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Binomial Posets, Möbius Inversion, and Permutation Enumeration*

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A unified method is presented for enumerating permutations of sets and multisets with various conditions on their descents, inversions, etc. We first prove several formal identities involving Möbius functions associated with binomial posets. We then show that for certain binomial posets these Möbius functions are related to problems in permutation enumeration. Thus, for instance, we can explain "why" the exponential generating function for alternating permutations has the simple form $(1 + \sin x)/(\cos x)$. We can also clarify the reason for the ubiquitous appearance of e^x in connection with permutations of sets, and of $\zeta(s)$ in connection with permutations of multisets.

1. BINOMIAL POSETS

We wish to show how the theory of binomial posets, as developed in [11] to formalize certain aspects of the theory of generating functions, can be used to unify and extend some results dealing with the enumeration of special classes of permutations. Although our results about permutations can be proved directly (i.e., without the use of binomial posets), binomial posets provide a means of handling in a routine way complicated recursions and identities involving permutations.

First let us recall the salient facts about posets in general, and binomial posets in particular. Through this paper we use the following notation: $\mathbb{C}$, complex numbers; $\mathbb{N}$, nonnegative integers; $\mathbb{P}$, positive integers; $[n]$, the set $\{1, 2, \ldots, n\}$, where $n \in \mathbb{P}$; $T \subset S$ or $S \supset T$, T is a subset of S, allowing $T = \varnothing$ and $T = S$.

Recall that a poset (or partially ordered set) P is *locally finite* if every interval $[x, y] = \{z \in P: x \leqslant z \leqslant y\}$ of P is finite. Let $\mathcal{S}(P)$ denote the set

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of all intervals of P, where by convention the null set $\varnothing$ is *not* an interval. If P is locally finite, define the *incidence algebra* $I(P)$ of P to be the vector space (over $\mathbb{C}$) of all functions $f: \mathscr{S}(P) \to \mathbb{C}$ endowed with the multiplication (convolution)

$$fg(x, y) = \sum_{z \in [x,y]} f(x, z)\, g(z, y),$$

where we write $f(x, z)$ for $f([x, z])$, etc. This makes $I(P)$ into an associative $\mathbb{C}$-algebra with identity δ given by $\delta(x, y) = \delta_{xy}$ (the Kronecker delta). A function $f \in I(P)$ is invertible if and only if $f(x, x) \neq 0$ for all $x \in P$. The *zeta function* $\zeta \in I(P)$ is defined by $\zeta(x, y) = 1$ for all $x \leqslant y$ in P. The *Möbius function* μ is defined to be ζ^{-1}, and is characterized by the recursion

$$\begin{aligned} \mu(x, x) &= 1, && \text{for all } x \text{ in } P \\ \sum_{z \in [x,y]} \mu(x, z) &= 0, && \text{for all } x < y \text{ in } P. \end{aligned} \tag{1}$$

Recall the result [17, Proposition 6, p. 346], known as "Philip Hall's theorem," that

$$\mu(x, y) = c_0 - c_1 + c_2 \cdots, \tag{2}$$

where c_i is the number of chains $x = x_0 < x_1 < \cdots < x_i = y$ (so $c_0 = 0$ unless $x = y$).

DEFINITION 1.1. A *binomial poset* is a partially ordered set P satisfying the following three conditions:

(a) P is locally finite and contains arbitrarily long finite chains. (A *chain* is a totally ordered subset of P.)

(b) For every interval $[x, y]$ of P, all maximal chains between x and y have the same length, which we denote by $\ell(x, y)$. If $\ell(x, y) = n$, then we call $[x, y]$ an *n-interval*. (The *length* of a chain is one less than its number of elements, so $\ell(x, x) = 0$.)

(c) For all $n \geqslant 0$, any two n-intervals contain the same number $B(n)$ of maximal chains. We call B the *factorial function* of P. Note $B(0) = B(1) = 1$. We write $B(P, n)$ for $B(n)$ if we wish to emphasize the binomial poset P.

Binomial posets were first defined in [11], where they were called "posets of full binomial type." (In [11] it was not assumed that P contains arbitrarily long finite chains, but we have added this condition for convenience in what follows.) Many examples of binomial posets are given in

[11] and [20]. We shall be concerned here mainly with the following class of examples.

EXAMPLE 1.2. Let q be a prime power, and let V_q be a vector space of infinite dimension over the finite field $GF(q)$. If $r \in \mathbb{P}$, define $L_r(V_q)$ to be the set of all r-tuples $(W_1, ..., W_r)$, where each W_i is a finite-dimensional subspace of V_q satisfying $\dim W_1 = \dim W_2 = \cdots = \dim W_r$. Order $L_r(V_q)$ by component-wise inclusion, i.e., $(W_1, ..., W_r) \leqslant (U_1, ..., U_r)$ if $W_i \subset U_i$ for $1 \leqslant i \leqslant r$. Then $L_r(V_q)$ is a binomial poset with factorial function $B(n) = [(1+q)(1+q+q^2) \cdots (1+q+q^2+\cdots+q^{n-1})]^r$. In particular, $L_1(V_q)$ is the lattice of finite dimensional subspaces of V_q. We make the convention (which is not uncommon in these situations) that V_1 is an infinite set whose "finite dimensional subspaces" are simply its finite subsets. Thus $L_r(V_1)$ consists of all r-tuples $(W_1, ..., W_r)$, where W_i is a finite subset of V_1 and $|W_1| = \cdots = |W_r|$, and $B(n) = n!^r$. In particular, $L_1(V_1)$ is the lattice of finite subsets of V_1.

EXAMPLE 1.3. Let P be a locally finite poset with a unique minimal element $\hat{O}$. Suppose that for all $x \leqslant y$ in P, all maximal chains of the interval $[x, y]$ have the same length $\ell(x, y)$. If $k \in \mathbb{P}$, define $P^{(k)}$ to be the subposet of P consisting of all $x \in P$ such that $\ell(\hat{O}, x)$ is divisible by k. Thus $\hat{O} \in P^{(k)}$, and $P^{(1)} = P$. Suppose P is binomial with factorial function $B(P, n)$. Then $P^{(k)}$ is binomial with factorial function

$$B(P^{(k)}, n) = B(P, kn)/B(P, k)^n. \tag{3}$$

If $P = L_r(V_q)$, then we write $P^{(k)} = L_r^{(k)}(V_q)$.

If P is a binomial poset, define $R(P)$ to be the subvector space of $I(P)$ consisting of all functions $f \in I(P)$ satisfying $f(x, y) = f(x', y')$ whenever $\ell(x, y) = \ell(x', y')$. If $f \in R(P)$, then we can write $f(n)$ for $f(x, y)$ when $n = \ell(x, y)$. The following theorem is the fundamental result linking binomial posets with generating functions. For a proof and some consequences and applications, see [11] or [20].

THEOREM 1.4. *Let P be a binomial poset with factorial function $B(n)$. Then $R(P)$ is a subalgebra of $I(P)$* (i.e., *is closed under convolution*). *Moreover, $R(P)$ is isomorphic to the ring $\mathbb{C}[[x]]$ of formal power series in the variable x over $\mathbb{C}$. This isomorphism is given by*

$$f \mapsto \sum_{n=0}^{\infty} f(n)\, x^n/B(n), \tag{4}$$

where $f \in R(P)$.

It follows from Theorem 1.4 that when $f \in R(P)$, the following three conditions are equivalent: (i) f^{-1} exists in $I(P)$, (ii) f^{-1} exists in $R(P)$, (iii) $f(0) \neq 0$. In particular, $\mu \in R(P)$.

