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Theory and Application of Plane Partitions. Part 2

By Richard P. Stanley

IV. Enumeration of column-strict plane partitions

14. *Part restrictions*

We are now ready to apply our theory of Schur functions to the enumeration of plane partitions. The first such results were obtained by MacMahon [9], using an entirely different technique.

If p_n is the number of plane partitions of n with a certain property, we say that the *generating function* for these plane partitions is the (formal) power series

$$\Sigma p_n x^n. \tag{46}$$

We will regard the plane partitions counted by (46) to be *enumerated* if an explicit expression can be found for (46). Only in rare cases can an explicit expression be found for p_n itself.

We will employ the notation

$$\begin{aligned}(\boldsymbol{k}) &= 1 - x^k \\ (\boldsymbol{k})! &= (\boldsymbol{1})(\boldsymbol{2})\dots(\boldsymbol{k})\end{aligned} \tag{47}$$

For instance, the generating function for plane partitions with ≤ 1 row (i.e., ordinary partitions) is $\prod_{n=1}^{\infty} (\boldsymbol{n})^{-1}$, a well-known result of Euler (see Hardy and Wright [6, Ch. 19]). The generating function for plane partitions with ≤ 1 row and ≤ 2 columns is $1/(\boldsymbol{2})!$, and here we have the explicit expression $p_n = \frac{1}{4}(2n + 3 + (-1)^n)$. In these examples, the generating functions can be determined by "inspection." For more general types of plane partitions, the generating functions still have a simple form, but there appears to be no "obvious" reason why this is so.

14.1. THEOREM. (Bender and Knuth [18]). *Let S be any subset of the positive integers. The generating function for column-strict plane partitions whose parts all lie in S is*

$$\prod_{i \in S} (\boldsymbol{i})^{-1} \prod_{\substack{i,j \in S \\ i<j}} (\boldsymbol{i+j})^{-1}.$$

Proof: By definition of the e_λ's, this generating function is obtained from $\sum_\lambda e_\lambda$ by setting $x_i = x^i$ if $i \in S$, $x_i = 0$ otherwise. The proof now follows from Corollary 8.3. □

The special cases $S = \{1, 2, 3, \ldots\}$ and $S = \{1, 3, 5, \ldots\}$ were first obtained by Gordon and Houten [31] before Bender and Knuth discovered the general case. In particular, the generating function for column-strict plane partitions with no part restrictions is

$$\prod_{k=1}^{\infty} (k)^{-[(k+1)/2]} = 1/(1)(2)(3)^2(4)^2(5)^3(6)^3 \ldots$$
$$= 1 + x + 2x^2 + 4x^3 + 7x^4 + 12x^5 + 21x^6 + 34x^7 + 56x^8 + \cdots \qquad (48)$$

Theorem 14.1 can be easily modified to keep track of not only the sum n of the parts, but also the number of parts, the number of columns of odd length (using Proposition 8.4), etc., and of course Corollary 8.3 itself keeps track of the number of occurrences of each part.

As a further application of Corollary 8.3, we prove a "stability theorem" for column-strict plane partitions. The proof illustrates an interesting technique involving generating functions.

14.2. Proposition. Let $b(n, p)$ be the number of column-strict plane partitions of n into exactly p parts. If $p \geq 2m$, then $b(m + p, p)$ is equal to the coefficient of x^m in the expansion of

$$\prod_{k=1}^{\infty} (k)^{-[(k+3)/2]} = 1 + 2x + 5x^2 + 11x^3 + 23x^4 + 45x^5 + 87x^6 + \cdots$$

(brackets denote integer part).

Proof: Define

$$F(q, x) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} b(n, p) x^n q^p.$$

By letting $x_i = qx^i$ in Corollary 8.3, we have

$$F(q, x) = 1 \Big/ \prod_{k=1}^{\infty} (1 - qx^k) \prod_{k=1}^{\infty} (1 - q^2 x^k)^{[(k-1)/2]}.$$

Let

$$F(q, x) = \sum_{p=0}^{\infty} x^p f_p(x) q^p.$$

It is clear that the expansion of $f_p(x)$ in powers of x has no negative exponents. Also define

$$G(q, x) = 1 \Big/ \prod_{k=1}^{\infty} (1 - q^2 x^k)^{[(k-1)/2]} = \sum_{p=0}^{\infty} x^{3p} g_p(x) q^{2p}.$$

It is also evident that the expansion of $g_p(x)$ has no negative exponents.

