COMBINATORICS AND INVARIANT THEORY

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ABSTRACT. Let $\rho: G \rightarrow \text{GL}(V)$ be a representation of a compact Lie group on a finite-dimensional complex vector space $V$. The action of $G$ extends to the polynomial ring $R = \mathbb{C}[V]$. If $\chi$ is an irreducible character of $G$, then let $R^G_\chi$ denote the module of $\chi$-invariants of $G$ over the ring $R^G$ of absolute invariants. Combinatorial techniques are used to investigate the Molien series (or Poincaré series) of $R^G_\chi$ and conditions for $R^G_\chi$ to be Cohen-Macaulay.

1. PRELIMINARIES. Invariant theory is the main part concerned with the following situation. Let $G$ be a group, and let $\rho: G \rightarrow \text{GL}(V)$ be a representation of $G$ as a group of linear transformations of a vector space $V$ of dimension $m$ over a field $k$. When $g \in G$ and $v \in V$, we write $\rho(g)(v)$ for $\rho(g)(v)$. Let $x_1, \ldots, x_m$ be a basis for $V$. Then $G$ acts on the polynomial ring $R = k[x_1, \ldots, x_m]$ by $g \cdot f(x_1, \ldots, x_m) = f(g \cdot x_1, \ldots, g \cdot x_m)$. A polynomial $f$ in $R$ is an invariant of $G$ if $g \cdot f = f$ for all $g \in G$. The set of all invariants forms a subalgebra $R^G$ of $R$ called the ring of invariants of $G$. We are concerned with the problem of "determining" $R^G$, or at least saying as much as possible about the structure of $R^G$.

In order to say anything nice one must put additional restrictions on $G$ and $\rho$. Here we will assume that $k = \mathbb{C}$, that $G$ is a compact Lie group, and that $\rho$ is continuous. (This latter assumption on $\rho$ will be automatically assumed without further comment.) Under these assumptions, the representation of $G$ on $R$ breaks up (uniquely) into irreducible representations, so we have a vector space direct sum $R = \bigoplus Q_i$, where each $Q_i$ is an irreducible $G$-invariant subspace. Each irreducible representation of $G$ is determined by its character $\chi: G \rightarrow \mathbb{C}$. Let $R^G_\chi$ denote the direct sum of those $Q_i$ which correspond to the irreducible character $\chi$. Note that if $\chi$ is the trivial character ($\chi(g) = 1$ for all $g \in G$), then $R^G = R^G_\chi$. It follows from Schur's lemma that each $R^G_\chi$
is a module over the ring \( R^G \), called the module of invariants relative to \( \chi \) or the module of \( \chi \)-invariants. Note that \( R = \bigoplus \nabla \chi^G \), where \( \chi \) ranges over all irreducible characters of \( G \). Note also that if \( \chi \) is a linear character of \( G \) (i.e., a homomorphism \( G \to \mathbb{C}^* = \mathbb{C} \cdot \{0\} \)), then \( R^G_{\chi} = \{ f \in R : g \cdot f = \lambda(f) g \text{ for all } g \in G \} \).

Let \( R_n \) denote the vector space of all polynomials in \( R \) (including 0) which are homogeneous of degree \( n \). Then \( R_n \) becomes a graded algebra, i.e., \( R_n = \bigoplus_{i+j=n} R_{i,j} \). Since \( R_n \) is clearly a \( G \)-invariant subspace, it follows that \( R^G \) has the structure of a graded algebra (viz., \( R^G_n = \bigoplus_{i+j=n} R^G_{i,j} \)) and that each \( R^G_{\chi} \) has the structure of a graded \( R^G \)-module, i.e., \( R^G_{\chi} = \bigoplus_{i+j=n} R^G_{\chi i,j} \). The Molien series (also called the Poincaré series, Hilbert series, or generating function) of \( R^G_{\chi} \) is the formal power series

\[
P_{\chi}(G, \lambda) = (\deg \chi) \sum_{n=0}^{\infty} \chi(n) \lambda^n \]

where \( \chi(n) \) is the multiplicity of \( \chi \) in the action of \( G \) on \( R_n \). Thus \( \dim R^G_{\chi} = (\deg \chi) \chi(n) \). When \( \chi \) is trivial we write \( P(G, \lambda) \) for \( P_{\chi}(G, \lambda) \).

Note that

\[
\sum_{\chi} P_{\chi}(G, \lambda) = (1-\lambda)^{-1}. 
\]

A theorem of Molien [9] [1,5227] gives an expression for \( P_{\chi}(G, \lambda) \) when \( G \) is finite. This result generalizes immediately to compact groups once the rudiments of the representation theory of such groups is known.