Let P be a binomial poset with factorial function $B(n)$. Suppose $[x, y]$ is an n-interval and $a_1, a_2, \ldots, a_k$ are nonnegative integers summing to n. Define

$$\begin{bmatrix} n \\ a_1, a_2, \ldots, a_k \end{bmatrix}$$

to be the number of chains $x = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_k = y$ such that $\ell(x_{i-1}, x_i) = a_i$, $1 \leqslant i \leqslant k$. It follows from Theorem 1.4, or is easily seen by a direct argument, that

$$\begin{bmatrix} n \\ a_1, a_2, \ldots, a_k \end{bmatrix} = \frac{B(n)}{B(a_1)\, B(a_2) \cdots B(a_k)}, \tag{5}$$

and hence $\left[{n \atop a_1, a_2, \ldots, a_k}\right]$ depends only on $n, a_1, a_2, \ldots, a_k$ (not on the particular n-interval $[x, y]$). If $k = 2$, we write $\left[{n \atop a_1}\right]$ instead of $\left[{n \atop a_1, a_2}\right]$, since in what follows there will be no confusion with the case $k = 1$. If we set $A(n) = \left[{n \atop 1}\right] = B(n)/B(n-1)$, then

$$B(n) = A(n)\, A(n-1) \cdots A(1). \tag{6}$$

Equation (6) explains why we call $B(n)$ a "factorial function." Equation (5) is the P-analog of the multinomial coefficient formula

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!\, a_2! \cdots a_k!}, \tag{7}$$

and indeed (5) reduces to (7) when $P = L_1(V_1)$, as defined in Example 1.2. Moreover, when $P = L_1(V_q)$, then (5) becomes the "q-multinomial coefficient," which we denote by

$$\binom{n}{a_1, a_2, \ldots, a_k}_q = \frac{B_q(n)}{B_q(a_1)\, B_q(a_2) \cdots B_q(a_k)}, \tag{8}$$

where

$$B_q(i) = (1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{i-1}). \tag{9}$$

2. Möbius Functions

In this section we will establish some formal results concerning the Möbius function of certain posets $[x, y]_S$ related to binomial posets. In the next section we will show that for certain binomial posets these Möbius functions are related to problems in permutation enumeration.

Let P be a binomial poset with factorial function $B(n)$. Let S be any subset of the positive integers $\mathbb{P}$. If $[x, y]$ is an interval of P, define $[x, y]_S$ to be the subposet of $[x, y]$ consisting of x, y, and all $z \in [x, y]$ satisfying $\ell(x, z) \in S$. Let μ_S denote the Möbius function of the poset $[x, y]_S$. Suppose $\ell(x, y) = n$ and $S \cap [n-1] = \{s_1, s_2, \ldots, s_{r-1}\}$, with $s_1 < s_2 < \cdots < s_{r-1}$. Set $s_0 = 0$, $s_r = n$. It then follows from (2) and (5) that

$$\mu_S(x, y) = \sum (-1)^i \begin{bmatrix} n \\ t_1, t_2, \ldots, t_i \end{bmatrix}, \tag{10}$$

where the sum is over all solutions to $a_1 + a_2 + \cdots + a_i = r$ in positive integers a_i, and where $t_j = s_{a_1+a_2+\cdots+a_j} - s_{a_1+a_2+\cdots+a_{j-1}}$. Hence $\mu_S(x, y)$ depends only on n and S (not on the particular n-interval $[x, y]$), so we write $\mu_S(n)$ for $\mu_S(x, y)$.

THEOREM 2.1. *We have*

$$(-1)^r \mu_S(n) = B(n) \cdot \left| \frac{1}{B(s_i - s_{j-1})} \right| = \left| \begin{bmatrix} n - s_{j-1} \\ s_i - s_{j-1} \end{bmatrix} \right|$$

where $|a_{ij}|$ *denotes the* $r \times r$ *determinant with* (i, j)*-entry* a_{ij} $(i, j \in [r])$, *with the convention* $1/B(-k) = 0$ *if* $k > 0$ *and* $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right] = 0$ *if* $j < 0$.

Proof. Let $a(i, j) = 1/B(s_i - s_{j-1})$. Then $a(i, j) = 0$ if $i < j - 1$, while $a(i, i+1) = 1$. Hence

$$\begin{aligned} &|a(i, j)| \\ &\quad = \sum_\pi (-1)^{\operatorname{sgn}(\pi)} a(1, \pi(1))\, a(2, \pi(2)) \cdots a(r, \pi(r)) \\ &\quad = \sum (-1)^{r-i} a(k_1, 1)\, a(k_2, k_1 + 1)\, a(k_3, k_2 + 1) \cdots a(r, k_{i-1} + 1), \end{aligned} \tag{11}$$

where the last sum is over all subsets $\{k_1, k_2, \ldots, k_{i-1}\} \subset [r-1]$, with $k_1 < k_2 < \cdots < k_{i-1}$. Now

$$\begin{aligned} &(-1)^{r-i} a(k_1, 1)\, a(k_2, k_1 + 1) \cdots a(r, k_{i-1} + 1) \\ &\quad = (-1)^{r-i}/B(s_{k_1})\, B(s_{k_2} - s_{k_1}) \cdots B(n - s_{k_{i-1}}) \\ &\quad = \frac{(-1)^{r-i}}{B(n)} \begin{bmatrix} n \\ t_1, t_2, \ldots, t_i \end{bmatrix} \end{aligned} \tag{12}$$

where $t_j = s_{a_1+a_2+\cdots+a_j} - s_{a_1+a_2+\cdots+a_{j-1}}$, with $a_t = k_t - k_{t-1}$ $(k_0 = 0,$

$k_i = r$), Hence, except for the factor $(-1)^r B(n)$, the summand in the right-hand side of (10) is equal to the right-hand side of (12). Thus $(-1)^r \mu_S(n) = B(n) \cdot |a(i,j)|$, as desired. Now

$$\begin{bmatrix} n \\ t_1, t_2, \dots, t_i \end{bmatrix} = \begin{bmatrix} n \\ t_1 \end{bmatrix}\begin{bmatrix} n - t_1 \\ t_2 \end{bmatrix}\begin{bmatrix} n - t_1 - t_2 \\ t_3 \end{bmatrix} \cdots \begin{bmatrix} n - t_1 - t_2 - \cdots - t_{i-1} \\ t_i \end{bmatrix}.$$

Hence

$$\begin{aligned} &B(n)\, a(k_1, 1)\, a(k_2, k_1 + 1) \cdots a(r, k_{i-1} + 1) \\ &\quad = \begin{bmatrix} n \\ s_{k_1} \end{bmatrix}\begin{bmatrix} n - s_{k_1} \\ s_{k_2} - s_{k_1} \end{bmatrix}\begin{bmatrix} n - s_{k_2} \\ s_{k_3} - s_{k_2} \end{bmatrix} \cdots \begin{bmatrix} n - s_{k_{i-1}} \\ n - s_{k_{i-1}} \end{bmatrix}. \\ &\quad = b(k_1, 1)\, b(k_2, k_1 + 1)\, b(k_3, k_2 + 1) \cdots b(n, k_{i-1} + 1), \end{aligned}$$

where

$$b(i,j) = \begin{bmatrix} n - s_{j-1} \\ s_i - s_{j-1} \end{bmatrix}.$$

It follows from (12) (putting $b(i,j)$ for $a(i,j)$) that

$$B(n) \cdot |a(i,j)| = |b(i,j)|,$$

completing the proof. ∎

THEOREM 2.2. *Let P be a binomial poset with factorial function $B(n)$, and let $S \subset \mathbb{P}$. Then*

$$-\sum_{n=1}^{\infty} \mu_S(n)\, x^n/B(n) = \left[\sum_{n=1}^{\infty} x^n/B(n)\right]\left[1 + \sum_{n \in S} \mu_S(n)\, x^n/B(n)\right].$$