Now

$$F(q, x) = G(q, x)/(1 - qx)(1 - qx^2) \ldots$$

so

$$\sum_{p=0}^{\infty} x^p f_p(x) q^p = \left(\sum_{p=0}^{\infty} x^{3p} g_p(x) q^{2p} \right) \cdot \sum_{p=0}^{\infty} \frac{x^p q^p}{(p)!}$$

(the latter sum being a well-known identity of Euler (cf. Hardy and Wright [6, Thm. 349])). Equating coefficients of q^p gives

$$x^p f_p(x) = \sum_{k=1}^{[p/2]} \frac{x^{3k} g_k(x) x^{p-2k}}{(p-2k)!},$$

so

$$f_p(x) = \sum_{k=0}^{[p/2]} \frac{x^k g_k(x)}{(p-2k)!} \qquad (49)$$

Using the notation $f(x) \equiv g(x) \pmod{x^{j+1}}$ to mean that the coefficients of x^i in $f(x)$ and $g(x)$ are the same, for $i = 0, 1, \ldots, j$, there follows from (49) that

$$f_p(x) \equiv \sum_{k=0}^{\infty} x^k g_k(x)/(1)(2)\ldots \pmod{x^{[p/2]+1}} \qquad (50)$$

The right-hand side of (50) is just

$$G\left(\frac{1}{x}, x\right) \Big/ (1)(2)\ldots = \prod_{k=1}^{\infty} (k)^{-[(k+3)/2]},$$

and the proof follows since

$$f_p(x) = \sum_{m=0}^{\infty} b(m+p, p) x^m. \quad \square$$

We conclude this section with an interesting application of Theorem 14.1, viz., the enumeration of *symmetric* plane partitions. A symmetric plane partition is a plane partition whose parts n_{ij} satisfy $n_{ij} = n_{ji}$.

14.3. Proposition (Gordon [70]). The generating function for symmetric plane partitions is

$$\prod_{k=1}^{\infty} (2k-1)^{-1}(2k)^{-[k/2]} = 1/(1)(3)(4)(5)(6)(7)(8)^2(9)(10)^2\ldots$$

$$= 1 + x + x^2 + 2x^3 + 3x^4 + 4x^5 + 6x^6 + 8x^7 + 12x^8 + 16x^9 + 22x^{10} + \cdots$$

Proof: We shall set up a one-to-one correspondence between column-strict plane partitions π of n into odd parts, and symmetric plane partitions $\bar{\pi}$ of n. The proof will then follow from Theorem 14.1 by taking S to be the set of all odd positive integers. Let π be a column-strict plane partition of n. There is a well-known correspondence between partitions of an integer k into distinct odd parts, and self-conjugate partitions of k [6, Thm. 347]. Apply this correspondence to each column of π, resulting in a new plane partition π_0 of n. Now replace each row of π_0 by its conjugate partition. We get a symmetric plane partition $\bar{\pi}$ of n, and it is easily seen that this establishes the desired one-to-one correspondence. $\square$

As an illustration of the above correspondence, we have

$$\begin{matrix} 7 & 5 & 5 & 3 \\ 5 & 3 & 1 & 1 \\ 1 & 1 & & \\ & & & \end{matrix} \qquad \begin{matrix} 4 & 3 & 3 & 2 \\ 4 & 3 & 2 & 2 \\ 3 & 3 & 1 & \\ 2 & & & \end{matrix} \qquad \begin{matrix} 4 & 4 & 3 & 1 \\ 4 & 4 & 2 & 1 \\ 3 & 2 & 2 & \\ 1 & 1 & & \end{matrix}$$

$$\pi \qquad\qquad \pi_0 \qquad\qquad \bar{\pi}$$

15. *Shape restrictions, hook lengths, and contents*

The generating function for column-strict plane partitions of shape λ whose parts lie in a set S is obtained from e_λ by setting $x_i = x^i$ if $i \in S$, $x_i = 0$ otherwise. By using the Jacobi–Trudi identity (Theorem 11.1), we can express this generating function as a determinant. As it stands, this determinant is not very enlightening, and we seek ways of simplifying it for suitable choices of S. We shall evaluate this determinant when $S = \{1, 2, \ldots, m\}$, i.e., we will find the generating function for column-strict plane partitions of shape λ and largest part $\le m$. This generating function will be denoted by $H_m(\lambda)$.

Our expression for $H_m(\lambda)$ involves two sets of numerical invariants of the partition λ, which we now define.

15.1. Definition. If λ is a partition of p, define the *hook lengths* of λ to be the integers $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, for $\lambda_i, \lambda'_j > 0$. We write $d_1, d_2, \ldots, d_p$ for the hook lengths of λ (in some order). Also define the *contents* of λ to be the integers $c_{ij} = j - i$, for $\lambda_i, \lambda'_j > 0$. We write $c_1, c_2, \ldots, c_p$ for the contents of λ (in some order).

