots > \mu_s > 0, \\ r &= \nu_1 > \nu_2 > \cdots > \nu_s > 0, \end{aligned}$$

with $\sum(\mu_i + \nu_i) = p + s$. This construction is defined by the conditions

$$\begin{aligned} \mu_i &= \lambda_i - i + 1 \qquad (\text{when} \quad \lambda_i - i + 1 > 0), \\ \nu_i &= \lambda_i' - i + 1 \qquad (\text{when} \quad \lambda_i' - i + 1 > 0). \end{aligned}$$

For instance, the linear partition $4 \geqslant 4 \geqslant 3 \geqslant 1 \geqslant 1 \geqslant 1$ corresponds to the pair of strict partitions

$$\begin{aligned} 4 > 3 > 1, \\ 6 > 2 > 1, \end{aligned}$$

as illustrated in Figure 1. Note that in this construction, $\lambda_i \geqslant i$ if and only if $\mu_i \geqslant 1$.

Bender and Knuth [1] generalize this construction straightforwardly to plane partitions as follows: If $\pi$ is a plane partition, then apply the construction of Frobenius to each column to get a pair of column-strict

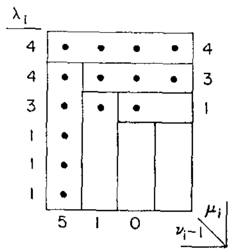


FIG. 1. A construction of Frobenius.

plane partitions $\pi_1$ and $\pi_2$ of the same shape. For instance, the plane partition

$$\begin{matrix} 4 & 4 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & & \\ 2 & & & \\ 2 & & & \end{matrix}$$

corresponds to the pair

$$\begin{matrix} 4 & 4 & 2 & 1 \\ 3 & 1 & 1 & \\ 2 & & & \end{matrix} \qquad \begin{matrix} 5 & 3 & 2 & 2 \\ 4 & 2 & 1 & \\ 1 & & & \end{matrix}$$

In this correspondence, the number $r$ of rows of $\pi$ equals the largest part $m_2$ of $\pi_2$, the largest part $m$ of $\pi$ equals the largest part $m_1$ of $\pi_1$, and the conjugate trace $t$ of $\pi$ equals the number of parts $p_1 = p_2$ of $\pi_1$ or $\pi_2$ (by the last sentence of the previous paragraph). Also if $\pi_i$ is a partition of $n_i$, then $\pi$ is a partition of $n_1 + n_2 - t$.

Thus $T^*_{rmt}(n)$ is equal to the number of pairs $\pi_1$, $\pi_2$ of column-strict plane partitions of the same shape satisfying:

(i) the largest part of $\pi_1$ is $\leqslant r$,

(ii) the largest part of $\pi_2$ is $\leqslant m$,

(iii) the number of parts of $\pi_1$ or $\pi_2$ is $t$,

(iv) the sum of the parts of $\pi_1$ and $\pi_2$ is $n + t$.

It follows from (K) that

$$\sum_{n=0}^{\infty} \sum_{t=0}^{\infty} T_{rmt}^{*}(n)\, q^t x^n = \prod_{i=1}^{m} \prod_{j=1}^{r} \sum_{a_{ij}=0}^{\infty} q^t x^n,$$

where

$$t = \sum_{i,j} a_{ij},$$

$$n + t = \sum_j j \sum_i a_{ij} + \sum_i i \sum_j a_{ij}.$$

The above product thus equals

$$\prod_{i=1}^{m} \prod_{j=1}^{r} \sum_{a_{ij}=0}^{\infty} q^{a_{ij}} x^{(i+j-1)a_{ij}} = \prod_{i=1}^{m} \prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}. \quad \blacksquare$$

## 3. The Trace of a Plane Partition

3.1. Definition. The *trace* of a plane partition $\pi$ with entries $n_{ij}$ is defined to be $\sum n_{ii}$. Let $T_{rt}(n)$ be the number of plane partitions of $n$ with $\leqslant r$ rows and trace $t$, and let

$$T_t(n) = \lim_{r\to\infty} T_{rt}(n).$$

Every plane partition $\pi$ has six *conjugates* (called *aspects* by MacMahon [6, Sect. 427]). One of these, call it $\pi'$, is obtained from $\pi$ by taking the conjugate partition of each row. For example

$$\begin{array}{cccc} 3 & 3 & 2 & 1 \\ 3 & 1 & & \\ 2 & 1 & & \\ & \pi & & \end{array} \qquad \begin{array}{ccc} 4 & 3 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & \\ & \pi' & \end{array}$$