Proof. Define a function $\chi : \mathbb{N} \to \{0, 1\}$ by $\chi(n) = 1$ if $n = 0$ or $n \in S$, $\chi(n) = 0$ otherwise. Then the recursion (1) for Möbius functions gives

$$\mu_S(n) = -\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} \mu_S(i)\, \chi(i), \qquad n \geqslant 1,$$

while $\mu_S(0) = 1$. Hence

$$-\mu_S(n)(1 - \chi(n)) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \mu_S(i)\, \chi(i), \qquad n \geqslant 1, \tag{13}$$

which translates into the generating function identity

$$-\sum_{n=0}^{\infty} \frac{\mu_S(n)\, x^n}{B(n)} + \sum_{n=0}^{\infty} \frac{\mu_S(n)\, \chi(n)\, x^n}{B(n)} = \left[\sum_{n=0}^{\infty} \frac{x^n}{B(n)}\right]\left[\sum_{n=0}^{\infty} \frac{\mu_S(n)\, \chi(n)\, x^n}{B(n)}\right] - 1.$$

This is clearly equivalent to the desired identity, completing the proof. ∎

Theorem 2.2 shows that the generating function $\sum_{n=1}^{\infty} \mu_S(n)\, x^n/B(n)$ can be computed in terms of $\sum_{n\in S} \mu_S(n)\, x^n/B(n)$. Thus it is natural to ask for what sets S can $\sum_{n\in S} \mu_S(n)\, x^n/B(n)$ be "explicitly" computed.

THEOREM 2.3. *Let $k \in \mathbb{P}$ and let $S = k\mathbb{P} = \{kn : n \in \mathbb{P}\}$. If P is a binomial poset with factorial function $B(n)$, then*

$$1 + \sum_{n\in S} \mu_S(n)\, x^n/B(n) = \left[\sum_{n=0}^{\infty} x^{kn}/B(kn)\right]^{-1}. \tag{14}$$

First Proof. Suppose $n \in S$ and $0 \leqslant i \leqslant n$. Then $i \in S$ if and only if $n - i \in S$. Moreover, if $n \notin S$ and $0 \leqslant i \leqslant n$, then $i \in S$ only if $n - i \notin S$. Hence from (13) we get for any $n \geqslant 0$,

$$\delta_{0n} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \mu_S(i)\, \chi(i)\, \chi(n - i),$$

where δ_{0n} is the Kronecker delta. Thus

$$1 = \left[\sum_{n=0}^{\infty} \frac{\chi(n)\, x^n}{B(n)}\right]\left[\sum_{n=0}^{\infty} \frac{\mu_S(n)\, \chi(n)\, x^n}{B(n)}\right]$$

$$= \left[\sum_{n=0}^{\infty} \frac{x^{kn}}{B(kn)}\right]\left[1 + \sum_{n\in S} \frac{\mu_S(n)\, x^n}{B(n)}\right],$$

completing the proof.

Second proof. Let $P^{(k)}$ be the binomial poset of Example 1.3, with factorial function $B'(n)$. If μ' is the Möbius function of $P^{(k)}$, then it follows from (3) and from Theorem 1.4 that

$$\sum_{n=0}^{\infty} \mu'(n)\, x^n/B'(n) = \left[\sum_{n=0}^{\infty} x^n/B'(n)\right]^{-1}.$$

But $\mu'(n) = \mu_S(kn)$ and by (3) we have $B'(n) = B(kn)/B(k)^n$. Hence

$$1 + \sum_{n=1}^{\infty} \mu_S(n)(B(k)\, x)^n/B(kn) = \left[\sum_{n=0}^{\infty} (B(k)\, x)^n/B(kn)\right]$$

If we put x^k for $B(k)\, x$, we get the desired result. ∎

Although our first proof of Theorem 2.3 is more elementary than our second proof, the second proof gives more insight into why such a simple formula as (14) holds. Namely, it asserts that the zeta and Möbius functions are inverses of each other in the poset $P^{(k)}$.

Combining Theorems 2.2 and 2.3, we obtain the following corollary.

COROLLARY 2.4. *Let P be a binomial poset with factorial function $B(n)$. Let $k \in \mathbb{P}$ and $S = k\mathbb{P}$. Then*

$$-\sum_{n=1}^{\infty} \mu_S(n)\, x^n/B(n) = \left[\sum_{n=1}^{\infty} x^n/B(n)\right]\left[\sum_{n=0}^{\infty} x^{kn}/B(kn)\right]^{-1}.$$

In the case $k = 1$ of Corollary 2.4, we have $S = \mathbb{P}$ and $\mu_S(n) = \mu(n)$, the Möbius function μ of P evaluated at any n-interval. Corollary 2.4 becomes

$$\sum_{n=0}^{\infty} \mu(n)\, x^n/B(n) = \left[\sum_{n=0}^{\infty} x^n/B(n)\right]^{-1}. \tag{15}$$

According to Theorem 1.4, Equation (15) asserts that the Möbius function μ of P is the inverse of the zeta function ζ. We now give a generalization of (15). Let P be any locally finite poset such that for all $x \leqslant y$ in P, any two maximal chains of the interval $[x, y]$ have the same length $\ell(x, y)$. Given $S \subset \mathbb{P}$, we can define $[x, y]_S$ and $\mu_S(x, y)$ exactly as we did for binomial posets. Let t be a variable, and define $g, h \in I(P)$ (the incidence algebra of P) by

$$\begin{aligned} g(x, y) &= \begin{cases} 1, & \text{if } x = y \\ (1 + t)^{n-1}, & \text{if } \ell(x, y) = n \geqslant 1 \end{cases} \\ h(x, y) &= \begin{cases} 1, & \text{if } x = y \\ \sum_S \mu_S(x, y)\, t^{n-1-s}, & \text{if } x < y, \end{cases} \end{aligned} \tag{16}$$

where $n = \ell(x, y)$, where S ranges over all subsets of $[n - 1]$, and where $s = |S|$.

LEMMA 2.5. *We have $g = h^{-1}$, as elements of $I(P)$.*

Proof. By (2), we have

$$\mu_S(x, y) = \sum (-1)^m,$$

where the sum is over all chains $x = x_0 < x_1 < \cdots < x_m = y$ such that for each $i \in [m - 1]$, $\ell(x, x_i) \in S$. Thus when $x < y$ we can write $h(x, y)$ as a double sum $h(x, y) = \sum_S \sum (-1)^m\, t^{n-1-s}$. Interchanging the order of summation, we have $h(x, y) = \sum (-1)^m \sum_S t^{n-1-s}$, where for a fixed chain $x = x_0 < x_1 < \cdots < x_m = y$, S ranges over all subsets of $[n-1]$ containing the $m-1$ numbers $\ell(x, x_i)$, $1 \leqslant i \leqslant m-1$. Hence by the binomial theorem $\sum_S t^{n-1-s} = (1+t)^{n-m}$. It follows that

$$h(x, y) = \sum (-1)^m f(x_0, x_1) f(x_1, x_2) \cdots f(x_{m-1}, x_m), \tag{17}$$

where the sum is over all chains $x = x_0 < x_1 < \cdots < x_m = y$ and where $f(u, v) = (1+t)^{\ell(u,v)-1}$, $u < v$. If we define $f(u, u) = 0$, then from (17) we have $h = \delta - f + f^2 - f^3 + \cdots = (\delta + f)^{-1}$, where δ is the multiplicative identity of $I(P)$. Since $\delta + f = g$, the proof is complete. ∎