1.1 Theorem. Let \( G \) be a compact Lie group acting on \( V \), and let \( \chi \) be an irreducible character of \( G \). Then

\[
P_{\chi}(G, \lambda) = (\deg \chi) \int_{G} \frac{\overline{\chi}(g) dg}{\det(1-\lambda g)}, \tag{1}
\]

where the integral is the Haar integral and the bar denotes complex conjugation.

For instance, when \( G \) is finite then (1) reduces to

\[
P_{\chi}(G, \lambda) = \frac{\deg \lambda}{|G|} \sum_{g \in G} \overline{\chi}(g) \det(1-\lambda g). \tag{2}
\]

A fundamental result of invariant theory states that when \( G \) is compact, \( R^G \) is a finitely-generated \( C \)-algebra (see [10] for a brief history of this problem). The same techniques can be used to show that \( R^G_{\chi} \) is a finitely-generated \( R^G \)-module. It follows from a standard result of commutative algebra [2.Ch.11] that \( P_{\chi}(G, \lambda) \) is a rational function of \( \lambda \). The Krull dimension of \( R^G_{\chi} \), denoted \( \dim R^G_{\chi} \), is defined to be the
order of the pole of $F_{X}(G, \lambda)$ at $\lambda = 1$. It follows from a well-known property of Krull dimension that $\dim R_{X}^{G} = \dim(R_{X}^{G}/\Ann R_{X}^{G})$, where $\Ann R_{X}^{G} = \{ f \in R_{X}^{G}: fR_{X}^{G} = 0 \}$. Clearly $\Ann R_{X}^{G} = 0$ if $R_{X}^{G}$ is non-void, so we have

$$\dim R_{X}^{G} = \dim R_{X}^{G}, \text{ if } R_{X}^{G} \neq \emptyset. \quad (3)$$

Let $d = \dim R_{X}^{G}$. It follows from the Noether normalization lemma that there exist homogeneous elements $\theta_{1}, \ldots, \theta_{d} \in R_{X}^{G}$, necessarily algebraically independent over $\mathbb{C}$, such that $R_{X}^{G}$ is a finitely-generated module over the polynomial ring $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$. The polynomials $\theta_{1}, \ldots, \theta_{d}$ are called a homogeneous system of parameters (h.s.o.p.) for $R_{X}^{G}$. From (3) it follows that if $R_{X}^{G} \neq \emptyset$, then $\theta_{1}, \ldots, \theta_{d}$ is an h.s.o.p. for $R_{X}^{G}$ if and only if $\theta_{1}, \ldots, \theta_{d}$ is an h.s.o.p. for $R_{X}^{G}$. A basic result of commutative algebra [11, p. IV-20, Thm.2] states that $R_{X}^{G}$ is a free $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$-module for some h.s.o.p. $\theta_{1}, \ldots, \theta_{d}$ if and only if $R_{X}^{G}$ is a free $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$-module for every h.s.o.p. $\theta_{1}, \ldots, \theta_{d}$. If $R_{X}^{G}$ is indeed a free $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$-module, then $R_{X}^{G}$ is called a Cohen-Macaulay module. (If $R_{X}^{G}$ itself is a free $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$-module, then $R_{X}^{G}$ is called a Cohen-Macaulay ring.) Suppose that $R_{X}^{G}$ is Cohen-Macaulay, that $\theta_{1}, \ldots, \theta_{d}$ is an h.s.o.p., and that $\eta_{1}, \ldots, \eta_{d}$ is a homogeneous basis for $R_{X}^{G}$ as a $\mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$-module. This may be written symbolically as

$$R_{X}^{G} = \bigoplus_{i=1}^{d} \mathbb{C}[\theta_{1}, \ldots, \theta_{d}] \eta_{i}. \quad (4)$$

One reason that it is nice for (4) to hold is that every element $f$ of $R_{X}^{G}$ can be put in a simple canonical form, viz., $f = \sum p_{i}(\theta_{1}, \ldots, \theta_{d}) \eta_{i}$, where $p_{i} \in \mathbb{C}[\theta_{1}, \ldots, \theta_{d}]$. Let $c_{i} = \deg p_{i}, e_{i} = \deg \eta_{i}$. A simple combinatorial argument shows that

$$F_{X}(G, \lambda) = \begin{bmatrix} c_{i} & e_{i} \\ \sum \lambda^{i} & 1 \end{bmatrix} \bigoplus (1-\lambda)^{-1} (1-\lambda)^{j} \cdot \quad (5)$$