It is easily seen that $\pi$ and $\pi'$ are plane partitions of the same integer $n$, that they have the same number of rows, and that the conjugate trace of $\pi$ is equal to the trace of $\pi'$. (This explains the terminology "conjugate trace.") There follows:

$$T_{rt}^{*}(n) = T_{rt}(n). \tag{5}$$

## 4. The Case $m = \infty$

Define generating functions $G_r(q, x)$ and $G(q, x)$ by

$$G_r(q, x) = \sum_{n=0}^{\infty} \sum_{t=0}^{\infty} T_{rt}^*(n)\, q^t x^n,$$

$$G(q, x) = \sum_{n=0}^{\infty} \sum_{t=0}^{\infty} T_t^*(n)\, q^t x^n.$$

Thus, by Theorem 2.2,

$$G_r(q, x) = \prod_{n=1}^{\infty} (1 - qx^n)^{-\min(r,n)},$$
$$G(q, x) = \prod_{n=1}^{\infty} (1 - qx^n)^{-n}. \tag{6}$$

Define $g_{rt}(x)$ and $g_t(x)$ by

$$G_r(q, x) = \sum_{t=0}^{\infty} \frac{g_{rt}(x)\, x^t q^t}{(\mathbf{t})!},$$
$$G(q, x) = \sum_{t=0}^{\infty} \frac{g_t(x)\, x^t q^t}{(\mathbf{t})!^2}. \tag{7}$$

4.1. Theorem. *$g_{rt}(x)$ is a polynomial in $x$ with integer coefficients given by*

$$x^{2t} g_{rt}(x) = (\mathbf{t})! \sum_{\lambda} F_r(\lambda)\, F(\lambda),$$

*where the sum is over all partitions $\lambda$ of $t$ (or over all partitions $\lambda$ of $t$ with $\leqslant r$ parts, since otherwise $F_r(\lambda)$ vanishes).*

*Proof.* By definition, $x^t g_{rt}(x)/(\mathbf{t})!$ is the generating function for plane partitions with $\leqslant r$ rows and conjugate trace (or trace) $t$. Hence, by the correspondence set up in the proof of Theorem 2.2,

$$\frac{x^t g_{rt}(x)}{(\mathbf{t})!} = \sum_{\lambda} F_r(\lambda)\, F(\lambda)\, x^{-t}.$$

It is easy to verify, using (2), that $F_r(\lambda)$ and $(\mathbf{t})!\, F(\lambda)$ are polynomials in $x$ with integer coefficients and are both divisible by $x^t$. Hence $g_{rt}(x)$ is a polynomial with integer coefficients. ∎

We remark that it can be shown, using techniques from this writer's thesis, that the coefficients of $F_r(\lambda)$ and $\mathbf{(t)!}\,F(\lambda)$ are non-negative, so the coefficients of $g_{rt}(\lambda)$ are also non-negative.

On the other hand, we can consider the polynomials $g_{rt}(x)$ from the point of view of (6). This gives:

4.2. THEOREM. *The polynomials $g_{rt}(x)$ satisfy the two recursions*

$$\text{(i)}\quad g_{rt}(x) = \sum_{k=1}^{r} (-1)^{k+1} x^{\binom{k}{2}} \times \binom{\mathbf{r}}{\mathbf{k}} (\mathbf{t}-\mathbf{k}+\mathbf{1})(\mathbf{t}-\mathbf{k}+\mathbf{2}) \cdots (\mathbf{t}-\mathbf{1})\, g_{r,t-k}(x),$$

$$\text{(ii)}\quad g_{rt}(x) = \sum_{k=0}^{t-1} x^k \binom{\mathbf{t}-\mathbf{k}+\mathbf{r}-\mathbf{1}}{\mathbf{r}-\mathbf{1}} (\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2}) \cdots (\mathbf{t}-\mathbf{1})\, g_{rk}(x).$$