When P is a binomial poset, Lemma 2.5 can be transformed into a power series identity via Theorem 1.4. Specifically, given a binomial poset P with factorial function $B(n)$, let t be an indeterminate and define for $n \in \mathbb{P}$,

$$h_n(t) = \sum_S \mu_S(n)\, t^{n-1-s}, \tag{18}$$

where S ranges over all subsets of $[n-1]$ and where $s = |S|$. Thus $h_n(t) = h(x, y)$ (as defined by (16)), where $[x, y]$ is an n-interval. For instance, $h_1(t) = -1$, $h_2(t) = -t + B(2) - 1$, $h_n(0) = \mu(n)$. Combining Lemma 2.5 and Theorem 1.4, there results:

COROLLARY 2.6. *We have*

$$1 + \sum_{n=1}^{\infty} h_n(t)\, x^n/B(n) = \left[1 + \sum_{n=1}^{\infty} (1+t)^{n-1} x^n/B(n)\right]^{-1}$$

$$= (1+t)\left[t + \sum_{n=0}^{\infty} \frac{(1+t)^n x^n}{B(n)}\right]^{-1}.$$

3. PERMUTATION ENUMERATION

In the previous section we developed some formal properties of binomial posets. We now give these results a more concrete significance by giving a combinatorial interpretation to the numbers $\mu_S(n)$ for certain binomial posets P.

Let $\pi = (a_1, a_2, \ldots, a_n)$ denote a permutation of $[n]$. An *ascent* (or *rise*) of π is a pair (a_i, a_{i+1}) with $a_i < a_{i+1}$. The *ascent set* $A(\pi)$ of π is defined by

$$A(\pi) = \{i \in [n-1] \colon a_i < a_{i+1}\}.$$

An *inversion* of π is a pair (i, j) with $i < j$ and $a_i > a_j$. Let $i(\pi)$ denote the number of inversions of π, so $0 \leqslant i(\pi) \leqslant \binom{n}{2}$.

THEOREM 3.1. *Let q be a prime power or $q = 1$, and let $r \in \mathbb{P}$. Let $L_r(V_q)$ be the binomial poset of Example 1.2. Finally let S be any subset of the positive integers. Then*

$$(-1)^m \mu_S(n) = \sum q^{i(\pi_1)+i(\pi_2)+\cdots+i(\pi_r)},$$

where the sum is over all r-tuples $(\pi_1, \pi_2, ..., \pi_r)$ of permutations of $[n]$ such that

$$A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) = [n-1] - S,$$

and where $m = 1 + |S \cap [n-1]|$.

Proof. Given $T \subset [n-1]$, let

$$f_r(T) = \sum q^{i(\pi_1)+i(\pi_2)+\cdots+i(\pi_r)}, \tag{19}$$

where the sum is over all r-tuples $(\pi_1, \pi_2, ..., \pi_r)$ of permutations of $[n]$ such that

$$A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) \supset T \tag{20}$$

Define $g_r(T)$ in the same way as $f_r(T)$ except (20) is replaced with $A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) = T$. Thus we wish to prove $g_r([n-1] - T) = (-1)^m \mu_T(n)$, where $m = 1 + |T|$. Suppose $[n-1] - T = \{s_1, s_2, ..., s_{k-1}\}$, with $s_1 < s_2 < \cdots < s_{k-1}$. Let $s_0 = 0$, $s_k = n$. (Thus $m + k = n + 1$.) We claim that

$$f_r(T) = \binom{n}{s_1 - s_0, s_2 - s_1, ..., s_k - s_{k-1}}_q^r, \tag{21}$$

where the right-hand side of (21) is the q-multinomial coefficient of (8), raised to the rth power. Now the sum in (19) ranges independently over each π_i subject to $A(\pi_i) \supset T$, since (20) holds if and only if each $A(\pi_i) \supset T$. It follows that $f_r(T) = f_1(T)^r$, so it suffices to prove (21) for $r = 1$. For simplicity we write $f(T) = f_1(T)$, $g(T) = g_1(T)$.

We prove (21) (when $r = 1$) by induction on k. The statement is trivial when $k = 1$, for then $T = [n-1]$, so $A(\pi) \supset T$ is equivalent to $\pi = (1, 2, ..., n)$. Thus $f(T) = 1 = \binom{n}{n}_q$, as desired.

Given $[n-1] - T = \{s_1, s_2, ..., s_{k-1}\}$, with $0 = s_0 < s_1 < s_2 < \cdots < s_{k-1} < s_k = n$, any $\pi = (a_1, a_2, ..., a_n)$ satisfying $A(\pi) \supset T$ can be obtained as follows. The last $n - s_{k-1}$ (or $s_k - s_{k-1}$) elements of π can be any of the $\binom{n}{n-s_{k-1}}$ choices C of $n - s_{k-1}$ elements from $[n]$, but they must be arranged in π in increasing order. The remaining s_{k-1} elements of π form a permutation π' of $[n] - C$ satisfying $A(\pi') \supset T'$, where $[s_{k-1} - 1] - T' = \{s_1, s_2, ..., s_{k-2}\}$. Let $C = \{b_1, b_2, ..., b_j\}$ ($j = n - s_{k-1}$) with $1 \leqslant b_1 < b_2 < \cdots < b_j \leqslant n$. Then

$$i(\pi) = i(\pi') + \sum_{c=1}^{j} (n - b_c - j + c).$$

Abbreviating $\sum_{c=1}^{j} (n - b_c - j + c)$ to $F(C)$, it follows by induction that

$$f(T) = \binom{s_{k-1}}{s_1 - s_0, s_2 - s_1, ..., s_{k-1} - s_{k-2}}_q \sum_C q^{F(C)}, \tag{22}$$

where the sum is over all subsets C of $[n]$ of cardinality $j = n - s_{k-1}$. Now as C ranges over all sequences $1 \leqslant b_1 < b_2 < \cdots < b_j \leqslant n$, we see that if $d_c = n - b_c - j + c$, then $d_1, d_2, ..., d_j$ ranges over all sequences satisfying $0 \leqslant d_j \leqslant d_{j-1} \leqslant \cdots \leqslant d_1 = n - j$. It follows that $\sum_C q^{F(C)} = \sum_r p_{j,n-j}(r)\, q^r$, where $p_{j,n-j}(r)$ is the number of partitions of r into at most j parts, with largest part at most $n - j$. But it is well-known (see, e.g., [13, Theorem 349] or [9, Example 8, p. 117]) that

$$\sum_r p_{j,n-j}(r)\, q^r = \binom{n}{j}_q.$$

Hence from (22) we get

$$\begin{aligned} f(T) &= \binom{s_{k-1}}{s_1 - s_0, s_2 - s_1, ..., s_{k-1} - s_{k-2}}_q \binom{n}{j}_q \\ &= \binom{n}{s_1 - s_0, s_2 - s_1, ..., s_k - s_{k-1}}_q. \end{aligned}$$

This proves (21).