Thus h_{ij} is the number of entries directly to the right or directly below the (i, j)-entry of the shape of λ, counting the (i, j)-entry itself once, thereby explaining the terminology "hook length." For instance, if λ is the partition 4, 3, 3, 2, then its hook lengths and contents are as follows:

$$\begin{matrix} 7 & 6 & 4 & 1 \\ 5 & 4 & 2 & \\ 4 & 3 & 1 & \\ 2 & 1 & & \end{matrix} \qquad\qquad \begin{matrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & -1 & 0 & \\ -3 & -2 & & \end{matrix}$$

hook lengths   contents

The notion of "hooks" were first introduced by Nakayama [48] in connection with modular representations of the symmetric group. Many of their basic combinatorial properties were derived by Frame, Robinson, and Thrall [28]. The definition of "contents" was first given in this writer's thesis [59, p. 163]; the terminology is derived from Robinson [12, Ch. 4].

In order to evaluate $H_m(\lambda) = e_\lambda(x, x^2, \ldots, x^m)$ using the Jacobi–Trudi identity, we need to evaluate $h_k(x, x^2, \ldots, x^m)$. This is a well-known result in the theory of partitions (see Hardy and Wright [6, Thm. 349]), viz.,

$$h_k(x, x^2, \ldots, x^m) = x^k \binom{m+k-1}{m-1}, \tag{51}$$

where $\binom{a}{b}$ denotes the *Gaussian coefficient*,

$$\binom{a}{b} = \frac{(a)!}{(b)!(a-b)!}. \tag{52}$$

There follows from the Jacobi–Trudi identity:

15.2. LEMMA. *If λ is given by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, then*

$$H_m(\lambda) = \left| x^{\lambda_i - i + j}\binom{m + \lambda_i - i + j - 1}{m-1} \right|, \qquad i, j = 1, \ldots, r. \quad \square$$

The main result of this section is an explicit evaluation of this determinant. The expression we give for it first appeared in this writer's thesis [59, Ch. V, Thm. 2.3]. An equivalent determinant was evaluated by Carlitz [20], though he does not give its combinatorial interpretation. Gordon and Houten [31] also evaluate this determinant in the case $m = \infty$, but neither they nor Carlitz give it as explicitly as we do. D. E. Littlewood [8, p. 124, Thm. I] proves a result essentially the same as the next theorem, though he relates it to plane partitions only when $x = 1$ [ibid., p. 189, Thm. VI]. The case $x = 1$ was first done by Frobenius [29, §3] in connection with a problem in group theory, and later a combinatorial proof was given by MacMahon [45].

15.3. THEOREM. *We have*

$$H_m(\lambda) = \frac{x^a(m + c_1)(m + c_2)\ldots(m + c_p)}{(d_1)(d_2)\ldots(d_p)}$$

where

$$a = \Sigma\binom{\lambda_i' + 1}{2} = \Sigma i\lambda_i,$$

and where the c_i's are the contents and d_i's the hook lengths of λ.

Proof: The proof is by induction on $\lambda_1 = c$. The theorem is trivially true for $\lambda_1 = 0$, for here the determinant of Lemma 15.2 is upper triangular with 1's on the main diagonal, so $H_m(\lambda) = 1$.

Now assume the theorem for plane partitions of shape λ^*, where λ^* is obtained from λ by removing the first column from its shape, so $\lambda_i^* = \lambda_i - 1$ (when $\lambda_i > 0$). Suppose the hook lengths of λ^* are $d_1, d_2, \ldots, d_{p-r}$ ($r = \lambda_1'$), and the contents are $c_1, c_2, \ldots, c_{p-r}$. Then the hook lengths of λ are $d_1, d_2, \ldots, d_{p-r}$, $\lambda_1 + r - 1$, $\lambda_2 + r - 2, \ldots, \lambda_r$, and the contents of λ are $c_1 + 1, c_2 + 1, \ldots, c_{p-r} + 1$, $0, -1, \ldots, -r + 1$. Hence our theorem will be proved by induction if we show

$$H_m(\lambda) = \frac{x^{\binom{r+1}{2}}(m)(m-1)\ldots(m-r+1)}{(\lambda_1 + r - 1)(\lambda_2 + r - 2)\ldots(\lambda_r)} H_{m+1}(\lambda^*), \tag{53}$$

where by Lemma 15.2,

$$H_{m+1}(\lambda^*) = \left| x^{\lambda_i - 1 - i + j}\binom{m + \lambda_i - i + j - 1}{m} \right|. \tag{54}$$

In the expression for $H_m(\lambda)$ in Lemma 15.2, divide $x^{r-j+1}(m-r+j)$ out of the j-th column $(j = 1, 2, \ldots, r)$ and multiply the i-th row by $(\lambda_i - i + r)$ $(i = 1, 2, \ldots, r)$. This gives

$$H_m(\lambda) = \frac{x^{\binom{r+1}{2}}(m)(m-1)\ldots(m-r+1)}{(\lambda_1+r-1)(\lambda_2+r-2)\ldots(\lambda_r)} \cdot \left| x^{\lambda_i - i + 2j - r - 1}\binom{m+\lambda_i-1+j-1}{m-1}\frac{(\lambda_i - i + r)}{(m-r+j)}\right|. \tag{55}$$

It remains to transform the determinant in (55) into the determinant (54). This is accomplished by a series of elementary column operations, somewhat similar to those used in proving the Jacobi–Trudi identity.

