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An Extremal Problem for Finite Topologies and Distributive Lattices*

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Let $(r_1, r_2, \ldots)$ be a sequence of non-negative integers summing to n. We determine under what conditions there exists a finite distributive lattice L of rank n with r_i join-irreducibles of rank i, for all $i = 1, 2, \ldots$. When L exists, we give explicit expressions for the greatest number of elements L can have of any given rank, and for the greatest total number of elements L can have. The problem is also formulated in terms of finite topological spaces.

Let X be a finite topological space with n elements $x_1, x_2, \ldots, x_n$. Let U_i be the smallest open set containing x_i. We shall determine the greatest number of open sets of a given cardinality m which X can have if the cardinality $|U_i|$ of each U_i is specified in advance. Our results will include the result of Sharp [3] and Stephen [5] that any topology on n points which is not discrete contains $\leqslant (\frac{3}{4})\, 2^n$ open sets. Another way of generalizing the result of Sharp and Stephen was given by Stanley [4]. For similar considerations involving finite topological groups, see Morris and Thompson [2].

Since X is a T_0-space if and only if the sets $U_1, U_2, \ldots, U_n$ are all distinct, there is no real loss of generality in restricting ourselves to T_0-spaces. It is well known that there is a one-to-one correspondence between finite T_0-spaces X with n elements, finite (partially) ordered sets P with n elements, and finite distributive lattices $L = J(P)$ of rank n. This correspondence seems first to have been considered by Alexandroff [1], and has been rediscovered several times. X, P, and L are related as follows: P is the set of U_i's ordered by inclusion and the set of join-irreducibles of L with the induced ordering; L is the set of open sets of X ordered by inclusion and

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the set of *ideals* (also called *order ideals* or *semi-ideals*) of P ordered by inclusion. An ideal of P is a subset I of P such that $x \in I$, $y \leqslant x$ implies $y \in I$. We say that an ideal I of P is *generated* by a subset $x_1, \ldots, x_m$ of P if the x_i's are the set of maximal elements of I, and we write $I = \langle x_1, \ldots, x_m \rangle$.

Thus the problem we are considering (restricted to T_0-spaces) can be reformulated as follows: Let $\mathbf{r} = (r_1, r_2, \ldots)$ be a sequence of non-negative integers summing to n. What is the greatest number $N(\mathbf{r}, m)$ of elements of rank m a distributive lattice L of rank n can have if L has exactly r_i join-irreducibles of rank i? We shall determine $N(\mathbf{r}, m)$ explicitly, and for a given choice of $\mathbf{r}$ we shall construct a distributive lattice $L(\mathbf{r}) = J(P(\mathbf{r}))$ which simultaneously achieves the values $N(\mathbf{r}, m)$ for all $m = 0, 1, 2, \ldots$. Hence if $N(\mathbf{r})$ denotes the largest total number of elements a distributive lattice can have if it has r_i join-irreducibles of rank i, then

$$N(\mathbf{r}) = \sum_{m=0}^{\infty} N(\mathbf{r}, m).$$

By summing this series, we also obtain an explicit expression for $N(\mathbf{r})$.

We proceed to the construction of the "extremal ordered set" $P(\mathbf{r})$. Its basic properties will then be verified. We assume $\mathbf{r} = (r_1, r_2, \ldots)$ where

$$\begin{gathered} r_1 + r_2 + \cdots + r_j \geqslant j, \quad \text{if} \quad 1 \leqslant j \leqslant n-1, \\ r_1 + r_2 + \cdots + r_n = n, \\ r_j = 0, \quad \text{if} \quad j > n. \end{gathered} \tag{1}$$

(Theorem 1 will show that this condition on $\mathbf{r}$ does not entail a loss of generality.)

Define $P_1(\mathbf{r})$ to be the ordered set consisting of r_1 disjoint points. Now suppose $r_j > 0$ and $P_j(\mathbf{r})$ has been defined. Let k be the least integer $> j$ satisfying $r_k > 0$. Then $P_k(\mathbf{r})$ is defined to be the ordered set obtained from $P_j(\mathbf{r})$ by inserting an additional r_k elements, all lying above the last $k - j$ *maximal* elements of $P_j(\mathbf{r})$ to be previously inserted. This process is continued until there is no remaining k satisfying $r_k > 0$.