1.2 Example. Let $G$ be the group of order 2 generated by $g = \diag(-1, -1)$ (with respect to some basis $x, y$ for $V$). Let $\chi$ be defined by $\chi(g) = -1$. Then we have

$$R_{X}^{G} = \mathbb{C}[x^{2}, y^{2}] (1 \oplus xy),$$

$$R_{X}^{G} = \mathbb{C}[x^{2}, y^{2}] (x \oplus y).$$

Hence $R_{X}^{G}$ and $R_{X}^{G}$ are Cohen-Macaulay, and we have $F_{X}(G, \lambda) = (1+\lambda^{2})/(1-\lambda^{2})^{2}$, $F_{X}(G, \lambda) = 2\lambda/(1-\lambda^{2})^{2}$. Note that $F(G, \lambda) = F_{X}(G, \lambda) = 1/(1-\lambda)^{2}$.

1.3 Example. Let $G$ be the one-dimensional torus $G = \{ g(u) = \diag(u, u, u^{2}, u^{-1}, u^{-2}) : |u| = 1 \}$. For $i \in \mathbb{Z}$, let $\chi_{i}$ be defined by $\chi_{i}(g(u)) = u^{i}$. Then it can be shown that $R_{X}^{G}$ is Cohen-Macaulay if and only if $i = -1, 0, \text{ or } 1$. The "if" part follows from Theorem 3.5 below.
It is not hard to compute that \( F_\chi^G(G, \lambda) = \mathcal{F}_\chi^G(G, \lambda) = (3\lambda^2 - \lambda^4)/(1 - \lambda^2)^3 \).

There is no way to write this in the form (5) (e.g., the numerator will always have the positive root \( \sqrt{3} \)). Hence \( R^G_\chi \) and \( R^G_{\chi - 2} \) are not Cohen-Macaulay. We omit the proof that \( R^G_\chi \) is not Cohen-Macaulay for \( |\chi| \geq 3 \).

Suppose that \( R^G_\chi \) is Cohen-Macaulay, so that it has a decomposition

\[
R^G_\chi = \bigoplus_{i=1}^t \mathbb{C}[\theta_1, \ldots, \theta_d] \eta_i.
\]

There is then a useful and important \( R^G_\chi \)-module \( \Omega_\chi \) associated with \( R^G_\chi \), which is a kind of "dual" module. The simplest description of \( \Omega_\chi \) is the following. Let \( B = \mathbb{C}[\theta_1, \ldots, \theta_d] \), and let \( \Omega(R^G_\chi) = \text{Hom}_B(R^G_\chi, B) \). This defines \( \Omega(R^G_\chi) \) as a \( B \)-module. The \( R^G_\chi \)-module structure is given by \((f \cdot g)(\phi) = f(\phi g)\) where \( f \in R^G_\chi \), \( \phi \in \Omega(R^G_\chi) \), \( g \in R^G_\chi \). It turns out that \( \Omega(R^G_\chi) \), considered as an \( R^G_\chi \)-module, is independent of the choice of the h.s.o.p. \( \theta_1, \ldots, \theta_d \). When \( \chi \) is trivial so \( R^G_\chi = R^G \), one calls \( \Omega(R^G) \) the canonical module of \( R^G \). See, e.g., [5] [15, \$7] for further information.

A basic combinatorial property of \( \Omega(R^G_\chi) \) which follows from the techniques of [14] is that \( \Omega(R^G_\chi) \) has a natural grading such that its Poincaré series \( P^\chi_\chi(G, \lambda) \) is given by

\[
P^\chi_\chi(G, \lambda) = (-1)^d_\chi \mathcal{Q}^\chi_\chi(G, 1/\lambda)
\]

for some \( \mathcal{Q} \in \mathbb{Z} \), where \( d = \dim R^G_\chi \). If the module \( \Omega(R^G_\chi) \) is a free \( R^G_\chi \)-module of rank one (i.e., isomorphic to \( R^G \) as an \( R^G_\chi \)-module), then \( R^G_\chi \) is called a Gorenstein ring. It follows from (6) that if \( R^G_\chi \) is Gorenstein then \( P(G, 1/\lambda) = (-1)^d_\chi \mathcal{Q}^\chi_\chi(G, \lambda) \) for some \( \mathcal{Q} \in \mathbb{Z} \). It follows from \[14, \text{Thm. 4.4} \] and Theorem 4.1 below that the converse is true:

1.4 THEOREM. A necessary and sufficient condition for \( R^G_\chi \) to be Gorenstein is that \( P(G, 1/\lambda) = (-1)^d_\chi \mathcal{Q}^\chi_\chi(G, \lambda) \) for some \( \mathcal{Q} \in \mathbb{Z} \).