First note that the entry in the i-th row and last column $(j = r)$ of the determinant in (55) is equal to

$$x^{\lambda_i - i + r - 1}\binom{m+\lambda_i - i + r - 1}{m}$$

and hence equals the corresponding entry in (54). Now multiply the last column of (55) by $(1)/x^2(m-1)$ and subtract it from the next-to-last $(j = r - 1)$ column. This transforms the $(i, r-1)$ entry into

$$\begin{aligned} &x^{\lambda_i - i + r - 3}\binom{m+\lambda_i - i + r - 2}{m-1}\frac{(\lambda_i - i + r)}{(m-1)} \\ &\qquad - x^{\lambda_i - i + r - 3}\binom{m+\lambda_i - i + r - 1}{m}\frac{(1)}{(m-1)} \\ &= x^{\lambda_i - i + r - 2}\binom{m+\lambda_i - i + r - 2}{m}. \end{aligned}$$

This expression agrees with the next-to-last column of (54).

Now multiply the column $j = r - 1$ of this transformed determinant by $(2)/x^3(m-2)$ and subtract it from the column $j = r - 2$. This transforms the $(i, r-2)$ entry into

$$\begin{aligned} &x^{\lambda_i - i + r - 5}\binom{m+\lambda_i - i + r - 3}{m-1}\frac{(\lambda_i - i + r)}{(m-2)} \\ &\qquad - x^{\lambda_i - i + r - 5}\binom{m+\lambda_i - i + r - 2}{m}\frac{(2)}{(m-2)} \\ &= x^{\lambda_i - i + r - 3}\binom{m+\lambda_i - i + r - 3}{m}, \end{aligned}$$

which agrees with the column $j = r - 2$ of (54).

Continuing in this way, at the k-th step multiplying the column $j = r - k + 1$ by $(k)/x^{k+1}(m-k)$ and subtracting it from the column $j = r - k$, we eventually transform the expression for $H_m(\lambda)$ in (55) into the expression in (54). Thus the theorem is proved. □

Obviously Lemma 15.2 can be extended *via* Proposition 12.2 to give a determinantal expression for the generating function $e_{\lambda/\mu}(x, x^2, \ldots, x^m)$ for skew column-strict plane partitions of shape λ/μ and largest part $\leq m$. It is not known in what cases this determinant can be evaluated in a manner analogous to Theorem 15.3, even in the limiting case $m \to \infty$.

16. *Row and column restrictions*

It is natural to consider the problem of enumerating column-strict plane partitions with $\leq c$ columns (respectively, $\leq r$ rows) and largest part $\leq m$. Clearly the generating function for this class of partitions is $\sum_\lambda H_m(\lambda)$, where the sum is over all partitions λ with largest part $\lambda_1 \leq c$ (respectively, number of parts $\lambda_1' \leq r$). By a series of intricate computations, Basil Gordon (unpublished) has succeeded in expressing the former in a simple form, as conjectured by Bender and Knuth [18]. The limiting case $m = \infty$ was earlier proved by Gordon and Houten [31] (after Gordon first did the case $c = 2$ [30]). We simply state Gordon's result here. It would be desirable to have a simpler proof of this result.

16.1. Proposition (Gordon). The generating function for column-strict plane partitions with $\leq c$ columns and largest part $\leq m$ is

$$\prod_{i=1}^{m} \prod_{j=i}^{m} \frac{(c+i+j-1)}{(i+j-1)}. \quad \square$$

One case of particular interest is $c = 2$, $m = \infty$. Here the above generating function reduces to $\prod_{i=1}^{\infty} (i)^{-1}$, which is the generating function for ordinary partitions. Hence the number of column-strict plane partitions of n with ≤ 2 columns is equal to the number of partitions of n. A combinatorial proof of this fact was given by Sudler [61].