Figure 1 illustrates the procedure if $n = 10$, $r_1 = 4$, $r_3 = 1$, $r_5 = 2$, $r_6 = 1$, $r_8 = 2$. To check that this construction is defined, we need to verify that $P_j(\mathbf{r})$ has at least $k - j$ maximal elements. Let

$$1 = i_1 < i_2 < \cdots < i_a = j < i_{a+1} = k < \cdots$$

be such that $r_{i_1}, r_{i_2}, \ldots$ are the nonzero r_i's. Upon forming $P_{i_s}(\mathbf{r})$, a new set of r_{i_s} maximal elements is inserted above a set of $i_s - i_{s-1}$ maximal

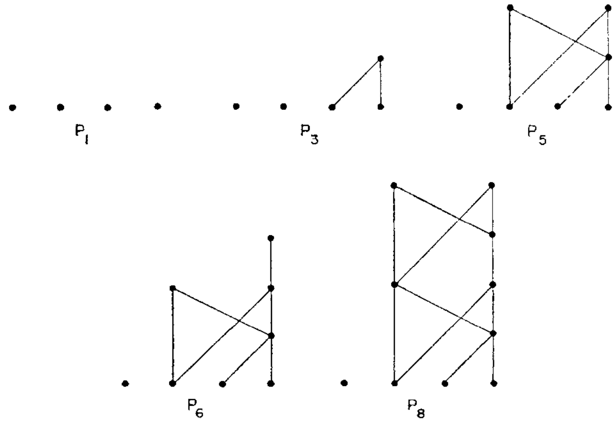

FIG. 1. Construction of $P(\mathbf{r})$ for $\mathbf{r} = (4, 0, 1, 0, 2, 1, 0, 2)$.

elements, resulting in a net gain (or loss) of $r_{i_s} - i_s + i_{s-1}$ maximal elements. Hence $P_j(\mathbf{r})$ has precisely

$$r_{i_1} + (r_{i_2} - i_2 + 1) + (r_{i_3} - i_3 + i_2) + \cdots + (r_{i_a} - i_a + i_{a-1})$$
$$= r_{i_1} + \cdots + r_{i_a} - i_a + 1 = r_1 + r_2 + \cdots + r_{k-1} - j + 1$$

maximal elements. But, by (1),

$$r_1 + r_2 + \cdots + r_{k-1} - j + 1 \geqslant (k-1) - j + 1 = k - j,$$

so $P(\mathbf{r})$ is defined as long as $\mathbf{r}$ satisfies (1).

LEMMA 1. *Each element inserted into $P_j(\mathbf{r})$ to form $P_k(\mathbf{r})$ generates an ideal of cardinality k. Hence $L(\mathbf{r}) = J(P(\mathbf{r}))$ contains exactly r_i join-irreducibles of rank i, for $i = 1, 2,..., n$.*

Proof. Induction on k. The statement is trivially true for $k = 1$. Now suppose it is true for all $i \leqslant j$, and that $P_k(\mathbf{r})$ is being formed from $P_j(\mathbf{r})$. Let $x_1 ,..., x_{k-j}$ be the last $k - j$ maximal elements that were inserted into $P_j(\mathbf{r})$, say x_1 last, x_2 next-to-last, etc. Then it suffices to show that the ideal $\langle x_1 ,..., x_{k-j}\rangle$ has cardinality $k - 1$. By the induction hypothesis, x_1 generates an ideal $\langle x_1\rangle$ of cardinality j. Now since $x_1 ,..., x_{k-j}$ are the *last* $k - j$ maximal elements of $P_j(\mathbf{r})$, it follows that, if the ideal $\langle x_i\rangle$ $(i = 2,..., k - j)$ contains t elements, then there is an $x \leqslant x_1$ such that the ideal $\langle x\rangle$ also contains t elements. But, by construction, x_i and x lie strictly above the same elements. Hence the only element of the ideal $\langle x_i \rangle$ not contained in $\langle x_1 \rangle$ is x_i itself. Thus

$$\langle x_1, x_2, \ldots, x_{k-j} \rangle = \langle x_1 \rangle \cup \{x_2, \ldots, x_{k-j}\},$$

so $\langle x_1, x_2, \ldots, x_{k-j} \rangle$ has cardinality $(j+1)+(k-j-1)=k$. □