2. FINITE GROUPS. We wish to describe two interesting properties of the modules \( R^G_\chi \) when \( G \) is finite. When \( G \) is not finite these properties need not hold, and in Sections 3 and 4 we will discuss some combinatorial techniques for verifying these properties in special cases.

Consider the following two properties of the pair \((G, \chi)\), where \( G \) is a compact Lie group acting on \( V \), and where \( \chi \) is an irreducible character of \( G \).

Property 1. The module \( R^G_\chi \) is Cohen-Macaulay, and the "dual module" \( \Omega(R^G_\chi) \) is isomorphic to \( R^G_\psi \), where \( \psi \) is the character defined by \( \psi(g) = \overline{\chi}(g)(\det g) \), the bars denoting complex conjugation. (By \( \det g \), we mean the determinant of the action of \( g \) on \( V \).)

Property 2. Let \( d = \dim R^G_\chi \), \( m = \dim V \), and let \( \psi \) be as above. Then

\[
P^\chi_\chi(G, 1/\lambda) = (-1)^d_\chi \mathcal{Q}^\chi_\chi(G, \lambda).
\]
Note that Property 2 yields an expression for the degree of $P_x(G, \lambda)$ (as a rational function), viz.,
$$\deg P_x(G, \lambda) = -(m+1),$$
where $k$ is the least degree of a $\psi$-invariant of $G$. This in turn implies that the largest $c_i$ in (5) is equal to $c_i + \ldots + c_d - m - 1$. For instance, if $\rho : G \to SL(V)$ then $\deg \Omega(G, \lambda) = -m$ when Property 2 holds.

It follows from (6) and an examination of the way in which $\Omega(R_x^G)$ is graded that Property 1 implies that $P_x(G, l/\lambda) = (-1)^{d-1} \lambda R_x^G(G, \lambda)$ for some integer $q$. However, in general it need not follow that $q = m$ (see Example 3.1). When Property 2 holds, this yields strong evidence for Property 1.

For finite groups it is relatively easy to verify Properties 1 and 2.

2.1 THEOREM. If $G$ is finite, then all the pairs $(G, \chi)$ satisfy Properties 1 and 2.

Proof. Hochster and Eagon [6, Prop.13] showed that $R^G_x$ is Cohen-Macaulay. For a relatively self-contained proof, see [15, §3]. The same techniques show $R^G_x$ is Cohen-Macaulay. Alternatively, we can use Lemma 3.2 below, together with the finite sum decomposition $R = \sum R_x^G$, to show that $R^G_x$ is Cohen-Macaulay. The computation of $\Omega(R_x^G)$ follows from the techniques of Watanabe [16] or by a direct argument shown to me by David Eisenbud. It remains to prove Property 2. In view of (2) this is a formal calculation. One immediately sees that
$$\frac{1}{\det(1-\lambda^{-1} g)} = (-1)^{m \cdot m} \frac{(\det g^{-1})}{\det(1-\lambda g^{-1})}.$$ Summing on $g^{-1}$ instead of $g$ gives
$$P_x(G, l/\lambda) = (-1)^{m \cdot m} \frac{(\det g)}{\det(1-\lambda g)} = (-1)^{m \cdot m} P_x(G, \lambda).$$ Since it is easy to see that $\dim R_x^G = m$ or $P_x(G, \lambda) = \chi(\lambda) = 0$, and since $\deg \chi = \deg \psi$, the proof is complete.

3. TORUS GROUPS. We now turn to the case where $G$ is an $s$-dimensional torus, i.e., isomorphic to the group $T = T^s = \{ u \mapsto \text{diag}(u_1, u_2, \ldots, u_s) : |u_i| = 1 \}$. Every continuous representation of $T$ of degree $m$ may be described (after a suitable choice of basis for $V$) by $m$ vectors $a_i \in \mathbb{Z}^s$, $1 \leq i \leq s$. If $\omega = (\omega_1, \ldots, \omega_s) \in \mathbb{Z}^s$, then write $u^\omega = u_1^{\omega_1} \ldots u_s^{\omega_s}$. Then the representation $\rho = \rho(u_1, \ldots, u_s)$ is defined by $\rho(u) \mapsto \text{diag}(u_1^{\omega_1}, \ldots, u_s^{\omega_s})$. The representation $\rho$ is faithful if and only if the greatest common divisor of the $s \times s$ minors of the matrix $\begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}$ is equal to one. We denote the image of $T$ in $GL(V)$ by $T(\mathbf{a}) = T(a_1, \ldots, a_m)$. Every irreducible character $\chi$ of $T^s$ is linear and may be described by a vector $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{Z}^s$. The character $\chi = \chi_\beta$ is given by $\chi(\rho(\mathbf{a})) = u_1^{\omega_1} \cdots u_s^{\omega_s}$. Conversely, if $\rho$ is faithful then any such $\beta$ defines a character. The
module $R^T_{\chi}$ has as a vector space basis all monomials $x_1^a_1 \cdots x_m^a_m$ such that $a_1 u_1 + \cdots + a_m u_m = \beta$.

