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# FINITE LATTICES AND JORDAN-HÖLDER SETS[1)]

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## 1. Introduction

In this paper we extend some aspects of the theory of 'supersolvable lattices' [3] to a more general class of finite lattices which includes the upper-semimodular lattices. In particular, all conjectures made in [3] concerning upper-semimodular lattices will be proved. For instance, we will prove that if $L$ is finite upper-semimodular and if $L'$ denotes $L$ with any set of 'levels' removed, then the Möbius function of $L'$ alternates in sign. Familiarity with [3] will be helpful but not essential for the understanding of the results of this paper. However, many of the proofs are identical to the proofs in [3] (once the machinery has been suitably generalized) and will be omitted.

## 2. Admissible labelings

Let $L$ be a finite lattice with bottom $\hat{0}$ and top $\hat{1}$, such that every maximal chain of $L$ has the same length $n$. Hence $L$ has a rank function $\varrho$ satisfying $\varrho(\hat{0})=0$, $\varrho(\hat{1})=n$, and $\varrho(y)=1+\varrho(x)$ whenever $y$ covers $x$ in $L$. We call $L$ a *graded* lattice.

Let $I$ denote the set of join-irreducible elements of $L$. A *labeling* $\omega$ of $L$ is any map $\omega: I \to \mathbf{P}$, where $\mathbf{P}$ denotes the positive integers. A labeling $\omega$ is said to be *natural* if $z, z' \in I$ and $z \leq z'$ implies $\omega(z) \leq \omega(z')$. If $x<y$ in $L$ and $\omega$ is a fixed labeling of $L$, define

$$\gamma(x, y)=\min\{\omega(z) \mid z \in I, x<x \vee z \leq y\}.$$

Thus, $\gamma(x, y)$ is the least label of a join-irreducible which is less than or equal to $y$ but not less than or equal to $x$. Note that $\gamma(x, y)$ is always defined since $y$ is a join of join-irreducibles. We are now able to make the key definition of this paper. A labeling $\omega$ is said to be *admissible* if whenever $x<y$ in $L$, there is a *unique* unrefinable chain $x=x_0<x_1<\cdots<x_m=y$ between $x$ and $y$ (so $m=\varrho(y)-\varrho(x)$) such that

$$\gamma(x_0, x_1) \leq \gamma(x_1, x_2) \leq \cdots \leq \gamma(x_{m-1}, x_m). \tag{1}$$

We then call the pair $(L, \omega)$ an *admissible* lattice. Our motivation for this definition is that admissibility seems to be the weakest condition for which Theorem 3.1 holds.

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The idea for this definition came from [3, Cor. 1.3] and its relation to [3, Thm. 2.1]. Our present Theorem 3.1 is a generalization of [3, Thm. 2.1].

We first note a simple property of admissible labelings.

2.1. PROPOSITION. *Let $\omega$ be an admissible labeling of a finite graded lattice $L$. Then $\omega$ is natural.*

*Proof.* Suppose $z, z' \in I$ with $z < z'$ and $\omega(z) > \omega(z')$. Since $z$ is join-irreducible, it covers a unique element $x$. Similarly $z'$ covers a unique element $y$. Hence any unrefinable chain between $x$ and $z'$ has the form $x < z = y_0 < y_1 < \cdots < y_m = y < z'$ (possibly $m = 0$ so $z = y$). Since $z$ is join-irreducible, it follows from the definition of $\gamma$ that $\gamma(x, z) = \omega(z)$. Similarly $\gamma(y, z') = \omega(z')$. Since $\omega(z) > \omega(z')$, $\omega$ cannot be admissible. □

We know of two main classes of admissible lattices. The first class is given by the next proposition.

First recall that a lattice $L$ of finite length is said to *upper-semimodular* if it is a graded lattice whose rank function $\varrho$ satisfies $\varrho(x) + \varrho(y) \geq \varrho(x \vee y) + \varrho(x \wedge y)$ for all $x, y \in L$. Equivalently, $L$ is upper-semimodular if whenever $x$ covers $y$, then $x \vee z$ covers or equals $y \vee z$, for all $x, y, z \in L$.

2.2. PROPOSITION. *Let $L$ be a finite upper-semimodular lattice and $\omega$ a natural labeling of $L$ such that whenever $z$ and $z'$ are incomparable join-irreducibles then $\omega(z) \neq \omega(z')$. (Such a labeling of $L$ is clearly possible; in fact, an injective natural labeling can always be found.) Then $\omega$ is admissible.*

To prove this result, we first need a lemma.