We now wish to express $g_r(S)$ in terms of the $f_r(T)$'s. Clearly,

$$f_r(S) = \sum g_r(T) \qquad (S \subset T \subset [n-1]).$$

Hence by the Principle of Inclusion–Exclusion,

$$g_r(S) = \sum (-1)^{|T-S|} f_r(T) \qquad (S \subset T \subset [n-1]). \tag{23}$$

Comparing (21) and (23) with (10), we conclude $g_r(S) = (-1)^m \mu_S(n)$, where $m = 1 + |S|$. This completes the proof. ∎

Remarks. We have given a straightforward, elementary proof of Theorem 3.1. It is possible to give more combinatorial proofs using lattice-theoretic techniques which, though less elementary, give more insight into the structure of the posets $L_r(V_q)$. For instance, the special case $r = 1$, $q = 1$ (so $L_1(V_1)$ is the lattice of finite subsets of an infinite set) follows from Theorem 9.1 and Proposition 14.1 of [18], in the special case that P is a disjoint union of points. The case $r = 1$ of Theorem 3.1 (but q arbitrary) follows from letting L be the lattice of subspaces of an n-dimensional vector space over $GF(q)$ in [19, Theorem 1.2].

We can substitute the expression for $\mu_S(n)$ given by Theorem 3.1 into our results in Section 2 to obtain explicit information about the enumeration of certain classes of permutations. We now discuss some of the resulting formulas in more detail.

COROLLARY 3.2. *Let q be an indeterminate, and let $S = \{s_1, s_2, ..., s_{m-1}\}$ be a subset of $[n-1]$, where $n \in \mathbb{P}$ and $s_1 < s_2 < \cdots < s_{m-1}$. Set $s_0 = 0$, $s_m = n$. Let $r \in \mathbb{P}$, and define*

$$\alpha = \sum q^{i(\pi_1)+i(\pi_2)+\cdots+i(\pi_r)},$$

where the sum is over all r-tuples of permutations of $[n]$ such that $A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) = [n-1] - S$. Then

$$\alpha = B_q(n)^r \cdot \left| \frac{1}{B_q(s_i - s_{j-1})^r} \right| = \left| \binom{n - s_{j-1}}{s_i - s_{j-1}}_q^r \right|, \qquad i, j \in [m],$$

where $B_q(s)$ and $\binom{k}{j}_q$ are given by (8) and (9), with the usual conventions $1/B(0) = 1$, $1/B(-s) = 0$ if $s > 0$, and $\binom{k}{j}_q = 0$ if $j < 0$.

The proof is immediate from Theorem 2.1 and Theorem 3.1. If we put $q = 1$ and $r = 1$ in Corollary 3.2, we get that the number of permutations π of $[n]$ with $A(\pi) = [n-1] - S$ is equal to

$$n! \mid 1/(s_i - s_{j-1})! \mid = \left| \binom{n - s_{j-1}}{s_i - s_{j-1}} \right|.$$

This is a well-known result of MacMahon [14, Vol. I, p. 190], rediscovered by Niven [15] and further studied by de Bruijn [2] and others.

COROLLARY 3.3. *Fix $k, r \in \mathbb{P}$. Let $n \in \mathbb{P}$, and let q be an indeterminate. Define*

$$f_{krq}(n) = \sum q^{i(\pi_1)+i(\pi_2)+\cdots+i(\pi_r)},$$

where the sum is over all r-tuples $(\pi_1, \pi_2, ..., \pi_r)$ of permutations of $[n]$ satisfying

$$A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) = [n-1] - k\mathbb{P}.$$

Then

$$\sum_{n=1}^{\infty} (-1)^{[(n-1)/k]} f_{krq}(n)\, x^n/B_q(n)^r$$
$$= \left[\sum_{n=1}^{\infty} x^n/B_q(n)^r\right]\left[\sum_{n=0}^{\infty} x^{kn}/B_q(kn)^r\right]^{-1}, \tag{24}$$

where $[(n-1)/k]$ *is the greatest integer symbol and where* $B_q(n) = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$. ∎

The proof follows from Corollary 2.4 and Theorem 3.1, upon observing that the set $k\mathbb{P} \cap [n-1]$ has $[(n-1/k)]$ elements. We can eliminate the unsightly factor $(-1)^{[(n-1)/k]}$ from (24) by the following device.

LEMMA 3.4. *Let* $k \in \mathbb{P}$, *and let* $F(x) = \sum_{n=1}^{\infty} (-1)^{[(n-1)/k]} f(n)\, x^n$ *be a power series over* $\mathbb{C}$. *Then*

$$\sum_{n=1}^{\infty} f(n)\, x^n = (2/k) \sum_{j=0}^{k-1} F(\zeta^{1+2j}x)/(\zeta^{1+2j}-1), \tag{25}$$

where $\zeta = e^{\pi i/k}$.

Proof. Let $1 \leqslant t \leqslant k$. Then $[(kn+t-1)/k] = n$. Hence

$$(1/k) \sum_{j=0}^{k-1} \zeta^{-2tj} F(\zeta^{2j}x) = \sum_{n=1}^{\infty} (-1)^n f(kn+t)\, x^{kn+t}.$$

Therefore

$$(\zeta^{-t}/k) \sum_{j=0}^{k-1} \zeta^{-2tj} F(\zeta^{2j+1}x) = \sum_{n=1}^{\infty} f(kn+t)\, x^{kn+t}. \tag{26}$$

Summing (26) for $1 \leqslant t \leqslant k$, we obtain

$$\sum_{n=1}^{\infty} f(n)\, x^n = \frac{1}{k} \sum_{t=1}^{k} \zeta^{-t} \sum_{j=0}^{k-1} \zeta^{-2tj} F(\zeta^{2j+1}x)$$
$$= \frac{1}{k} \sum_{j=0}^{k-1} F(\zeta^{2j+1}x) \sum_{t=1}^{k} \zeta^{-t(1+2j)}$$
$$= \frac{1}{k} \sum_{j=0}^{k-1} F(\zeta^{2j+1}x) \frac{\zeta^{-(1+2j)}(1-\zeta^{-k(1+2j)})}{1-\zeta^{-(1+2j)}}$$
$$= \frac{2}{k} \sum_{j=0}^{k-1} \frac{F(\zeta^{2j+1}x)}{(\zeta^{1+2j}-1)}.$$

This completes the proof. ∎

Combining Corollary 3.3 and Lemma 3.4, we obtain:

COROLLARY 3.5. *Let $f_{krq}(n)$ and $B_q(n)$ be as in Corollary* 3.3, *and let $F(x)$ equal the right-hand side of* (24). *Then*

$$\sum_{n=1}^{\infty} f_{krq}(n)\, x^n/B_q(n)^r = (2/k) \sum_{j=0}^{k-1} F(\zeta^{2j+1}x)/(\zeta^{1+2j} - 1), \tag{27}$$

where $\zeta = e^{\pi i/k}$. ∎

We now turn to special cases of Corollaries 3.3 and 3.5. If we set $k = 1$, it follows from (24) or (27) that

$$1 + \sum_{n=1}^{\infty} f_{1rq}(n)\, x^n/B_q(n)^r = \left[\sum_{n=0}^{\infty} (-1)^n\, x^n/B_q(n)^r\right]^{-1}. \tag{28}$$

When in addition $r = q = 1$, then $f_{111}(n)$ is equal to the number of permutations π of $[n]$ such that $A(\pi) = \varnothing$. Clearly $\pi = (n, n-1, \ldots, 1)$, so $f_{111}(n) = 1$. This agrees with (28), which reduces to $e^x = (e^{-x})^{-1}$. If we put $q = 1$ and $r = 2$ in (28), then we get a recent result of Carlitz, Scoville, and Vaughan [8, Theorem 1]. Equation (28) is thus a q-generalization of [8, Theorem 1], and also a generalization to arbitrary r-tuples of permutations.