3.1 EXAMPLE. Let $s = 2$, $m = 3$, $a_1 = (1,1)$, $a_2 = (1,-1)$, $a_3 = (-1,0)$, $\beta = (1,0)$, so $T(\alpha) = \{\text{diag}(uv, uv^{-1}, u^{2i}) : |u| = |v| = 1\}$. The module $R^T_{\chi}$ is spanned (as a vector space) by all monomials $x^y z^c$ with $a+b+c=1$ and $a-b=0$. Hence $R^T_{\chi} = \langle [xyz^2]xyz \rangle$.

We have seen in Example 1.3 that $R^T_{\chi}$ need not be Cohen-Macaulay. A result of Hochster [7] states that $R^T_{\chi}$ is always Cohen-Macaulay. This result is generalized in [8] and also proved in [4]. We will use Hochster's result to give a sufficient (but not necessary) condition for $R^T_{\chi}$ to be Cohen-Macaulay. First we require a simple result from commutative algebra, whose proof is omitted.

3.2 LEMMA. Let $A = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be a Cohen-Macaulay graded algebra over the field $k = \Lambda_0$. Suppose that $A = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_{r-1}$, where $\Lambda_0 = \mathbb{B}$ is a graded subalgebra of $A$ and each $\Lambda_i$ is a graded $\mathbb{B}$-module. Then each $\Lambda_i$ is a Cohen-Macaulay $\mathbb{B}$-module.

3.3 EXAMPLE. Let $r$ be a positive integer and let $0 \leq i \leq r$. Given $A = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$, let $\Lambda_i = \Lambda_1 \oplus \Lambda_{r+1} \oplus \Lambda_{2r+1} \oplus \cdots$. The assumptions of Lemma 3.2 clearly hold, so each $\Lambda_i$ is Cohen-Macaulay if $A$ is Cohen-Macaulay. The subalgebra $\mathbb{B} = \Lambda_0$ is called a Veronese subalgebra of $A$, and we may call $\Lambda_i$ a Veronese module.

3.4 DEFINITION. Let $T(\alpha)$ denote the torus $\text{diag}(u_1^{a_1}, \ldots, u_m^{a_m})$. Define a character $\chi$ of $T(\alpha)$ to be critical if the system of linear equations $z_1 a_1 + \cdots + z_m a_m = \beta$ has a rational solution $(z_1, \ldots, z_m) = (a_1, \ldots, a_m)$ with the following two properties:

(i) $a_i < 0$ for $1 \leq i \leq m$

(ii) If $(b_1, \ldots, b_m)$ is an integer solution to $z_1 a_1 + \cdots + z_m a_m = \beta$ such that $b_i \geq a_i$ for all $i$, then $b_i > 0$ for all $i$.

We now come to the main result of this section.

3.5 THEOREM. If $\chi$ is a critical character of $T$, then $R^T_{\chi}$ is Cohen-Macaulay.

Proof. Suppose in the notation of Definition 3.4 that $a_i = -p_i/q_i$ for integers $p_i > 0$ and $q_i > 0$. Let $\mu$ be the least common multiple of $q_1, q_2, \ldots, q_m$, and define $a'_i = (\mu/q_i) a_i$. Let $T'$ be the torus $T' = T(a'_1, \ldots, a'_m)$. For any vector $v = (v_1, \ldots, v_m)$ of integers satisfying $\sum_{i=1}^m c_i q_i$, let $R'(v)$ be the subspace of $R^{T'}$ spanned by all monomials $x_1^{c_1} \cdots x_m^{c_m}$ such that $c_i \equiv v_i \pmod{q_i}$. Clearly $R'(0,0,\ldots,0) = \mathbb{B}$ is a subalgebra of $R^{T'}$, each
$R'(v)$ is a $B$-module, and $R'^T = \bigcup R'_v$. (The modules $R'_v$ are "generalized Veronese modules.") Let $\mathcal{V}_q$ be the least non-negative residue of $p_i$ modulo $q_i$, and let $s = (q_1, \ldots, q_m)$. We claim that $B \cong R'^T$ and $R'(c) \cong R'^T_{X_B}$. Since $X'^T$ is Cohen-Macaulay by Hochster's result [7], it will follow from Lemma 3.2 that $R'^T_{X_B}$ is a Cohen-Macaulay $R'^T$-module.