2.3. LEMMA. *Let $(L, \omega)$ satisfy the hypotheses of Proposition 2.2, and let $x < y$ in $L$. Let $z$ be a minimal element of the set $J$ of all join-irreducibles $z'$ of $L$ satisfying $\omega(z') = \gamma(x, y)$ and $x < x \vee z' \leq y$. ($J$ is not empty by definition of $\gamma(x, y)$.) Then $x \vee z$ covers $x$.*

*Proof.* Let $I$ denote as before the set of join-irreducibles of $L$. Let $I' \subseteq I$ be the set of all $z' \in I$ satisfying $z' < z$. Let $z' \in I'$. Since $z' < z$, $x \leq x \vee z' \leq y$. Since $\omega$ is natural, $\omega(z') \leq \omega(z)$. If $\omega(z') < \omega(z)$, then by definition of $\gamma(x, y)$ we cannot have $x < x \vee z' \leq y$, so $x = x \vee z'$. On the other hand, if $\omega(z') = \omega(z)$, then by hypothesis we cannot have $x < x \vee z' \leq y$, so once again $x = x \vee z'$. Thus $x = x \vee z'$ for all $z' \in I'$. Let $w = \bigvee_{z' \in I'} z'$. Since $z$ is join-irreducible, $w < z$. Since $x = x \vee z'$ for all $z' \in I'$, we have $x \vee w = x$.

Now if $z$ doesn't cover $w$, then $w < w' < z$ for some $w' \in L$. But then there is a new join-irreducible $v < z$ such that $w < w \vee v \leq w'$, contradicting the definition of $w$. Hence $z$ covers $w$. But by upper-semimodularity, if $z$ covers $w$, then $x \vee z$ covers or equals $x \vee w = x$. By assumption, $x < x \vee z$, so $x \vee z$ covers $x$. □

*Proof of Proposition* 2.2. Let $x < y$ in $L$, and let $m = \varrho(y) - \varrho(x)$. We first show the existence of an unrefinable chain $x = x_0 < x_1 < \cdots < x_m = y$ between $x$ and $y$ satisfying (1). Let $z_1$ be a minimal element of the set $J_1$ of join-irreducibles $z$ satisfying $\omega(z)=\gamma(x_0, y)$ and $x<x\vee z\leq y$. Let $x_1=x_0\vee z_1$. By Lemma 2.3, $x_1$ covers $x_0$, while by definition $x_1\leq y$.

If $m=1$, we are done, so assume $m\geq 2$. Let $z_2$ be a minimal element of the set $J_2$ of join-irreducibles $z$ satisfying $\omega(z)=\gamma(x_1, y)$ and $x_1<x_1\vee z\leq y$. Let $x_2=x_1\vee z_2$. Once again by Lemma 2.3 $x_2$ covers $x_1$, while again by definition $x_2\leq y$. Now by definition of $\gamma(x_0, y)$ we have $\omega(z_1)=\gamma(x_0, y)\leq\omega(z_2)=\gamma(x_1, y)$. Continuing in this way, after $m$ steps we get an unrefinable chain $x=x_0<x_1<\cdots<x_m=y$ satisfying $\gamma(x_0, y)\leq\gamma(x_1, y)\leq\cdots\leq\gamma(x_{m-1}, y)$. But clearly by definition of $\gamma$ and the $x_i$'s, $\gamma(x_i, y)=\gamma(x_i, x_{i+1})$. Hence we have constructed a chain $C$ satisfying (1).

It remains to show the uniqueness of $C$. We shall prove the following two results:

(i) If $x'\in L$ is such that $x'$ covers $x$, $x'\leq y$, and $\gamma(x, x')=\gamma(x, y)$, then $x'=x_1$;

(ii) If $x=x'_0<x'_1<\cdots<x'_m=y$ is any unrefinable chain satisfying (1), then $\gamma(x'_1, x)=\gamma(x, y)$.

Thus (i) and (ii) imply that $x'_1$ is uniquely determined, viz., $x'_1=x_1$ (where $x_1=x_0\vee z_1$ as defined above). Hence the proof of the proposition follows by induction on $m$.

*Proof of* (i). Suppose $x''\neq x'$ also is such that $x''$ covers $x$, $x''\leq y$, and $\gamma(x, x'')=\gamma(x, y)$. Thus there exist $z', z''\in I$ such that $\omega(z')=\omega(z'')=\gamma(x, y)$, $x\vee z'=x'$, $x\vee z''=x''$. Since $x'$ and $x''$ both cover $x$, they are incomparable. Hence $z'$ and $z''$ are incomparable. Thus by hypothesis $\omega(z')\neq\omega(z'')$, a contradiction. Hence $x''$ cannot exist.