*Proof.* From (6) we have

$$G_r(q, x) = (1-q)(1-qx) \cdots (1-qx^{r-1})\, G_r(q/x, x). \tag{8}$$

According to a well-known identity of Euler (e.g., [3, Th. 348]),

$$(1-q)(1-qx) \cdots (1-qx^{r-1}) = \sum_{k=0}^{r} (-1)^k x^{\binom{k}{2}} \binom{\mathbf{r}}{\mathbf{k}} q^k.$$

Thus equating coefficients of $q^t/\mathbf{(t)!}$ in (8) gives

$$x^t g_{rt}(x) = \sum_{k=0}^{r} (-1)^k x^{\binom{k}{2}} \binom{\mathbf{r}}{\mathbf{k}} (\mathbf{t}-\mathbf{k}+\mathbf{1})(\mathbf{t}-\mathbf{k}+\mathbf{2}) \cdots (\mathbf{t})\, g_{r,t-k}(x).$$

Moving the term $k = 0$ to the left and dividing by $-\mathbf{(t)}$ gives (i).

Similarly (ii) is obtained from (8) using

$$1/(1-q)(1-qx) \cdots (1-qx^{r-1}) = \sum_{k=0}^{\infty} \binom{\mathbf{k}+\mathbf{r}-\mathbf{1}}{\mathbf{r}-\mathbf{1}} q^k. \quad \blacksquare$$

4.3. COROLLARY. (i) $g_{rt}(1) = r^t$,

$$\text{(ii)}\quad g_{rt}(x) \equiv \frac{\prod_{k=r}^{\infty} \mathbf{(k)}}{\prod_{k=1}^{\infty} \mathbf{(k)}^{\min(k,r)}} \pmod{x^{t+1}},$$

(*in particular,* $g_{rt}(0) = 1$), *where the notation* $f(x) \equiv g(x) \pmod{x^{t+1}}$ *signifies that the coefficients of* $x^k$ *in* $f(x)$ *and* $g(x)$ *are the same, for* $k = 0, 1,\ldots, t$.

(iii) *$T^*_{rt}(n+t)(=T_{rt}(n+t))$ is the coefficient of $x^n$ in the expansion of*

$$\prod_{k=1}^{\infty} (\mathbf{k})^{-\min(k+1,r)}$$

*if* $0 \leqslant n \leqslant t$.

*Proof.* (i) Straightforward induction on $t$, using Theorem 4.2.

(ii) We have the following congruences (mod $x^{t+1}$) for $k = 0, 1, 2,\ldots$

$$(k=0) \qquad \binom{\mathbf{t}+\mathbf{r}-\mathbf{1}}{\mathbf{r}-\mathbf{1}} (\mathbf{t}-\mathbf{1})! - x^t \equiv \frac{(\mathbf{1})(\mathbf{2})\cdots}{(\mathbf{r}-\mathbf{1})!},$$

$(1 \leqslant k \leqslant t-1)$

$$x^k \binom{\mathbf{t}-\mathbf{k}+\mathbf{r}-\mathbf{1}}{\mathbf{r}-\mathbf{1}} (\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2})\cdots(\mathbf{t}-\mathbf{1}) \equiv \frac{x^k(\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2})\cdots}{(\mathbf{r}-\mathbf{1})!},$$

$$(k=t) \qquad x^t \equiv \frac{x^t(\mathbf{t}+\mathbf{1})(\mathbf{t}+\mathbf{2})\cdots}{(\mathbf{r}-\mathbf{1})!},$$

$$(k>t) \qquad 0 \equiv \frac{x^k(\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2})\cdots}{(\mathbf{r}-\mathbf{1})!}.$$

Multiply the $k$-th congruence by $g_{rk}(x)$ and sum over all $k$. This gives

$$\sum_{k=0}^{t-1} \binom{\mathbf{t}-\mathbf{k}+\mathbf{r}-\mathbf{1}}{\mathbf{r}-\mathbf{1}} (\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2})\cdots(\mathbf{t}-\mathbf{1})\, g_{rk}(x) - x^t g_{r0}(x) + x^t g_{rt}(x)$$

$$\equiv \sum_{k=0}^{\infty} \frac{x^k(\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2})\cdots}{(\mathbf{r}-\mathbf{1})!} g_{rk}(x). \tag{9}$$