For the case of column-strict plane partitions with $\leq r$ rows and largest part $\leq m$, Gordon [33] obtains generating functions $C_r(x)$ in the case $m = \infty$. These generating functions are built up inductively as r increases using the "false theta-functions" of L. J. Rogers [54], and are not nearly as elegant as those of Proposition 16.1. The form of these generating functions imply, however, that no significant simplification is possible. For instance,

$$C_2(x) = \frac{(1-x) \sum_{n=0}^{\infty} (-1)^n x^{n(n+3)/2}}{(1)^2(2)^2(3)^2 \cdots}$$

$$C_3(x) = \frac{(1-x)\left(2 - (1+x+2x^2) \sum_{n=0}^{\infty} (-1)^n x^{1/n(n+5)/2}\right)}{(1)^3(2)^3(3)^3 \ldots}.$$

17. *Young tableaux, ballot problems, and Schensted's theorem*

A *Young tableau* is a plane partition with the p parts $1, 2, \ldots, p$. Usually in the definition of a Young tableau, the parts are required to be in *increasing* order in every row and column; however, the transformation $i \to p - i + 1$ shows that we get an equivalent theory by requiring the parts to be in decreasing order. Let f^λ denote the number of Young tableaux of shape λ. For instance, if $\lambda = (3, 2)$, then $f^\lambda = 5$, corresponding to the five Young tableaux

5 4 3   5 4 2   5 4 1   5 3 2   5 3 1
2 1   3 1   3 2   4 1   4 2

The numbers f^λ appear in several contexts in combinatorial theory, which we shall discuss. First we give Frame's elegant result [28] on computing f^λ

17.1. PROPOSITION. If $d_1, d_2, \ldots, d_p$ are the hook lengths of λ, then

$$f^\lambda = p!/d_1 d_2 \ldots d_p.$$

Proof: Let μ be the partition $\langle 1^p \rangle$. By definition of e_λ, f^λ is the coefficient of k_μ in e_λ, i.e., $f^\lambda = (e_\lambda, h_1^p)$. Consider the expansion of e_λ by the Jacobi–Trudi identity (Theorem 11.1). It is easily seen that the coefficient of k_μ in a term $h_{\lambda_1 - 1 + t_1} h_{\lambda_2 - 2 + t_2} \cdots h_{\lambda_r - r + t_r}$ is the multinomial coefficient

$$(\Sigma(\lambda_i - i + t_i))!/\Pi(\lambda_i - i + t_i)! = p!/\Pi(\lambda_i - i + t_i)!,$$

since $t_1, \ldots, t_l$ is a permutation of $1, \ldots, r$. It follows that

$$f^\lambda = p!|1/(\lambda_s - s + t)!|.$$

The above determinant is in fact a special case of the determinant for $H_m(\lambda)$ given by Lemma 15.2. Namely, let $H(\lambda) = \lim_{m \to \infty} H_m(\lambda)$. Then by Lemma 15.2,

$$\begin{aligned} H(\lambda) &= |x^{\lambda_i - i + j}/(\lambda_i - i + j)!| \\ &= (1)^{-p}|x^{\lambda_i - i + j}/(1 + x + x^2 + \cdots + x^{\lambda_i - i + j - 1})|. \end{aligned} \tag{56}$$

On the other hand, by Theorem 15.3,

$$H(\lambda) = x^a/(1)^p \Pi(1 + x + x^2 + \cdots + x^{d_i - 1}). \tag{57}$$

Comparing (56) and (57) as $x \to 1$ proves the proposition. □

We now consider the relation of f^λ to Knuth's correspondence $A \xrightarrow{K} (\pi, \sigma)$ (Theorem 6.1). It is clear that π and σ will both be Young tableaux with p parts if and only if A is a $p \times p$ permutation matrix. Thus the total number of ordered pairs (π, σ) of Young tableaux of the same shape and p parts is equal to the number of $p \times p$ permutation matrices. There follows:

17.2 PROPOSITION. $\sum_{\lambda \vdash p} (f^\lambda)^2 = p!$ □

Moreover, by Proposition 8.1, $A \xrightarrow{K} (\pi, \pi)$ where π is a Young tableaux with p parts, if and only if A is a symmetric permutation matrix, i.e., an element of the symmetric group S_p whose square is the identity. There follows:

17.3 PROPOSITION. Let t_p be the number of elements x in S_p satisfying $x^2 = 1$. Then

$$\sum_{\lambda \vdash p} f^\lambda = t_p. \quad \square$$

Propositions 17.2 and 17.3 were known (at least implicitly) since the early days of group representation theory, in the context of the following result of Young [66].

17.4. PROPOSITION. Let $\lambda \vdash p$. Then f^λ is the degree of the irreducible (ordinary) representation of the symmetric group S_p corresponding to the partition λ.