*Proof of* (ii). Let $x=x'_0<x'_1<\cdots<x'_m=y$ be an unrefinable chain satisfying (1). Hence $\gamma(x, x'_1)\geq\gamma(x, y)$. Suppose $\gamma(x, x'_1)>\gamma(x, y)$. Let $z\in I$ satisfy $\omega(z)=\gamma(x, y)$ and $x<x\vee z\leq y$. Let $i$ be the least positive integer for which $x\vee z\leq x'_i$. (Clearly $i$ exists since $x\vee z\leq x'_m$.) Then $x'_{i-1}\vee z=x'_i$, so $\gamma(x'_{i-1}, x'_i)=\gamma(x, y)<\gamma(x, x'_1)$. Thus (1) cannot hold. □

The second main class of admissible lattices are the *supersolvable lattices* [3]. If $L$ is a finite lattice and $\Delta$ a maximal chain of $L$, we call the pair $(L, \Delta)$ a supersolvable lattice (or *SS*-lattice) if the sublattice of $L$ generated by $\Delta$ and any chain in $L$ is distributive. It is easily seen that if $(L, \Delta)$ is an *SS*-lattice, then $L$ is graded (cf. [3, § 1]).

2.4. PROPOSITION. *Let $(L, \Delta)$ be an SS-lattice with $\Delta$ given by $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$. Define a labeling $\omega: I\to\mathbf{P}$ by letting $\omega(z)$ be the least positive integer $t$ for which $z\leq x_t$. Then $\omega$ is admissible*

*Proof.* Recall that an interval $[u, v]$ of a lattice is *prime* if it contains exactly two elements, i.e., if $v$ covers $u$. In a distributive lattice $D$, two prime intervals $[x, y]$ and $[u, v]$ are said to be *projective* if there is a unique join-irreducible $z$ such that $y=x\vee z$ and $v=u\vee z$. This is easily seen to be equivalent to the usual definition of projectivity (e.g., [1, p. 14]) if one thinks of $D$ as being coordinatized by a ring of sets.

If $y$ covers $x$ in $L$, then it is easily seen [3, p. 198] that there is a unique positive integer $t$, which we denote by $\gamma'(x, y)$, for which the prime intervals $[x, y]$ and $[x_{t-1}, x_t]$ are projective in the distributive lattice $D_{xy}$ generated by $\Delta$ and $\{x, y\}$. In [3, Cor. 1.3], it was shown that for any $x' < y'$ in $L$, there is a unique unrefinable chain $x' = x'_0 < x'_1 < \cdots < x'_m = y$ between $x'$ and $y'$ such that

$$\gamma'(x'_0, x'_1) < \gamma'(x'_1, x'_2) < \cdots < \gamma'(x'_{m-1}, x'_m).$$

Hence it suffices to prove that $\gamma(x, y) = \gamma'(x, y)$ whenever $y$ covers $x$.

We shall need the following elementary facts concerning projectivity in a finite distributive lattice $D$. The proofs are immediate from the above definition of projectivity.

(a) The prime intervals $[x, y]$ and $[u, v]$ are projective in $D$ if and only if $x = (x \vee u) \wedge y$ and $y = (x \vee v) \wedge y$.

(b) If $[x, y]$ and $[u, v]$ are projective prime intervals in $D$, then $y \nleq u$.

(c) Suppose $[w, z]$ is a prime interval in $D$ and $z$ is join-irreducible. If $y$ covers $x$ in $D$ and $z \leq y$, $z \nleq x$, then $[w, z]$ and $[x, y]$ are projective.

We proceed to prove that if $y$ covers $x$ in $L$, then $\gamma(x, y) = \gamma'(x, y)$. By definition of $\gamma(x, y)$, there is a join-irreducible $z$ satisfying $x \vee z = y$ and $\omega(z) = \gamma(x, y)$. Let $w$ be the unique element of $L$ covered by $z$, and set $s = \omega(z)$. By (c), $[w, z]$ and $[x_{s-1}, x_s]$ are projective in the distributive lattice $D_{wz}$ generated by $\Delta$ and $\{w, z\}$, so $\gamma'(w, z) = s$. If $z'$ is a join-irreducible of $L$ such that $z' < z$, then it follows from (b) (taking $D$ to be generated by $\Delta$ and $\{z, z'\}$) that $\omega(z') \neq \omega(z)$. Since $\omega(z') \leq \omega(z)$, thus $\omega(z') < \omega(z)$.

We claim that $w \leq x$. It suffices to prove $z' \leq x$ for all join-irreducibles $z' \leq w$. If $z'$ is such a join-irreducible, then by the above $\omega(z') < \omega(z)$. Hence by the definition of $z$, $x \vee z' = y$. But $x \vee z' \leq y$ since $z \leq y$. Since $y$ covers $x$, we must have $z' \leq x$. Hence $w \leq x$.