LEMMA 2. *If the ideal $\langle x_1, x_2, \ldots, x_{r_k} \rangle$ generated by the r_k elements inserted into $P_j(\mathbf{r})$ to form $P_k(\mathbf{r})$ is removed from $P_k(\mathbf{r})$, then the remaining ordered set Q consists of $r_1 + \cdots + r_{k-1} - k + 1$ disjoint points.*

Proof. It follows from Lemma 1 that Q has $r_1 + \cdots + r_{k-1} - k + 1$ elements; we show that these are all maximal elements of $P_k(\mathbf{r})$. Let $x \in Q$. Suppose $y \in P_k(\mathbf{r})$ and $y > x$. If the ideal $\langle y \rangle$ has cardinality t and if some other ideal $\langle z \rangle$ has cardinality t, then, by the construction of $P_k(\mathbf{r})$, we also have $z > x$. But, also by construction, the elements $x_1, x_2, \ldots, x_{r_k}$ lie above *some* element z such that $\langle z \rangle$ has cardinality t (since $r_t \neq 0$). Hence $x_i > x$, contradicting $x \in Q$. □

LEMMA 3. *The number of ideals of $P(\mathbf{r})$ of cardinality m is*

$$\binom{r_1 - 1}{m} + \binom{r_1 + r_2 - 2}{m-1} + \binom{r_1 + r_2 + \cdots + r_3 - 3}{m-2} + \cdots$$
$$+ \binom{r_1 + r_2 + \cdots + r_m - m}{1} + 1.$$

Equivalently, this is the number of elements of rank m in $L(\mathbf{r}) = J(P(\mathbf{r}))$.

Proof. In the process of constructing $P_k(\mathbf{r})$ from $P_j(\mathbf{r})$, every ideal of $P_j(\mathbf{r})$ of cardinality m remains an ideal of cardinality m. We count the number of new ideals I of cardinality m. Such an ideal must contain $i \geqslant 1$ of the elements $x_1, \ldots, x_{r_k}$ inserted into $P_j(\mathbf{r})$. By Lemma 1, these i elements generate an ideal J of cardinality $k - 1 + i$. The remaining $m - k - i + 1$ elements of I must form an ideal in $P_k(\mathbf{r}) - \langle x_1, \ldots, x_{r_k} \rangle$. By Lemma 2, $P_k(\mathbf{r}) - \langle x_1, \ldots, x_{r_k} \rangle$ is a disjoint union of points of cardinality $r_1 + \cdots + r_{k-1} - k + 1$. Hence the total number of new ideals I of cardinality m is

$$\binom{r_1 + \cdots + r_k - k + 1}{m - k + 1} - \binom{r_1 + \cdots + r_{k-1} - k + 1}{m - k + 1}.$$

Summing on k and using the identity $\binom{a+1}{b} - \binom{a}{b} = \binom{a}{b-1}$ gives the result. □

LEMMA 4. *The total number of ideals of $P(\mathbf{r})$ (or total number of elements of $L(\mathbf{r})$) is*

$$\sum_{k=1}^{n-1} 2^{r_1+r_2+\cdots+r_k-k} + 2^{r_1+r_2+\cdots+r_n-n+1}.$$

Proof. Sum the result of Lemma 3 on m, taking care to count each term exactly once. □

Note, e.g., that when $\mathbf{r} = (n, 0, 0,...)$, then the result of Lemma 3 reduces to $\binom{n}{m}$ and of Lemma 4 to 2^n.

We are now in a position to evaluate the numbers $N(\mathbf{r}, m)$ and $N(\mathbf{r})$.