Let us now consider the case k arbitrary and $q = r = 1$. Then $f_{k11}(n)$ is the number of permutations $(a_1, a_2, \ldots, a_n)$ of $[n]$ such that $a_i > a_{i+1}$ if and only if k divides i. The function $f_{k11}(n)$ was considered by Carlitz in [4], and his equation (1.11) is equivalent to the case $r = q = 1$ of (24). Note that $f_{211}(n)$ is the number of "alternating permutations" of $[n]$, i.e., permutations $(a_1, a_2, \ldots, a_n)$ satisfying $a_1 < a_2 > a_3 < a_4 > \cdots$. Now we have that

$$\sum_{n=0}^{\infty} x^{kn}/B_1(kn) = \sum_{n=0}^{\infty} x^{kn}/(kn)! = (1/k) \sum_{j=0}^{k-1} e^{\rho^j x},$$

where $\rho = e^{2\pi i/k}$. It follows (using (27)) that the generating function $\sum_{n=1}^{\infty} f_{k11}(n)\, x^n/n!$ can be explicitly expressed in terms of exponential functions. In theory this expression can be "rationalized" so that all non-real numbers disappear and only the functions exp, sin, and cos appear. For instance, when $k = 2$ we obtain

$$1 + \sum_{n=1}^{\infty} f_{211}(n)\, x^n/n! = \sec x + \tan x \tag{29}$$

a famous result of André [1] (see also [9, p. 258; 12, p. 89; 7, Eq. (3.9)]). When $k = 3$ some computation reveals

$$1 + \sum_{n=1}^{\infty} \frac{f_{311}(n)\, x^n}{n!} = \frac{3 + 2(3)^{1/2}\, e^{x/2} \sin(x(3)^{1/2}/2)}{e^{-x} + 2e^{x/2} \cos(x(3)^{1/2}/2)} . \tag{30}$$

Note that Corollary 3.5 immediately provides us with q-analogs of (29) and (30). For instance, the q-analog of (29) can be written in the form

$$1 + \sum_{n=1}^{\infty} \frac{f_{21q} x^n}{B_q(n)} = \frac{1 + \sum_{n=0}^{\infty} (-1)^n\, x^{2n+1}/B_q(2n+1)}{\sum_{n=0}^{\infty} (-1)^n\, x^{2n}/B_q(2n)} .$$

We now turn to the combinatorial significance of Corollary 2.6. Let P be the binomial poset $L_r^{(k)}(V_q)$ of Examples 1.2 and 1.3. Let μ be the Möbius function of P and let μ' be that of $L_r(V_q)$. If $S \subset [n-1]$, then it follows immediately from the definition of μ_S that $\mu_S(n) = \mu'_{kS}(kn)$, where $k_S = \{ki : i \in S\}$. Now define

$$G_{nkrq}(t) = \sum_{s=0}^{n-1} g_{nksrq} t^s,$$

where

$$g_{nksrq} = \sum q^{i(\pi_1) + i(\pi_2) + \cdots + i(\pi_r)},$$

where the sum is over all r-tuples $(\pi_1, \pi_2, ..., \pi_r)$ of permutations of $[kn]$ such that

(i) $A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) \supset [kn-1] - k[n-1]$,

and

(ii) $|A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r)| = s + (k-1)\, n$.

It follows from Theorem 3.1 that

$$G_{nkrq}(t) = (-1)^n\, h_n(-t),$$

where h_n is defined by (18) (with respect to the poset P). Hence by Corollary 2.6 there follows:

COROLLARY 3.6. We have

$$1 + \sum_{n=1}^{\infty} G_{nkrq}(t)\, x^n/B_q(kn)^r = \left[1 - \sum_{n=1}^{\infty} (t-1)^{n-1}\, x^n/B_q(kn)^r\right]^{-1}.$$

We briefly discuss some special cases of Corollary 3.6. If we put $t = 1$, then

$$G_{nkrq}(1) = \sum q^{i(\pi_1)+i(\pi_2)+\cdots+i(\pi_r)},$$

where the sum is over all r-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of permutations of $[kn]$ such that $A(\pi_1) \cap A(\pi_2) \cap \cdots \cap A(\pi_r) \supset [kn-1] - k[n-1]$. From (21) it follows that

$$G_{nkrq}(1) = \binom{nk}{k, k, \ldots, k}_q^r = B_q(nk)^r / B_q(k)^{nr}.$$

This agrees with Corollary 3.6, which asserts that

$$1 + \sum_{n=1}^{\infty} \frac{G_{nkrq}(1)\, x^n}{B_q(kn)^r} = \left(1 - \frac{x}{B_q(k)^r}\right)^{-1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{B_q(nk)^r}{B_q(k)^{nr}} \frac{x^n}{B_q(nk)^r}.$$

Now consider the case $k = 1$. Here condition (i) above is vacuous. Thus for instance when $q = r = 1$, g_{n111} is simply the number of permutations of $[n]$ with exactly s ascents. Hence g_{n111} is an Eulerian number $A(n, s+1)$ and $G_{n11}(t)$ is the Eulerian polynomial $A_n(t)$ (see, e.g., [9, Section 6.5; 12; 16]), and Corollary 3.6 is equivalent to the well-known result (e.g., [9, p. 244; 12, p. 215; 12, p. 68])

$$1 + \sum_{n=1}^{\infty} \frac{A_n(t)}{t} \frac{x^n}{n!} = \frac{1-t}{e^{x(t-1)} - t} = \left[1 - \sum_{n=1}^{\infty} \frac{(t-1)^{n-1} x^n}{n!}\right]^{-1}.$$

It follows that $G_{n11q}(t)$ is a q-generalization of the Eulerian polynomials and that g_{n111q} is a q-generalization of the Eulerian numbers. This generalization differs from that of Carlitz [2] but agrees with the generalization alluded to in [19, pp. 208–209]. The generating function in Corollary 3.6 can be expressed in terms of exponential functions whenever $r = q = 1$ (k arbitrary). For instance,

$$1 + \sum_{n=1}^{\infty} \frac{G_{n211}(t)\, x^n}{(2n)!} = \frac{t-1}{t - \cosh\,(x(t-1))^{1/2}} \tag{31}$$

Thus, to be explicit, the coefficient of $t^m x^n/(2n)!$ in (31) is the number of permutations $(a_1, a_2, \ldots, a_{2n})$ of $[2n]$ such that $a_{2i-1} < a_{2i}$ for all $i \in [n]$ and $a_{2j} < a_{2j+1}$ for exactly m values of $j \in [n-1]$. Finally we mention that the special case $k = q = 1$, $r = 2$ of Corollary 3.6 is equivalent to [8, Theorem 2].