We need to show $\omega(z) = t$, i.e., $s = t$. By (a) and (c) this is equivalent to $w = (w \vee x_{t-1}) \wedge z$ and $z = (w \vee x_t) \wedge z$. Since $\gamma'(x, y) = t$, we know by (a) that

$$x = (x \vee x_{t-1}) \wedge y \tag{2}$$

$$y = (x \vee x_t) \wedge y. \tag{3}$$

Since $w \leq x$, $z \nleq x$, $z \leq y$, and $z$ covers $w$, from (2) we get $w = x \wedge z = (x \vee x_{t-1}) \wedge z$. Thus since $w \leq x$ and $w \leq z$, $w \leq (w \vee x_{t-1}) \wedge z \leq (x \vee x_{t-1}) \wedge z = w$ so $w = (w \vee x_{t-1}) \wedge z$ as desired. To prove the other equality $z = (w \vee x_t) \wedge z$, we need to show $w \vee x_t \geq z$. Since $w$ is the only element which $z$ covers, this is equivalent to $x_t \nleq w$. But if $x_t \leq w$, then $x_t \leq x$ since $w \leq x$. From (3) this would imply $y = x \wedge y = x$, a contradiction. □

It follows from Proposition 2.4 that the theory of $SS$-lattices, as developed in [3], is a special case of the theory of admissible lattices. A large class of examples of $SS$-lattices, some of which are not semimodular, is given in [3, §2].

### 3. Jordan-Hölder sequences

Let $(L, \omega)$ be an admissible finite graded lattice. Let $x \leq y$ in $L$, and suppose $K$ is an unrefinable chain in $L$ between $x$ and $y$ given by $x = x_0 < x_1 < \cdots < x_m = y$. Define the *Jordan-Hölder sequence* (or *J-H sequence*) associated with $K$ to be the sequence $a_1, a_2, \ldots, a_m$ of positive integers given by $a_i = \gamma(x_{i-1}, x_i)$. We shall denote this sequence by $\pi_K$ and shall write

$$\pi_K = (a_1, a_2, \ldots, a_m).$$

In [3] $\pi_K$ was called a '*J-H* permutation' but here repetitions among the $a_i$ are possible.

Now define the *Jordan-Hölder set* (or *J-H set*) $\mathscr{J}_{xy}(L, \omega)$ of $(L, \omega; x, y)$ (denoted $\mathscr{J}_{xy}$ for short) to be the set of all *J-H* sequences $\pi_K$, *including repetitions*, as $K$ ranges over all unrefinable chains between $x$ and $y$. It follows from the definition of an admissible labeling that there is a unique element $\pi_K = (a_1, \ldots, a_m)$ of $\mathscr{J}_{xy}$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_m$. If $x = \hat{0}$ and $y = \hat{1}$, we denote $\mathscr{J}_{xy}(L, \omega)$ simply by $\mathscr{J}(L, \omega)$ or just $\mathscr{J}$, and call it the *J-H set of* $(L, \omega)$.

If $k \in \mathbf{P}$, let $\mathbf{k}$ denote the set $\{1, 2, \ldots, k\}$. We also write $S = \{m_1, m_2, \ldots, m_s\}_<$ to signify that $S = \{m_1, m_2, \ldots, m_s\}$ and $m_1 < m_2 < \cdots < m_s$. Suppose $L$ is a finite graded lattice and $[x, y]$ is an interval of $L$ of length (rank) $m$, i.e., $\varrho(y) - \varrho(x) = m$. If $\{m_1, \ldots, m_s\}_< = S \subseteq \mathbf{m-1}$, define $\alpha_{xy}(S)$ to be the number of chains

$$x < y_1 < \cdots < y_s < y$$

in $L$ satisfying $\varrho(y_i) - \varrho(x) = m_i$, $i = 1, 2, \ldots, s$. Thus if $S = \{k\}$, then $\alpha_{xy}(S)$ is the number of elements $z$ of $[x, y]$ of rank $k$ in $[x, y]$ (i.e., $\varrho(z) - \varrho(x) = k$). Moreover, $\alpha_{xy}(\phi) = 1$ and $\alpha_{xy}(\mathbf{m-1})$ is the total number of unrefinable chains in $L$ between $x$ and $y$. Now define for $S \subseteq \mathbf{m-1}$,

$$\beta_{xy}(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_{xy}(T),$$

so by the Principle of Inclusion-Exclusion [2],

$$\alpha_{xy}(S) = \sum_{T \subseteq S} \beta_{xy}(T).$$

As mentioned in [3, p. 198], if $L_{xy}(S)$ denotes the partially ordered set of all $z \in L$ satisfying either (a) $z = x$; (b) $z = y$; or (c) $x < z < y$ and $\varrho(z) - \varrho(x) \in S$, then

$$\mu_S(x, y) = (-1)^{s+1} \beta_{xy}(S), \tag{4}$$

where $\mu_S$ is the Möbius function of $L_{xy}(S)$ and $|S| = s$. For this reason we call the function $\beta_{xy}(\cdot)$ the *rank-selected Möbius invariant* of the interval $[x, y]$.