Now $x^t g_{r0}(x) \equiv x^t$ and $x^t g_{rt}(x) \equiv x^t$, so, by Theorem 4.2(ii), the left-hand side of (9) is congruent to $g_{rt}(x)$ (mod $x^{t+1}$). By (7), the right-hand side is equal to

$$\left(\prod_{k=r}^{\infty} (\mathbf{k})\right) \sum_{k=0}^{\infty} \frac{x^k g_{rk}(x)}{(\mathbf{k})!} = \left(\prod_{k=r}^{\infty} (\mathbf{k})\right) \prod_{k=1}^{\infty} (\mathbf{k})^{-\min(r,k)}$$

and the proof follows.

(iii) We have

$$\sum_{n=0}^{\infty} T_{rt}(n+t)\,x^n = g_{rt}(x)/(\mathbf{t})!$$

$$\equiv \frac{(\prod_{k=r}^{\infty}(\mathbf{k}))\prod_{k=1}^{\infty}(\mathbf{k})^{-\min(r,k)}}{(\mathbf{1})(\mathbf{2})(\mathbf{3})\cdots} \pmod{x^{t+1}}$$

$$= \prod_{k=1}^{\infty}(\mathbf{k})^{-\min(k+1,r)}$$

and the proof follows. ∎

The recursion of Theorem 4.2 can also be used to compute the degrees of $g_{rt}(x)$. For instance,

$$\deg g_{2,2t-1}(x) = t^2,$$
$$\deg g_{2,2t}(x) = t(t+1).$$

Some small values for $r = 2$ are:

$$g_{20}(x) = 1,$$
$$g_{21}(x) = 1 + x,$$
$$g_{22}(x) = 1 + x + 2x^2,$$
$$g_{23}(x) = 1 + x + 2x^2 + 3x^3 + x^4,$$
$$g_{24}(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 2x^5 + 2x^6.$$

Note that the coefficient of $x^n$ in $g_{2t}(x)$ is $p(n)$, the number of partitions of $n$, for $0 \leqslant n \leqslant t$, in accordance with Corollary 4.3(ii).

4.4. Corollary. *For any $r$, we have*

$$t!\sum_{\lambda} \frac{(r+c_1)(r+c_2)\cdots(r+c_t)}{h_1{}^2h_2{}^2\cdots h_t{}^2} = r^t,$$

*where the sum is over all partitions $\lambda$ of $t$, and the $h_i$'s and $c_i$'s are the hook lengths and contents of $\lambda$.*

*Proof.* The left-hand side is obtained by putting $x = 1$ in Theorem 4.1 and using (2). The right-hand side is obtained from Corollary 4.3(i). ∎

Corollary 4.4 should be compared with the known result

$$\sum_{\lambda}\left(\frac{t!}{h_1h_2\cdots h_t}\right)^2 = t!, \tag{10}$$

which is a consequence of the fact that the numbers $t!/h_1h_2 \cdots h_t$ are the degrees of the irreducible ordinary representations of the symmetric group on $t$ letters (cf. [7, eq. 2.37]). In fact, (10) follows from Corollary 4.4 by equating coefficients of $r^t$. (10) may also be regarded as a sort of limiting case of Corollary 4.4 as $r \to \infty$; see the next section.

## 5. THE CASE $r = m = \infty$

An analysis similar to the preceding can be made of the generating function $G(q, x)$ and of the $g_t(x)$'s. We state results analogous to those in Section 4. Those proofs which are straightforward modifications of the above proofs, or which follow from the above by letting $r \to \infty$, will be omitted.