Proof: We have $f^\lambda = (e_\lambda, h_1^p) = (e_\lambda, s_1^p)$. Thus by Frobenius' theorem (Theorem 13.2), $f^\lambda = \chi^\lambda_{\langle 1^p \rangle}$. But $\chi^\lambda_{\langle 1^p \rangle}$ is just the character χ^λ evaluated at the identity element of S_p, which is the degree of the representation corresponding to λ. □

Thus Proposition 17.2 is seen to be a special case of the theorem that the sum of the squares of the degrees of the irreducible (ordinary) representation of a finite group is equal to the order of the group (see, e.g., M. Hall [5, Thm. 16.5.5]). On the other hand, Proposition 17.3 follows from Proposition 17.4 and the following result of Frobenius and Schur (see [69, p. 197]): The sum of the degrees of the (ordinary) irreducible representations of a finite group G is equal to the number of elements x in G satisfying $x^2 = 1$ if and only if every representation of G is equivalent to a real representation. Since it was known to Frobenius and Young that every representation of S_p is in fact equivalent to a *rational* representation, Proposition 17.3 follows. Proposition 17.3 seems to have been first stated explicitly by Robinson [53, part 1, p. 755], in the context of the representation theory of the symmetric group. In a purely combinatorial form, Proposition 17.3 was essentially first observed by Schützenberger [57, Prop. 2], and was stated explicitly by Bender and Knuth [18] and by Knuth [7, vol. 3, §52.4].

The numbers t_p were first considered by H. A. Rothe [72], and later studied by Chowla, Herstein, and Moore [25], Moser and Wyman [46], and others. It is known, for instance, that

$$t_p = t_{p-1} + (p-1)t_{p-2}, \qquad t_0 = t_1 = 1$$

$$\sum_{p=0}^{\infty} t_p x^p/p! = e^{x+x^2/2}$$

$$t_p \sim (p/e)^{p/2} e^{\sqrt{p}}/\sqrt{2}e^{1/4}$$

Kreweras [43] has studied hook lengths, the f^λ's, and related topics from a lattice-theoretical point of view. He defines the *Young lattice* $\boldsymbol{T}$ (*trellis de Young*) to be the set of all partitions of all integers, ordered by defining $\lambda \le \mu$ if $\lambda_i \ge \mu_i$ for all i. This lattice $\boldsymbol{T}$ has the interesting characterization of being the unique locally finite distributive lattice with 1 such that if an element X is covered by n elements, then X covers $n+1$ elements (for terminology, see Birkhoff [2]). The number of maximal chains between a partition $\lambda \in \boldsymbol{T}$ and the top element 1 (corresponding to the void partition (0)) is f^λ. More generally, if $\lambda \le \mu$ in $\boldsymbol{T}$, then the number $f^{\lambda/\mu}$ of maximal chains between λ and μ is equal to the number of Young tableaux of shape λ/μ (see Section 12), so by Proposition 12.4, $f^{\lambda/\mu} = (e_{\lambda/\mu}, h_1^q) = (e_\lambda, e_\mu h_1^q)$, where q is the number of parts in the shape λ/μ. Using this definition of f^λ, Kreweras gives interesting combinatorial proofs of Propositions 17.1, 17.2, and 17.3, by induction arguments on elements of $\boldsymbol{T}$. A more general lattice-theoretical approach to partitions is considered by Stanley [59].

Let us now turn to a well-known result of Erdös and Szekeres [26]: any permutation of the integers $1, 2, \ldots, n^2+1$ either contains an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$. Schensted [55] has obtained a considerable strengthening of this result *via* Proposition 8.5. Note that when Proposition 8.5 is applied to the permutation matrix A of the permutation $\begin{pmatrix} 1 & 2 & \ldots & m \\ j_1 & j_2 & \ldots & j_m \end{pmatrix}$, we get that $A \xrightarrow{K} (\pi, \sigma)$, where π and σ are Young tableaux of the same shape whose number of rows is equal to the length of the longest increasing subsequence of the permutation $j_1\, j_2, \ldots, j_m$, and whose number of columns is equal to the length of the longest decreasing subsequence of $j_1, j_2, \ldots j_m$. There follows from the definition of f^λ:

17.5. Proposition (Schensted [55]). The number of permutations of $1, 2, \ldots, m$ with longest increasing subsequence of length c and longest decreasing subsequence of length r is equal to $\sum_\mu (f^\mu)^2$, where the sum is over all partitions μ of m satisfying $\mu'_1 = c$, $\mu_1 = r$. $\square$

Note that the result of Erdös and Szekeres follows trivially from this proposition, since any partition μ of $n^2 + 1$ necessarily satisfies $\mu_1 > n$ or $\mu'_1 > n$. As a further application, Stanley [58] found the number of permutations of $1, 2, \ldots, n^2$ with longest increasing and decreasing subsequences both of length n.

Another type of problem closely related to the f^λ's are *ballot problems*. The "generalized ballot problem" may be stated as follows. After an election among r candidates $A_1, \ldots, A_r$, the number of votes received by A_i is λ_i, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. What is the probability $P(\lambda)$ that A_i never trails A_{i+1} during the voting, for all $i = 1, 2, \ldots, r-1$, if the votes are cast consecutively?

The "classical ballot problem," or Bertrand's ballot problem [19] is the case $r = 2$. Bertrand obtained the probability $P(\lambda) = (\lambda_1 - \lambda_2 + 1)/(\lambda_1 + 1)$. The generalized ballot problem was first solved by MacMahon [9, Section 103]. We shall now indicate a solution, expressed somewhat differently from MacMahon. Let us call a sequence of votes satisfying the conditions of the ballot problem an *admissible sequence corresponding to* λ. (Called by MacMahon a *lattice permutation*.)