If $\pi=(a_1, a_2, \ldots, a_m)$ is a finite sequence of integers, then a pair $a_j > a_{j+1}$ is called a *descent* of $\pi$, and the set

$$D(\pi)=\{j: a_j > a_{j+1}\}$$

is called the *descent set* of $\pi$. We can now state the fundamental combinatorial property of *J-H* sets. This result is a direct generalization of [3, Thm. 1.2]. The proof is identical to the proof of [3, Thm. 1.2], except that here the definition of an admissible lattice plays the role of Lemma 3.1 of [3]. Thus no condition is needed about distributive sublattices of $L$.

3.1. THEOREM. *Let $(L, \omega)$ be an admissible lattice, and let $[x, y]$ be an interval of $L$ of length $m$. If $S \subseteq \mathbf{m-1}$, then the number of sequences $\pi$ in the J-H set $\mathcal{J}_{xy}(L, \omega)$ with descent set $D(\pi)=S$ is equal to $\beta_{xy}(S)$. (The reader is reminded that $\mathcal{J}_{xy}(L, \omega)$ contains one sequence $\pi$ for each maximal chain of $[x, y]$, so that repeated sequences are taken into account.)*

3.2. COROLLARY. *Let $(L, \omega)$ be an admissible lattice. If $[x, y]$ is an interval of $L$ of length $m$ and if $S \subseteq \mathbf{m-1}$, then $\beta_{xy}(S) \geq 0$.* □

In view of (4), Corollary 3.2 may be restated as follows:

3.2′. COROLLARY. *Let $(L, \omega)$ be an admissible lattice of length $n$, and let $S \subseteq \mathbf{n-1}$. Then the Möbius function $\mu_S$ of the rank-selected partially ordered set $L(S)$ alternates in sign; i.e., if $[x, y]$ is an interval in $L(S)$ of length $k$, then*

$$(-1)^k \mu_S(x, y) \geq 0. \quad \square$$

Since by Proposition 2.2 every finite upper-semimodular lattice has an admissible labeling, Corollary 3.2′ applies to all such lattices, and in particular, to finite geometric lattices.

3.3. COROLLARY. *Let $(L, \omega)$ be an admissible lattice and $[x, y]$ an interval of $L$ of length $m$. Let $\mu$ denote the Möbius function of $L$. Then $(-1)^m \mu(x, y)$ is equal to the number of unrefinable chains $x=x_0<x_1<\cdots<x_m=y$ between $x$ and $y$ satisfying*

$$\gamma(x_0, x_1)>\gamma(x_1, x_2)>\cdots>\gamma(x_{m-1}, x_m).$$

*Proof.* Let $S=\mathbf{m-1}$ in Theorem 3.1, and use (4). □

## 4. Applications

We shall state those results in [3] proved for *SS*-lattices which remain true for admissible lattices. The proofs are exactly the same as in the *SS*-case once suitable analogues are given for two concepts in [3]. First, the role of the 'induced $M$-chain $A_{xy}$ between $x$ and $y$' is replaced by the unique unrefinable chain $x=x_0<x_1<\cdots<x_m=y$ between $x$ and $y$ satisfying $\gamma(x_0, x_1)\leq\gamma(x_1, x_2)\leq\cdots\leq\gamma(x_{m-1}, x_m)$. Secondly, we need a replacement for statement (A) in the proof of Theorem 5.2 of [3]. Although a direct analogue of (A) can be given, it is simpler to use the following fact:

4.1. LEMMA. *If $[x, y]$ is an interval of an upper-semimodular admissible lattice $(L, \omega)$ such that $y$ is the join of atoms of $[x, y]$, then there is an unrefinable chain $x=x_0<x_1<\cdots<x_m=y$ between $x$ and $y$ such that $\gamma(x_0, x_1)>\gamma(x_1, x_2)>\cdots>\gamma(x_{m-1}, x_m)$.*

*Proof.* Recall that a *geometric lattice* is an upper-semimodular lattice whose join-irreducibles are its atoms. If $L'$ denotes the partially ordered set of all elements of $[x, y]$ which are a join of atoms of $[x, y]$ (including $x$ as the void join), then $L'$ has the structure of a geometric lattice (though $L'$ is not necessarily a sublattice of $L$). If $\mu$ denotes the Möbius function of $L$ and $\mu'$ that of $L'$, then from [2, Cor. on p. 349] we conclude $\mu(x, y)=\mu'(x, y)$. Hence by [2, §7, Thm. 4], $\mu(x, y)\neq 0$. The desired result now follows from Corollary 3.3. □

The reader can now verify that the proofs of the following results are the same as the analogous results for $SS$-lattices given in [3].