4. Dirichlet Posets

A Dirichlet poset is a generalization of a binomial poset; Dirichlet posets are associated with power series in more than one variable in the same way that binomial posets are associated with power series in one variable. We shall consider here only the simplest kinds of Dirichlet posets, viz., those which are products of binomial posets. For a more general discussion, see [11, Section 7]. If $\{P_i\}$ is a collection of posets, where i ranges over some index set J, and where each P_i has a unique minimal element $\hat{0}_i$, define the (restricted) direct product $P = \Pi P_i$ as follows. The elements of P are the "J-tuples" $(x_i)_{i \in J}$, such that $x_i = \hat{0}_i$ for all but finitely many i. P is given the usual product ordering, i.e., $(x_i) \leqslant (y_i)$ if $x_i \leqslant y_i$ for all $i \in J$. Now suppose that each P_i is a binomial poset with factorial function $B_i(n)$ (not to be confused with the $B_q(n)$ of (9)). Let $I(P)$ be the incidence algebra of $P = \Pi P_i$, and define a subset $D(P)$ of $I(P)$ as follows. $D(P)$ consists of all functions $f \in I(P)$ such that $f((x_i), (y_i)) = f((x_i'), (y_i'))$ (where $(x_i) \leqslant (y_i)$ and $(x_i') \leqslant (y_i')$ in P) if for all i we have $\ell(x_i, y_i) = \ell(x_i', y_i')$. Hence if $f \in D(P)$, then $f((x_i), (y_i))$ depends only on the J-tuple (n_i) of nonnegative integers given by $n_i = \ell(x_i, y_i)$. Thus we write $f(n_i) = f(n_i)_{i \in J}$ for $f((x_i), (y_i))$. The fundamental theorem about the set $D(P)$, a generalization of Theorem 1.4, is the following (see also [11, Proposition 7.2], for a somewhat different point of view):

Theorem 4.1. *The set $D(P)$ is a subalgebra of $I(P)$. Moreover, $D(P)$ is isomorphic to the ring $\mathbb{C}[[x_i]]$ of formal power series in the variables x_i, $i \in J$, over $\mathbb{C}$. This isomorphism is given by*

$$f \mapsto \sum_{(n_i)} f(n_i) \frac{x_1^{n_1}}{B_1(n_1)} \frac{x_2^{n_2}}{B_2(n_2)} \cdots,$$

where the sum is over all J-tuples (n_i) of nonnegative integers such that $n_i = 0$ for all but finitely many i.

Example 4.2. The archetypal example of a Dirichlet poset, and the one that explains the terminology "Dirichlet poset", is the following. Let J be the set of all positive integers, and for each $i \in J$ let $P_i = \mathbb{N}$, the nonnegative integers with their usual order. Each P_i is a binomial poset with $B_i(n) = 1$. Thus $P = \Pi P_i$ is a Dirichlet poset. P is isomorphic in a natural way to the lattice of positive integers ordered by divisibility. Indeed, if $(x_i) \in P$ (so each $x_i \in \mathbb{N}$), then we associate (x_i) with the positive integer $\Pi p_i^{x_i}$, where p_i is the ith prime. This establishes the desired isomorphism, so henceforth we shall identify P with the positive integers $\mathbb{P}$. Then $D(P)$ consists of all functions $f \in I(P)$ such that $f(m, n) = f(m', n')$ whenever $n/m = n'/m'$. Thus we can write $f(n/m)$ for $f(m, n)$, and the correspondence of Theorem 4.1 takes the form

$$f \mapsto \sum_{n \in \mathbb{P}} f(n)\, x_1^{\nu_1} x_2^{\nu_2} \cdots,$$

where $n = p_1^{\nu_1} p_2^{\nu_2} \cdots$. If we put p_i^{-s} for x_i, then we get $f \mapsto \sum_{n \in \mathbb{P}} f(n)\, n^{-s}$. Hence $D(P)$ is isomorphic, in a natural way, to the algebra of formal Dirichlet series over $\mathbb{C}$. This explains the terminology "Dirichlet poset."

In order to apply Dirichlet posets to permutation enumeration, we need an analog of Theorem 3.1 for special Dirichlet posets. Such an analog is provided by the next theorem. First we must extend our terminology on permutations to the case of multisets. A *finite multiset* M of positive integers may be regarded as a collection of positive integers with repetitions allowed, such that the total number of elements appearing in the collection is finite. We write $M = \{1^{\nu_1}, 2^{\nu_2}, \ldots\}$ to indicate that i is repeated ν_i times, so $\sum \nu_i < \infty$ since M is finite. We write $\Omega = \sum \nu_i = |M|$. A *permutation* π of M is a linear arrangement $(a_1, a_2, \ldots, a_\Omega)$ $(\Omega = |M|)$ of the elements of M. For instance, there are $7!/2!3!2! = 210$ permutations of $M = \{1^2, 2^3, 4^2\}$, of which $(4, 2, 2, 1, 4, 1, 2)$ is one. A *descent* of $\pi = (a_1, a_2, \ldots, a_\Omega)$ is a pair (a_i, a_{i+1}) with $a_i > a_{i+1}$, and the *descent set* $D(\pi)$ of π is defined by $D(\pi) = \{i \in [\Omega - 1]\colon a_i > a_{i+1}\}$. For instance, $D(4, 2, 2, 1, 4, 1, 2) = \{1, 3, 5\}$.

THEOREM 4.3. *Let P be the positive integers ordered by divisibility, and let $[m, n]$ be an interval of P, so $m \mid n$. Suppose $n/m = p_1^{\nu_1} p_2^{\nu_2} \cdots$, and let $\Omega = \Omega(n) = \nu_1 + \nu_2 + \cdots$. Let S be any subset of $\mathbb{P}$, and let $t = 1 + |S \cap [\Omega - 1]|$. Finally let μ be the Möbius function of P. Then $(-1)^t \mu_S(m, n)$ is equal to the number of permutations π of the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots\}$ satisfying $D(\pi) = S \cap [\Omega - 1]$.* ∎

We will omit the proof of Theorem 4.3, since it can be proved in a manner analogous to Theorem 3.1. It is also equivalent to combining [18, Theorem 9.1] and [18, Proposition 14.1] in the special case that P is a disjoint union of chains and ω is a natural labeling (as defined in [18]).

We can now state multiset analogs of Corollary 3.3 and Corollary 3.6.

COROLLARY 4.4. *Fix $k \in \mathbb{P}$. If $n = p_1^{\nu_1} p_2^{\nu_2} \cdots$ and $\Omega = \sum \nu_i$, define $f_k(n)$ to be the number of permutations π of the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots\}$ satisfying $D(\pi) = [\Omega - 1] \cap k\mathbb{P}$. Also define $\epsilon_k(n) = (-1)^{[(\Omega-1)/k]}$. Then*

$$\sum_{n=2}^{\infty} \epsilon_k(n) f_k(n)\, n^{-s} = (\zeta(s) - 1)\Big(\sum_m m^{-s}\Big)^{-1}, \tag{32}$$

where m ranges over all positive integers $p_1^{\nu_1}p_2^{\nu_2}\cdots$ *satisfying* $k \mid \Omega(n)$ *(where as usual* $\Omega(n) = \sum \nu_i$*). Here* $\zeta(s) = \sum_1^\infty n^{-s}$ *is the Riemann zeta function.* ∎

The proof is analogous to the proof of Corollary 3.3 and will be omitted.

Let us examine more closely the special cases $k = 1$ and $k = 2$ of Corollary 4.4. When $k = 1$ we seek permutations π of $\{1^{\nu_1}, 2^{\nu_2},...\}$ satisfying $D(\pi) = [\Omega - 1]$. Such a permutation will exist (in which case it will be unique) if and only if each $\nu_i \leqslant 1$, i.e., if and only if n is squarefree. Hence $f_1(n) = |\mu(n)|$ (where μ is the ordinary number-theoretic Möbius function), and $\epsilon_k(n) f_k(n) = (-1)^{\Omega-1} |\mu(n)| = -\mu(n)$. Thus (32) becomes

$$-\sum_{n=2}^{\infty} \mu(n)\, n^{-s} = (\zeta(s) - 1)\, \zeta(s)^{-1},$$

which agrees with the well-known formula $\sum_1^\infty \mu(n)\, n^{-s} = \zeta(s)^{-1}$.