To prove the claim, first note that $B \cong R'^T$ is clear, since it follows from the statement that $(z_1, \ldots, z_m)$ is a solution to $\sum z_i q_i = 0$ in non-negative integers $z_i$ if and only if $(q_1 z_1, \ldots, q_m z_m)$ is a solution to $\sum y_i q_i = 0$ in non-negative integers $y_i$ with $y_i \equiv 0 \mod q_i$. Now suppose $(q_1^{-1} c_1, \ldots, q_m^{-1} c_m + s)$ is a solution to $\sum y_i c_i = 0$ in non-negative integers. Hence $\sum (q_i^{-1} c_i + s) y_i c_i = 0$, which is equivalent to

$$\sum (q_i^{-1} c_i + s) y_i c_i = 0.$$  

Now $z_i = \frac{p_i - c_i}{q_i} \geq -\frac{(p_i/q_i)}{c_i} = a_i$. Hence by assumption $z_i - \frac{p_i - c_i}{q_i} \geq 0$.

It follows that the linear transformation $R'(v) \to R'^T$ defined by $x_1^{c_1} \ldots x_m^{c_m} \mapsto x_1^{c_1/q_i} a_1 \ldots x_m^{c_m/q_i} a_m$ is an isomorphism of $R'^T$-modules, so the proof is complete.

Note that the condition on $(a_1, \ldots, a_m)$ in Definition 3.4 is automatically satisfied if $-1 < a_i < 0$. In other words, $X_B$ is critical if $\beta$ lies in the interior of the convex polytope (actually a zonotope) $\Delta_T = \{x \in \mathbb{R}^m : \lambda x < 1, \lambda x > 0\}$, called the critical zonotope of $T$. For instance, if

$T$ is the one-dimensional torus $\{u \in \mathbb{R}^m : a_i \alpha_i, a_j \beta_j \}$ with $a_i > 0$ and $\beta_j > 0$, then an integer $\beta$ belongs to $\Delta_T$ if and only if $-\alpha_i < \beta < \beta_j$. In general, it can be shown that the number of integer vectors $\beta$ in the interior of $\Delta_T$ is given by $\sum (-1)^{s} |X'| h(X)$, where $X$ ranges over all linearly independent subsets of $\{a_1, \ldots, a_m\}$ and where $h(X)$ is equal to the greatest common divisor of the $s$-wise minors of the matrix whose rows are the elements of $X$. For instance, if $s=1$, $a_1=1$, $a_2=-1$, $a_3=1$, $a_4=-1$ (as in Example 1.3), then we obtain $1+1+1+1 = 4$. If $s=2$, $a_1=(1,0)$, $a_2=(-1,1)$, $a_3=(-2,-4)$, $a_4=(0,1)$, then we obtain $1+4+1+6+1+2+1+2+1+1+11 = 37$.

**Remark.** It should be possible to prove Theorem 3.5 using the techniques of [18], but the proof we have given is certainly more elementary.

3.6 Example. Let $s=1$, $a_1=6$, $a_2=-2$, $a_3=-3$, $s=6$. Then $\beta$ is critical (e.g., let $(a_1, a_2, a_3) = (0, \frac{1}{2}, -1)$) but $\beta \notin \Delta_T$.

3.7 Example. Let $s=1$, $a_1=1$, $a_2=-1$, $\beta = 1$. Then $(T, X_B)$ satisfies Property 1 but not Property 2.
3.9 **EXAMPLE.** [13, Ex.8.6] can be used to produce a pair \((T, x_8)\) which satisfies Property 2 but for which \(R^T_x\) is not Cohen-Macaulay. For this example one has \(s = \dim T = 7\), \(\dim R^T_x = 4\), and \(m = 11\).

We now state a strengthening of Theorem 3.5.

3.10 **THEOREM.** Let \(x_8\) be a critical character of the torus \(T\). Then \((T, x_8)\) satisfies Properties 1 and 2.

**Sketch of proof.** Property 2 can be deduced from [13, Thm. 10.2]. Property 1 follows from Theorem 3.5 and the techniques used to prove [14, Thm. 6.7].