17.6. Proposition. The number of admissible sequences corresponding to λ is f^λ.

Proof: We need to set up a one-to-one correspondence between admissible sequences and Young tableaux of shape λ. Suppose $\lambda \vdash p$. If the ith voter votes for candidate A_j, then we require the integer $p - i + 1$ to appear in the jth row of the Young tableau. This is easily seen to set up the desired correspondence. $\square$

Thus the desired probability is

$$P(\lambda) = f^\lambda \bigg/ \binom{p}{\lambda_1, \lambda_2, \ldots, \lambda_r},$$

where $\binom{p}{\lambda_1, \lambda_2, \ldots, \lambda_r}$ is a multinomial coefficient. Hence by Proposition 17.1, we finally get

$$P(\lambda) = \frac{\lambda_1 ! \, \lambda_2 ! \ldots \lambda_r !}{d_1 d_2 \ldots d_p},$$

where the d_i's are the hook lengths of λ.

For further aspects of ballot problems, see Feller [4, Ch. III], Tákacs [14, Ch. 1], and the excellent survey article by Barton and Mallows [68].

In conclusion, we mention a result of Philip Hall [35] connected with finite abelian p-groups.

17.7. Proposition (P. Hall). Let $C_\lambda(p)$ be the number of composition series of a finite abelian p-group of type $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then for fixed λ, $C_\lambda(p)$ is a polynomial in p of degree $\Sigma(i-1)\lambda_i = \Sigma\binom{\lambda'_i}{2}$ and leading coefficient f^λ. $\square$

V. Enumeration of ordinary plane partitions

18. *Row, column, and part bounds*

We proceed to the enumeration of ordinary plane partitions (i.e., no strictness conditions). The general idea is to set up correspondences between column-strict plane partitions and plane partitions. There are two basic such correspondences known, leading to two classes of generating functions for plane partitions. The first such generating function was the main result of MacMahon on plane partitions [9, Section 495]; his method of proof is entirely different from ours. Other proofs have been given by Chaundy [23] and Carlitz [21].

18.1 THEOREM. The generating function $G_m(r, c)$ for plane partitions with $\leq r$ rows and $\leq c$ columns, with largest part $\leq m$, is given by

$$G_m(r, c) = \frac{\binom{m+r}{r}\binom{m+r+1}{r}\cdots\binom{m+r+c-1}{r}}{\binom{r}{r}\binom{r+1}{r}\cdots\binom{r+c-1}{r}}. \tag{58}$$

Proof: Let π be a column-strict plane partition of shape λ, where $\lambda = \langle c^r \rangle$. Then the transformation $n_{ij} \to n_{ij} - r + i - 1$ (defined for all parts $n_{ij} > 0$ of π) sets up a one-to-one correspondence between column-strict plane partitions of n shape λ with largest part $\leq m$ and plane partitions of $n - c\binom{r+1}{2}$ with $\leq r$ rows and $\leq c$ columns, with largest part $\leq m - r$. It follows that $G_m(r, c) = x^{-c\binom{r+1}{2}} H_{m+r}(\lambda)$. It is a simple matter to evaluate the hook lengths and contents of λ to obtain *via* Theorem 15.3 an explicit expression for this generating function, which is then easily transformed into the form (58). □

By letting $c \to \infty$, $m \to \infty$ in the above theorem, we get the beautiful generating functions of MacMahon [9, Section 422] (see also Chaundy [23], Carlitz [21], and for $r \leq 3$, Cheema and Gordon [24]):

18.2. COROLLARY. *The generating function for plane partitions with $\leq r$ rows is*

$$\sum_{k=1}^{\infty} (k)^{-\min(k,r)}.$$

In particular, the generating function for (unrestricted) plane partitions is

$$\prod_{k=1}^{\infty} (k)^{-k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + 86x^7$$
$$+ 160x^8 + 282x^9 + 500x^{10} + \cdots \quad □$$

Though the above generating functions are very simple in form, no really easy proof of Corollary 18.2 is known.