4.2. PROPOSITION. (Generalizes [3, Prop. 3.3]). *Let $(L, \omega)$ be an admissible lattice, and let $[x, y]$ be an interval of L length m. Let $S\subseteq \mathbf{m-1}$. If $\beta_{xy}(S)>0$ and $T\subseteq S$, then $\beta_{xy}(T)>0$.* □

Suppose $L$ is a finite geometric lattice. Then $L$ is upper-semimodular, so by Proposition 2.2 $L$ possesses an admissible labeling. Moreover, every interval of $L$ is a geometric lattice, and the Möbius function of $L$ is never 0. It follows from (4) and Proposition 4.2 that Corollary 3.2′ can be strengthened in the case of geometric lattices as follows:

4.3. COROLLARY. *Let $L$ be a finite geometric lattice of rank $n$, and let $S\subseteq \mathbf{n-1}$. Then the Möbius function $\mu_S$ of the rank-selected partially ordered set $L(S)$ strictly alternates in sign; i.e., if $[x, y]$ is an interval in $L(S)$ of length $k$, then*

$$(-1)^k \mu_S(x, y)>0. \quad \square$$

For some related properties of geometric lattices, see the next section.

Recall [3, §5] that a *Loewy chain* between $x$ and $y$ in a lattice $L$ of finite length is a chain $x=x_0<x_1<\cdots<x_r=y$ such that each $x_i$, $i\in\mathbf{r}$, is the join of the atoms of the interval $[x_{i-1}, x_i]$.

4.4. PROPOSITION. (Generalizes [3, Lemma 5.1]). *Let $(L, \omega)$ be an admissible* *lattice with $[x, y]$ an interval of length $m$. Let $K$ be an unrefinable chain in $L$ between $x$ and $y$:*

$$K:\ x=y_0<y_1<\cdots<y_m=y.$$

*Let $0<m_1<m_2<\cdots<m_r=m$. Then the subchain*

$$x=y_0<y_{m_1}<y_{m_2}<\cdots<y_{m_r}=y$$

*of $K$ is a Loewy chain between $x$ and $y$ if*

$$\mathbf{m-1}-D(\pi_K)\subseteq\{m_1, m_2, \ldots, m_{r-1}\}. \quad \square$$

4.5. THEOREM. (Generalizes [3, Thm. 5.2]). *Let $L$ be a finite upper-semimodular lattice with $[x, y]$ an interval of $L$ of length $m$. Let $S=\{m_1, m_2, \ldots, m_s\}_< \subseteq \mathbf{m-1}$. There exists a chain $C$,*

$$C:\ x=y_0<y_1<\cdots<y_s<y_{s+1}=y$$

*satisfying the two conditions*

(i) *$\varrho(y_i)-\varrho(x)=m_i$, $1\leq i\leq s$ (where $\varrho$ as usual is the rank function of $L$);*

(ii) *$C$ is a Loewy chain between $x$ and $y$,*

*if and only if $\beta_{xy}((\mathbf{m-1})-S)>0$.*

Now recall [3, §6] that if $q$ is a fixed positive integer, then a *$q$-lattice* is a lattice $L$ of finite length with the property that every interval $[x, y]$ of $L$ for which $y$ is the join of atoms of $[x, y]$ is isomorphic to the lattice of subspaces of a projective geometry of degree $q$ (or to a Boolean algebra if $q=1$). Such a lattice is necessarily upper-semimodular [3, pp. 213–214] and hence possesses an admissible labeling. A $q$-lattice, however, need not be supersolvable, so the next proposition is strictly stronger than the corresponding Lemma 6.4 of [3]. For instance, let $L'$ be the lattice of subgroups of a finite abelian $p$-group of type (3,3). Let $L$ be $L'$ truncated above rank 3, i.e., identify all elements of $L'$ of rank at least 4. Then $L$ is a $p$-lattice but is not supersolvable.

4.6. PROPOSITION. (Replaces [3, Lemma 6.4]). *Let $(L, \omega)$ be an admissible $q$-lattice of rank $n$. Let $S\subseteq\mathbf{n-1}$, with $(\mathbf{n-1})-S=\{j_1, j_2, \ldots, j_{t-1}\}_<$. Also let $j_0=0$, $j_t=n$. Define $N(S)$ to be the number of maximal chains $K$ of $L$ satisfying $D(\pi_K)\supseteq S$, where $D(\pi_K)$ is the descent set of the J-H sequence $\pi_K$. Then $N(S)=q^k M$, where*

$$k=\sum_{n=1}^{t}\binom{j_r-j_{r-1}}{2} \tag{5}$$

*and where $M$ is the number of Loewy chains*

$$\hat{0}=y_0<y_1<\cdots<y_t=\hat{1} \tag{6}$$

*such that $\varrho(y_i)=j_i$, $0\leq i\leq t$.*

Since Proposition 4.6 is not a strict analogue of [3, Lemma 6.4], we shall give a proof.

*Proof.* If $K$ is a maximal chain of $L$ such that $D(\pi_K)\supseteq S$, then by Proposition 4.4 the subchain $C$ of $K$ consisting of all $x\in K$ such that $\varrho(x)=j_i (0\le i\le t)$ is a Loewy chain. Hence it suffices to prove that if we have a Loewy chain (6) with $\varrho(y_i)=j_i$, then the number of refinements of $C$ to a maximal chain $K$ satisfying $D(\pi_K)\supseteq S$ is equal to $q^k$, where $k$ is given by (5).