5.1. THEOREM. *$g_t(x)$ is a polynomial in $x$ with non-negative integer coefficients given by*

$$x^{2t}g_t(x) = (\mathbf{t})!^2 \sum F(\lambda)^2,$$

*where the sum is over all partitions $\lambda$ of $t$.* ∎

5.2. THEOREM. *The polynomials $g_t(x)$ satisfy the two recursions*

$$\text{(i)}\quad g_t(x) = \sum_{k=0}^{t-1} (-1)^{t-k+1} x^{\binom{t-k}{2}} \binom{\mathbf{t}}{\mathbf{k}} (\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2}) \cdots (\mathbf{t}-\mathbf{1})\, g_k(x),$$

$$\text{(ii)}\quad g_t(x) = \sum_{k=0}^{t-1} x^k \binom{\mathbf{t}}{\mathbf{k}} (\mathbf{k}+\mathbf{1})(\mathbf{k}+\mathbf{2}) \cdots (\mathbf{t}-\mathbf{1})\, g_k(x). \quad ∎$$

5.3. COROLLARY. (i) $g_t(1) = t!$,

$$g_t(-1) = 2^{[t/2]}[t/2]!$$

(*brackets denote the integer part*),

$$g_t'(1) = \tbinom{t}{2}\, t!.$$

(ii) $\deg g_t(x) = t(t-1)$.

(iii) $x^{t(t-1)}g_t(1/x) = g_t(x)$.

(iv) $g_t(x) \equiv \prod_{k=1}^{\infty} (\mathbf{k})^{-(k-1)} = 1 + x^2 + 2x^3 + 4x^4 + 6x^5 + 12x^6 + \cdots$ (mod $x^{t+1}$).

(v) $T_t^*(n+t)$ $(= T_t(n+t))$ *is the coefficient of* $x^n$ *in the expansion of*

$$\prod_{k=1}^{\infty} (\mathbf{k})^{-(k+1)} = 1 + 2x + 6x^2 + 14x^3 + 33x^4 + 70x^5 + 149x^6 + \ldots,$$

*if* $0 \leqslant n \leqslant t$.

*Proof of* (iii). This can be proved using either Theorem 5.1 or Theorem 5.2. From Theorem 5.1, $g_t(x)$ is the sum of terms of the form

$$f(x) = x^{-2t}(\mathbf{t})!^2(F(\lambda)^2 + F(\lambda')^2)$$

(divided by 2 if $\lambda$ is self-conjugate). By (2),

$$f(x) = x^{-2t}(\mathbf{t})!^2(x^\alpha + x^\beta)/(\mathbf{h}_1)^2(\mathbf{h}_2)^2 \cdots (\mathbf{h}_t)^2,$$

where $\alpha = \sum \lambda_i(\lambda_i + 1)$, $\beta = \sum \lambda_i'(\lambda_i' + 1)$.

Now it is easily verified that, for any partition $\lambda$ of $t$,

$$t + \sum h_i = \sum \binom{\lambda_i + 1}{2} + \sum \binom{\lambda_i' + 1}{2}.$$

This is precisely the relation we need to conclude $x^{t(t-1)}f(1/x) = f(x)$. Summing over all $f(x)$'s gives the result.

One can also prove this result from Theorem 5.2 by induction on $t$, transforming the recursion of Theorem 5.2(i) into the recursion of Theorem 5.2(ii). We omit the details. ▮

When we put $x = 1$ in Theorem 5.1 and evaluate the left-hand side by Corollary 5.3(i) and the right-hand side by (2), we get the formula (10). Thus Theorem 5.1 is a kind of "$q$-generalization" of (10) (though we have been using the variable $x$ instead of $q$).

Some small values of $g_t(x)$ are:

$$g_0(x) = 1,$$

$$g_1(x) = 1,$$

$$g_2(x) = 1 + x^2,$$

$$g_3(x) = 1 + x^2 + 2x^3 + x^4 + x^6,$$

$$g_4(x) = 1 + x^2 + 2x^3 + 4x^4 + 2x^5 + 4x^6 + 2x^7 + 4x^8 + 2x^9 + x^{10} + x^{12},$$

$$\begin{aligned} g_5(x) = {} & 1 + x^2 + 2x^3 + 4x^4 + 6x^5 + 7x^6 + 8x^7 + 12x^8 + 12x^9 + 14x^{10} \\ & + 12x^{11} + 12x^{12} + 8x^{13} + 7x^{14} + 6x^{15} + 4x^{16} + 2x^{17} + x^{18} \\ & + x^{20}. \end{aligned}$$

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