THEOREM 1. *Let $\mathbf{r} = (r_1, r_2,...)$ with $\sum r_i = n$. There exists a distributive lattice of rank n with exactly r_i join-irreducibles of rank i if and only if $\mathbf{r}$ satisfies* (1).

Proof. Lemma 1 shows that $L(\mathbf{r})$ has the desired property. Conversely, suppose $L = J(P)$ has r_i join-irreducibles of rank i. Let $r_j \neq 0$ and let k be the least integer $>j$ such that $r_k \neq 0$. A join-irreducible of rank k in L corresponds to an element $x \in P$ which lies strictly above $k - 1$ elements. Hence

$$\begin{aligned} r_1 + r_2 + \cdots + r_j &= r_1 + r_2 + \cdots + r_{j+1} \\ &= \cdots = r_1 + r_2 + \cdots + r_{k-1} \geqslant k - 1, \end{aligned}$$

and the proof follows. □

We remark that, for a given n, it is well known that the number of sequences $\mathbf{r}$ satisfying (1) is the *Catalan number*

$$\frac{1}{n+1}\binom{2n}{n}.$$

THEOREM 2. *Let $\mathbf{r} = (r_1, r_2,...)$ satisfy* (1). *Let $N(\mathbf{r}, m)$ be the greatest number of elements of rank m a distributive lattice L of rank n can have if it has r_i join-irreducibles of rank i. Then*

$$\begin{aligned} N(\mathbf{r}, m) = \binom{r_1 - 1}{m} + \binom{r_1 + r_2 - 2}{m-1} + \binom{r_1 + r_2 + r_3 - 3}{m-2} + \cdots \\ + \binom{r_1 + r_2 + \cdots + r_m - m}{1} + 1. \end{aligned}$$

Proof. Lemma 3 shows that $L(\mathbf{r})$ achieves the desired number of elements of rank m ($m = 0, 1,..., n$). Conversely, assume that L has r_i

join-irreducibles of rank i. Let $L = J(P)$, and define P_k to be the ordered subset of P consisting of those $x \in P$ such that $|\langle x\rangle| \leqslant k$. Thus $\phi = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$, and $| P_k - P_{k-1} | = r_k$. Let $P_k - P_{k-1} = \{x_1, ..., x_{r_k}\}$ and define $J = \langle x_1, ..., x_{r_k}\rangle$. Now any ideal of cardinality m in P_{k-1} is also an ideal of P_k. The number of new ideals I of P_k of cardinality m which intersect $P_k - P_{k-1}$ in a fixed set $x_1, ..., x_i$ is just the number of ideals of $P_k - J$ of cardinality $m - j$, where $j = |\langle x_1, ..., x_i\rangle|$. Since $|\langle x_i\rangle| = k$,

$$| J | \geqslant k + r_k - 1 \qquad \text{and} \qquad | P_k - J | \leqslant r_1 + \cdots + r_{k-1} - k + 1.$$

Thus $P_k - J$ cannot contain more ideals of cardinality $m - j$ than does a disjoint union of points of cardinality $r_1 + \cdots + r_{k-1} - k + 1$. If we take $P = P(\mathbf{r})$ so $P_k = P_k(\mathbf{r})$, then, by Lemma 2, $P_k(\mathbf{r}) - J$ is in fact a disjoint union of $r_1 + \cdots + r_{k-1} - k + 1$ points. Thus $P(\mathbf{r})$ has $N(\mathbf{r}, m)$ ideals of cardinality m. The proof follows from Lemma 3. □

Since $P(\mathbf{r})$ has $N(\mathbf{r}, m)$ ideals of cardinality m for *all* m, there follows from Lemma 4:

Corollary. *Let* $\mathbf{r} = (r_1, r_2, ...)$ *satisfy* (1). *Let* $N(\mathbf{r})$ *be the greatest total number of elements a distributive lattice* L *of rank* n *can have if it has* r_i *join-irreducibles of rank* i. *Then*

$$N(\mathbf{r}) = \sum_{k=1}^{n-1} 2^{r_1+r_2+\cdots+r_k-k} + 2^{r_1+r_2+\cdots+r_n-n+1}.$$

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