4. **COMPACT GROUPS.** We now turn to the consideration of arbitrary compact groups. First we state what is perhaps the deepest known result in invariant theory.

4.1 **THEOREM (Hochster and Roberts [8]).** Let the compact group \(G\) act on a finite-dimensional vector space \(V\). Then \(R^G_V\) is a Cohen-Macaulay ring.

**REMARK.** Hochster and Roberts state their result for linearly reductive linear algebraic groups, but this easily yields the result for compact groups.

As we did for tori, we can ask for a generalization of Theorem 4.1 to \(\chi\)-invariants. We do not know how to prove an analogue of Theorem 3.5 for arbitrary compact \(G\), but by combinatorial reasoning we can give a plausible conjecture. To do so, we now consider Property 2. Recall (Theorem 2.1) that Property 2 was verified for finite groups simply by substituting \(1/\lambda\) for \(\lambda\) in (2). Unfortunately the same proof does not work for arbitrary compact groups because the operation of substituting \(1/\lambda\) for \(\lambda\) does not commute with the integral in (1). In fact, we know such a proof cannot work because Property 2 need not hold (Examples 1.3 and 3.7). We can use Theorem 3.10, however, to give a sufficient condition for Property 2 to hold. First we need to review some facts concerning integration on compact groups.

Let \(G\) be compact and connected, and let \(T\) be a maximal torus of \(G\). Thus \(T\) is isomorphic to \(\{\text{diag} (u_1, u_2, \ldots, u_n) : |u_1| = \ldots = |u_n| = 1\}\). Suppose we have an action \(\rho : G \to GL(V)\). With respect to a suitable basis for \(V\), the image \(\rho(T)\) will be of the form \(T(a) = t_{a_1} \cdots t_{a_m}\), with \(a_1, \ldots, a_m \in \mathbb{Z}^S\). (The vectors \(a_1, \ldots, a_m\) are the "weights" of \(\rho\) with respect to an appropriate basis.) Then there exist non-zero vectors \(e_1, \ldots, e_k \in \mathbb{Z}^S\) depending only on \(G\) and not on \(\rho\) (the "roots" of \(G\) with respect to an appropriate basis – the roots are the non-zero weights of the adjoint representation of \(G\)) such that for any irreducible character \(\chi\) of \(G\) we have

\[
F_x(G, \lambda) = \frac{\deg \chi}{|W|} \int_{T(a)} \frac{(1-\chi_{\rho_k} (g)) \cdots (1-\chi_{\rho_1} (g)) \chi(g) dg}{\det (1-ag)} \quad (7)
\]
where $W$ is the Weyl group of $G$. Equation (7) is an immediate consequence of the Weyl integration formula \cite{[17, Thm. 7.4, D.]} \cite{[1, Thm. 6.1]} and Theorem 1.1. For an example of the use of (7) in computing $F_{\chi}(G, \lambda)$, see \cite{[12, Appendix]}. If $G$ is not connected then there is a straightforward generalization of (7) involving a sum over the components of $G$. For simplicity’s sake we will assume henceforth that $G$ is connected, though our results can be extended to arbitrary compact $G$.

The character $\chi$ when restricted to $T$ breaks up into irreducible characters of $T$, say

$$\chi(g) = \chi_{\gamma_1}(g) + \cdots + \chi_{\gamma_t}(g).$$

We now define $\chi$ to be a critical character of the representation $\rho: G \to GL(V)$ if for all $1 \leq i \leq t$ and all subsets $S$ of $\{\gamma_1, \ldots, \gamma_t\}$, the character $\chi_S$ of the torus $T(\alpha)$ defined by $\omega = \gamma_1 - \cdots - \gamma_j$ is a critical character of $T(\alpha)$.

4.2 EXAMPLE. Let $G = SU(2, \mathbb{C})$. Then $s=1$, $r=2$, $\beta_1 = 1$, $\beta_2 = 2$. For each positive integer $m$ there is a unique irreducible representation $\rho_m$ of degree $m$, and $\alpha_1 = -m+1, \alpha_2 = -m+3, \alpha_3 = -m+5, \ldots, \alpha_m = -1$. Take, for instance, the case $m=6$ and let $\chi$ have degree 8. Then $\alpha_1 = -5, \alpha_2 = -3, \alpha_3 = -1, \alpha_4 = 1, \alpha_5 = 3, \alpha_6 = 5, \gamma_1 = -7, \gamma_2 = -5, \gamma_3 = -3, \gamma_4 = -1, \gamma_5 = 1, \gamma_6 = 3, \gamma_7 = 5, \gamma_8 = 7$. Let $\omega = 7 - (-2) = 9$. Now $\chi_{\omega}$ is not a critical character of the torus diagonal $(u^{-5}, u^{-3}, u^{-1}, u, u^3, u^5)$, so $\chi$ is not a critical character of $\rho_6$. However, any character $\chi$ of $SU(2, \mathbb{C})$ of degree <8 is a critical character of $\rho_6$. More generally, any irreducible character $\chi$ of $SU(2, \mathbb{C})$ of degree

$$\leq \frac{1}{4} m^2 - 2$$

is a critical character of $\rho_m$.