We now turn to the case $k = 2$ of Corollary 4.4. If $n = p_1^{\nu_1}p_2^{\nu_2}...$, then $f_2(n)$ is the number of "weakly alternating" permutations of the multiset $M = \{1^{\nu_1}, 2^{\nu_2},...\}$, i.e., the number of permutations $\pi = (a_1, a_2, ..., a_\Omega)$ of M satisfying $a_1 \leqslant a_2 > a_3 \leqslant a_4 > \cdots$. Set $\gamma(s) = \sum_m m^{-s}$, where m ranges over all positive integers satisfying $2 \mid \Omega(m)$. It is well-known and easily proved that $\zeta(2s)/\zeta(s) = \sum_1^\infty (-1)^{\Omega(n)}\, n^{-s}$. Hence $\gamma(s) = (1/2)\, [\zeta(s) + (\zeta(2s)/\zeta(s))]$ and (32) becomes

$$\sum_{n=2}^{\infty} \epsilon_2(n) f_2(n)\, n^{-s} = \frac{\zeta(s) - 1}{\gamma(s)} = \frac{2\zeta(s)(\zeta(s) - 1)}{\zeta(s)^2 + \zeta(2s)}. \tag{33}$$

This implies the identity

$$\sum_{d \mid n} \epsilon_2(d) f_2(d) \left[\frac{\tau(n/d) + 1}{2}\right] = \tau(n) - 1,$$

where by convention $f_2(1) = 0$, where $\tau(n)$ is the number of divisors of n, and where brackets denote the integer part. We can rewrite (33) in the form

$$1 - \sum_{n=2}^{\infty} \epsilon_2(n) f_2(n)\, n^{-s} = \frac{1}{\gamma(s)} - \frac{\zeta(s) - \gamma(s)}{\gamma(s)} \tag{34}$$

$$= \frac{2}{\Pi(1 - p^{-s})^{-1} + \Pi(1 + p^{-s})^{-1}} - \frac{\Pi(1 - p^{-s})^{-1} - \Pi(1 + p^{-s})^{-1}}{\Pi(1 - p^{-s})^{-1} + \Pi(1 + p^{-s})^{-1}},$$

where the products range over all primes p. A little thought shows that we can eliminate the unsightly $\epsilon_2(n)$ from (34) as follows:

$$1+\sum_{n=2}^{\infty} f_2(n)\, n^{-s} = \frac{2+(1/i)[\Pi(1-ip^{-s})^{-1} - \Pi(1+ip^{-s})^{-1}]}{\Pi(1-ip^{-s})^{-1}+\Pi(1+ip^{-s})^{-1}}$$

$$= \frac{2\Pi(1+p^{-2s})+(1/i)[\Pi(1+ip^{-s})-\Pi(1-ip^{-s})]}{\Pi(1+ip^{-s})+\Pi(1-ip^{-s})}. \tag{35}$$

Equation (35) is equivalent to a result of Carlitz [5] (corrected in [6]). It can be written in the equivalent form

$$1+\sum_{n=2}^{\infty} f_2(n)\, n^{-s} = \Big[1+\sum_m (-1)^{\frac{1}{2}(\Omega(m)-1)}\, m^{-s}\Big]\Big[\sum_j (-1)^{\frac{1}{2}\Omega(j)}\, j^{-s}\Big]^{-1}, \tag{36}$$

where m ranges over all positive integers such that $\Omega(m)$ is odd, while j ranges over all positive integers such that $\Omega(j)$ is even. Equations (35) or (36) can be extended to arbitrary k, but the formulas become messy. We do have, however, the relatively simple result

$$1+\sum_n f_k(n)\, n^{-s} = \Big[1+\sum_n (-1)^{\Omega(n)/k}\, n^{-s}\Big]^{-1}, \tag{37}$$

where in both sums n ranges over all integers satisfying $n \geqslant 2$ and $k \mid \Omega(n)$. The easy deduction of (37) from (32) is left to the reader.

We now give a multiset analog of Corollary 3.6. Fix $k \in \mathbb{P}$, and let $n = p_1^{\nu_1} p_2^{\nu_2} \cdots$ where $k \mid \Omega(n)$. Define g_{nks} to be the number of permutations π of the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots\}$ satisfying

(i) $D(\pi) \subset k\mathbb{P}$,

and

(ii) $|D(\pi)| = (\Omega(n)/k) - 1 - s$

Now let t be an indeterminate and define

$$G_{nk}(t) = \sum_{s=0}^{(\Omega(n)/k)-1} g_{nks} t^s,$$

COROLLARY 4.5. *We have*

$$1+\sum_n G_{nk}(t)\, n^{-s} = \Big[1-\sum_n (t-1)^{(\Omega(n)/k)-1}\, n^{-s}\Big]^{-1},$$

where in both sums n ranges over all integers satisfying $n \geqslant 2$ and $k \mid \Omega(n)$.

The proof is analogous to the proof of Corollary 3.6 and will be omitted. Note that if we put $t = 0$ in Corollary 4.5, we obtain (37). The case $k = 1$ of Corollary 4.5 is equivalent to a result of Dillon and Roselle [10, (1.4)–(1.6)].

References

1. D. André, Developpements de sec x et de tang x, *C. R. Acad. Sci. Paris* **88** (1879), 965–967.
2. N. G. deBruijn, Permutations with given ups and downs, *Nieuw Arch. Wisk.* (3) **18** (1970), 61–65.
3. L. Carlitz, q-Bernoulli and Eulerian numbers, *Trans. Amer. Math. Soc.* **76** (1954), 332–350.
4. L. Carlitz, Permutations with prescribed pattern, *Math. Nachr.* **58** (1973), 31–53.
5. L. Carlitz, Enumeration of up-down sequences, *Discrete Math.* **4** (1973), 273–286.
6. L. Carlitz, Addendum, *Discrete Math.* **5** (1973), 291.
7. L. Carlitz, Permutations and sequences, *Advances in Math.* **14** (1974), 92–120.
8. L. Carlitz, R. Scoville, and T. Vaughan, Enumeration of pairs of permutations and sequences, *Bull. Amer. Math. Soc.* **80** (1974), 881–884.
9. L. Comtet, "Advanced Combinatorics," Reidel, Dordrecht/Boston, 1974.
10. J. F. Dillon and D. P. Roselle, Simon Newcomb's problem, *SIAM J. Appl. Math.* **17** (1969), 1086–1093.
11. P. Doubilet, G.-C. Rota, and R. P. Stanley, On the foundations of combinatorial theory (VI): The idea of generating function, *in* "Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II: Probability Theory," University of California (1972), 267–318.
12. D. Foata and M. P. Schützenberger, "Théorie Géométrique des Polynômes Eulériens," Lecture Notes in Mathematics, **138**, Springer-Verlag, Berlin, 1970.
13. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 4th ed., Oxford, 1960.
14. P. A. MacMahon, "Combinatory Analysis," Vols. 1–2, Cambridge, 1916; reprinted by Chelsea, New York, 1960.
15. I. Niven, A combinatorial problem of finite sequences, *Nieuw Arch. Wisk.* (3) **16** (1968), 116–123.
16. J. Riordan, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.
17. G.-C. Rota, On the foundations of combinatorial theory, I. Theory of Möbius functions, Z. Wahrscheinlichkeitstheorie, **2** (1964), 340–368.
18. R. P. Stanley, Ordered structures and partitions, *Mem. Amer. Math. Soc.* **119**, 1972.
19. R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.
20. R. P. Stanley, Generating functions, MAA Studies in Combinatorics (G.-C. Rota, ed.), to appear.