Assume we have such a Loewy chain $C$. Since $L$ is a $q$-lattice, each interval $[y_{r-1}, y_r]$ $(1\le r\le t)$ is a projective geometry of degree $q$ (or a Boolean algebra if $q=1$). Hence $\mu(y_{r-1}, y_r)=(-1)^b q^{k_r}$, where $b=j_r-j_{r-1}$ and $k_r=\binom{j_r-j_{r-1}}{2}$. Now by Corollary 3.3 the number of maximal chains $y_{r-1}=z_0<z_1<\cdots<z_b=y_r$ of the interval $[y_{r-1}, y_r]$ such that

$$\gamma(z_0, z_1)>\gamma(z_1, z_2)>\cdots>\gamma(z_{b-1}, z_b)$$

is just $(-1)^b\mu(y_{r-1}, y_r)=q^{k_r}$. Hence the total number of refinements of $C$ to a maximal chain $K$ satisfying $D(\pi_K)\supseteq S$ is equal to $q^{k_1}q^{k_2}\ldots q^{k_t}=q^k$, and the proof follows. □

4.7. COROLLARY. (Generalizes [3, Corollary 6.5]). *Let $L$ be a $q$-lattice of rank $n$, and let $S\subseteq \mathbf{n-1}$, with $(\mathbf{n-1})-S=\{j_1, j_2, \ldots, j_{t-1}\}_<$ and $j_0=0, j_t=n$. Then $\beta(S)$ is divisible by $q^k$, where $k$ is given by* (6).

The derivation of Corollary 4.7 from Proposition 4.6 is not quite as trivial as the derivation of [3, Corollary 6.5] from [3, Lemma 6.4], so we shall give a proof.

*Proof.* Fix an admissible labeling $\omega$ of $L$. By Theorem 3.1, $\beta(S)$ is equal to the number of maximal chains $K$ of $L$ satisfying $D(\pi_K)=S$. Hence if $N(S)$ is defined as in Proposition 4.6, we have

$$N(S)=\sum_{T\supseteq S}\beta(T),$$

so

$$\beta(S)=\sum_{T\supseteq S}(-1)^{|T-S|}N(T). \tag{7}$$

Suppose we have $\mathbf{n-1}\supseteq T\supseteq S$ where $(\mathbf{n-1})-T=\{i_1, i_2, \ldots, i_{s-1}\}_<$ and $(\mathbf{n-1})-S=\{j_1, j_2, \ldots, j_{t-1}\}_<$. and $i_0=j_0=0$, $i_s=j_t=n$. An easy computation shows that

$$\sum_{r=1}^{s}\binom{i_r-i_{r-1}}{2}\ge\sum_{r=1}^{t}\binom{j_r-j_{r-1}}{2}.$$

It follows from Proposition 4.6 that each term $N(T)$ appearing in (7) is divisible by $q^k$, so the proof follows. □

4.8. THEOREM. (Generalizes [3, Theorem 6.6]). *Let L be a q-lattice of rank n, and let* $S \subseteq \mathbf{n-1}$ *with* $|S| = s$. *Then* $\beta(S)$ *is divisible by* $q^{Q(n,s)}$, *where*

$$Q(n, s) = \frac{1}{2}\left[\frac{n}{n-s}\right]\left(n+s-(n-s)\left[\frac{n}{n-s}\right]\right)$$

*(brackets denote the integer part). This result is best possible in the sense that given n and* $0 \leq s \leq n-1$, *there exists a q-lattice (which can even be chosen to be modular) of rank n and a set* $S \subseteq \mathbf{n-1}$ *of cardinality s such that* $\beta(S) = q^{Q(n,s)}$ (see [3, p. 216]). □

## 5. The broken circuit theorem

In this section we shall point out the connection between our work and the so-called 'broken circuit theorem' of G.-C. Rota [2, Prop. 1, p. 358], which generalizes to arbitrary finite geometric lattices a result of Whitney on graphs. The reader should be warned that [2, Prop. 1, p. 358] is *false* when $k > 1$. However, the proof is valid when $k = 1$, and this is the case which will concern us here.

We proceed to describe the broken circuit theorem. Let $L$ be a finite geometric lattice of rank $n$, and let $a_1, a_2, \ldots, a_t$ be an ordering of the atoms $A$ of $L$. A subset $C$ of $A$ is called a *circuit* if the rank of the join of the elements of $C$ is $|C| - 1$, while the rank of the join of the elements of any proper subset $C'$ of $C$ is $|C'|$. A subset $B = \{a_{i_1}, a_{i_2}, \ldots, a_{i_j}\}$ of $A$ is called a *broken circuit* if there exists an atom $a_m$ such that $m > i_r$ for $r = 1, 2, \ldots, j$, and such that $B \cup \{a_m\}$ is a circuit. Note that the notion of a circuit depends only on $L$, while that of a broken circuit also depends on the ordering chosen for the elements of $A$.