4.3 THEOREM. Let $\rho: G \to GL(V)$ be a representation of the compact connected Lie group $G$ on $V$. If $\chi$ is a critical character of $\rho$, then $(G, \chi)$ has Property 2.

Proof. Write the numerator $(1 - \chi_{\beta_1}(g)) \cdots (1 - \chi_{\beta_t}(g))\overline{\chi}(g)$ of the integrand of (7) as a linear combination of characters of $T$. Thus $F_{\chi}(G, \lambda)$ is a linear combination of terms of the form

$$F_{\chi}(\lambda) = \int_{T(\alpha)} \frac{\overline{\chi}(g) dg}{\det(I - \lambda g)}.$$

The definition of critical character insures that each $\chi_{\gamma}$ is a critical character of $T(\alpha)$. Hence by Theorem 3.10 we have

$$F_{\chi}(1/\lambda) = (-1)^{d_{\gamma}} \int_{T(\alpha)} \frac{\chi_{\gamma}(g) (\det g) dg}{\det(I - \lambda g)},$$

where $d_{\gamma} = \dim R_{\chi_{\gamma}}$. Since by (3) the numbers $d_{\gamma}$ are all equal (say to $d$)
whenever \( R^T \) is non-empty, we get

\[
P_{\chi}(G, 1/\lambda) = (\deg \chi) (-1)^d \lambda^m \int \frac{1}{\det \lambda \varphi} \prod (1 - \alpha_i \varphi(g)) \chi(g) (\det \varphi) \, dg
\]

since a well-known property of roots states that \( \xi \) is a root if and only if \( -\xi \) is a root. Since \( \dim R^\chi = \dim R^\varphi \) (the above formula showing that \( R^\chi \) is void if and only if \( R^\varphi \) is void), it follows from our original definition of Krull dimension that \( (-1)^d = (-1)^{\dim R^\chi} \), completing the proof.

Remark: It is not necessarily true that \( d = \dim R^\chi \), but when \( \chi \) is critical we have shown that \( d = \dim R^\chi \) (mod 2). If \( \chi \) is not critical then this congruence need not hold, e.g., for the adjoint representation of \( SU(2, \mathbb{C}) \) when \( \chi \) is trivial.

Theorem 4.3, together with (6), suggest the following conjecture.

CONJECTURE. Let \( \varphi \) be as in Theorem 4.3, and let \( \chi \) be a critical character of \( \varphi \). Then \((G, \chi)\) satisfies Property 1.

There is one special case for which we can verify the above conjecture. This is when \( \chi \) is trivial and \( \varphi: G \to SL(V) \). In this case, \( \chi = \varphi \) and Theorem 4.3 states that when \( \chi \) is critical we have

\[
P(G, 1/\lambda) = \pm \lambda^m P(G, \lambda).
\]

Then by Theorem 1.4 \( R^G \) is Gorenstein, so \( D(R^G) = R^G \) and \( R^G \) has Property 1. This writer and independently M. Hochster conjecture that for any \( \varphi: G \to SL(V) \), the ring \( R^G \) is Gorenstein. Theorems 1.4 and 4.3 establish this for finite \( G \) and toroidal \( G \). In these cases independent algebraic proofs can be given (4) [16]. Hochster and Roberts [8, Cor. 1.9] show that \( R^G \) is Gorenstein when \( G \) is semisimple, although (8) need not hold. For instance, for the representations \( \alpha_m \) of \( SU(2, \mathbb{C}) \) defined above, we have (8) for \( m \geq 4 \) by Theorem 4.3. However, when \( m = 2 \) we have \( P(G, 1/\lambda) = P(G, \lambda) \), and when \( m = 3 \) we have \( P(G, 1/\lambda) = -\lambda^2 P(G, \lambda) \).

Presumably algebraic techniques will be required to resolve the above conjecture, but at least combinatorial reasoning has led to its formulation and enhanced its believability.

**BIBLIOGRAPHY**