BROKEN CIRCUIT THEOREM (G.-C. Rota). *Let L be a finite geometric lattice of rank n with an ordering* $a_1, a_2, \ldots, a_t$ *of the atoms of L. Let* $\mu$ *be the Möbius function of L. Then* $(-1)^n \mu(\hat{0}, \hat{1})$ *is equal to the number of sets of n atoms of L not containing any broken circuit.* □

Given an ordering $a_1, a_2, \ldots, a_t$ of the atoms of a finite geometric lattice $L$ of rank $n$, define a labeling $\omega$ of $L$ by $\omega(a_i) = t - i + 1$, so $i < j$ implies $\omega(a_i) > \omega$. By Proposition 2.2, $(L, \omega)$ is an admissible lattice. Let $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ be a maximal chain $K$ in $L$ satisfying

$$\gamma(x_0, x_1) > \gamma(x_1, x_2) > \cdots > \gamma(x_{n-1}, x_n). \tag{8}$$

We know by Corollary 3.3 that the number of such maximal chains $K$ is $(-1)^n \mu(\hat{0}, \hat{1})$. We would like to relate this fact to the Broken Circuit Theorem by constructing an explicit one-to-one correspondence $\lambda$ between maximal chains $K$ satisfying (8) and sets of $n$ atoms of $L$ containing no broken circuit.

This correspondence $\lambda$ is defined as follows. Given a maximal chain $K$ satisfying (8), let $\lambda(K)$ be the set $\{b_1, b_2, \ldots, b_n\}$ of those $n$ atoms defined by $\omega(b_j)=\gamma(x_{j-1}, x_j)$.

5.1. PROPOSITION. *The function $\lambda$ defines a one-to-one correspondence between maximal chains $K$ of $L$ satisfying* (8), *and sets of $n$ atoms of $L$ containing no broken circuit.*

*Proof.* We first prove that $\lambda(K)$ contains no broken circuits. Suppose $B=\{b_{i_1}, b_{i_2}, \ldots, b_{i_s}\}$ is a broken circuit contained in $\lambda(K)$ with $i_1<i_2<\cdots<i_s$, i.e., $\omega(b_{i_1})>\omega(b_{i_2})>\cdots>\omega(b_{i_s})$. By definition of broken circuit, there exists an atom $a$ of $L$ such that $\omega(b_{i_r})>\omega(a)$ for $r=1, 2, \ldots, s$ and $B\cup\{a\}$ is a circuit. By definition of the $b_i$'s and $\gamma$, $x_{i_s-1}\vee b_{i_s}=x_{i_s}$ and $b_{i_r}\leq x_{i_s-1}$ for $r=1, 2, \ldots, s-1$. Hence since $B\cup\{a\}$ is a circuit, $x_{i_s-1}\vee a=x_{i_s}$. By definition of $\gamma$, this means $\omega(b_{i_s})<\omega(a)$, a contradiction. Hence $\lambda(K)$ contains no broken circuit.

Now let $B=\{b_1, b_2, \ldots, b_n\}$ be a set of $n$ atoms containing no broken circuit, with $\omega(b_1)>\omega(b_2)>\cdots>\omega(b_n)$. Recall that a *basis* of $L$ is a set of $n$ atoms $c_1, c_2, \ldots, c_n$ of $L$ such that $\varrho(c_1\vee c_2\vee\cdots\vee c_n)=n$. Equivalently, a basis is a set of $n$ atoms containing no circuit. Now note that $B$ is a basis, since if it contained a circuit it would contain a broken circuit. If $\lambda(K)=B$, then $K$ must be given by $x_j=b_1\vee b_2\vee\cdots\vee b_j$, so $\lambda$ is injective. It remains to prove that these $x_j$'s satisfy $\gamma(x_{j-1}, x_j)=\omega(b_j)$, which shows $\lambda$ is surjective. By definition of the $x_i$'s, $x_{j-1}\vee b_j=x_j$. Suppose $a$ is an atom such that $x_{j-1}\vee a=x_j$ and $\omega(a)<\omega(b_j)$. Thus the set $\{b_1, b_2, \ldots, b_j, a\}$ contains a circuit $C$. Moreover, $a\in C$ since the $b_i$'s are independent. Since $\omega(b_1)>\omega(b_2)>\cdots>\omega(b_j)$ and $\omega(b_j)>\omega(a)$, $\omega(a)<\omega(b_i)$ for $1\leq i\leq j$. Hence $C-\{a\}$ is a broken circuit, a contradiction. This completes the proof. $\square$

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