We remark that Bender and Knuth [18] give a mild generalization of the $c = \infty$ case of Theorem 18.1, viz., the generating function for plane partitions with $\leq r$ rows and largest part $\leq m$, and with exactly k parts in the rth row is

$$x^{rk} \prod_{j=1}^{k} (m+j-1)(j)^{-1} \cdot \prod_{i=1}^{m} \prod_{j=1}^{r-1} (i+j-1)^{-1}.$$

Various persons have considered the problem (or obviously equivalent problems) of enumerating the number of $r \times c$ rectangular arrays of the integers $0, 1, 2, \ldots, m$ which decrease in every row and column (e.g., Carlitz [21], or for

$r = 2$, Carlitz and Riordan [22]). Clearly the desired number of such arrays is $G_m(r, c)_{x=1}$. Thus Theorem 18.1 gives this number explicitly, since

$$\begin{pmatrix} a \\ b \end{pmatrix}_{x=1} = \binom{a}{b}.$$

It is natural to ask for the generating function for plane partitions of a given shape λ and largest part $\leq m$. Unfortunately the technique used in proving Theorem 18.1 does not extend to this case. MacMahon [9, Section 495] obtains a fairly simple expression for this generating function involving a determinant; but it seems unlikely that this determinant can be simplified significantly, even in the cases $m = \infty$ or $x = 1$.

It is possible, however, to extend the proof of Theorem 18.1 to the case of *reverse plane partitions* of shape λ, when $m = \infty$. A reverse plane partition of shape λ is an array of positive integers of shape λ which *increases* in every row and column. We state some results on reverse plane partitions which follow from Theorem 15.3, referring the reader to Stanley [59, Ch. V, Section 2] for the details.

18.3. Proposition. Let $\lambda \vdash p$. The generating function for reverse plane partitions of shape λ is

$$x^p/(d_1)(d_2)\ldots(d_p),$$

where the d_i's are the hook lengths of λ. □

18.4. Proposition. Let $\lambda \vdash p$. The generating function for column-strict reverse plane partitions with largest part $\leq m$ is $H_m(\lambda)$. □.

18.5. Proposition. Let $\lambda \vdash p$. The generating function for row *and* column-strict reverse plane partitions of shape λ is

$$x^b/(d_1)(d_2)\ldots(d_p),$$

where the d_i's are the hook lengths of λ and $b = \Sigma(\binom{\lambda_i'}{2} + i\lambda_i) = p + \Sigma d_i$.

19. *The conjugate trace and trace of a plane partition*

If we let $c \to \infty$ in (58), we see that the generating function for plane partitions with $\leq r$ rows and largest part $\leq m$ is

$$\prod_{i=1}^{m} \prod_{j=1}^{r} (i + j - 1)^{-1} \tag{59}$$

In view of (59), it is natural to ask whether there is some combinatorial interpretation of the coefficient of $q^t x^n$ in the expansion of

$$\prod_{i=1}^{m} \prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}.$$

An affirmative answer was given by Stanley [60], using a second correspondence between column-strict plane partitions and plane partitions. In order to state Stanley's result, we require some definitions.

19.1. Definition. Let π be a plane partition. The *conjugate trace* of π is defined to be the number of parts n_{ij} of π satisfying $n_{ij} \geq i$. The *trace* of π is defined to be Σn_{ii}.

We write $T^*_{rmt}(n)$ (respectively, $T_{rmt}(n)$) for the number of plane partitions of n with $\leq r$ rows and largest part $\leq m$, and with conjugate trace t (respectively, trace t). Also define

$$T^*_{rt}(n) = \lim_{m\to\infty} T^*_{rmt}(n),\ T^*_t(n) = \lim_{r\to\infty} T^*_{rt}(n)$$

$$T_{rt}(n) = \lim_{m\to\infty} T_{rmt}(n),\ T_t(n) = \lim_{r\to\infty} T_{rt}(n).$$

Thus, e.g., $T_{rt}(n)$ is equal to the number of plane partitions of n with $\leq r$ rows and trace t.

Every plane partition π has six *conjugates* (called *aspects* by MacMahon [9, Section 427]). One of these, call it π', is obtained from π by taking the conjugate partitions of each row. For example,

$$\begin{array}{cccc} 3 & 3 & 2 & 1 \\ 3 & 1 & & \\ 2 & 1 & & \\ & \pi & & \end{array} \qquad\qquad \begin{array}{ccc} 4 & 3 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & \\ & \pi' & \end{array}$$

It is easily seen that π and π' are plane partitions of the same integer n, that they have the same number of rows, and that the conjugate trace of π is equal to the trace of π'. (This explains the terminology "conjugate trace.") There follows:

19.2. PROPOSITION. $T^*_{rt}(n) = T_{rt}(n)$. □

We now come to Stanley's result:

19.3. THEOREM. *We have*

$$\sum_{n=0}^{\infty}\sum_{t=0}^{\infty} T^*_{rmt}(n) q^t x^n = \prod_{i=1}^{m}\prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}.$$

Proof: Frobenius [29] (cf. also Sudler [61], Littlewood [8, p. 602]) has constructed a one-to-one correspondence between linear partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ of p and pairs of strict partitions μ and ν of the form

$$\lambda_1 = \mu_1 > \mu_2 > \cdots > \mu_s > 0$$

$$r = \nu_1 > \nu_2 > \cdots > \nu_s > 0,$$

with $\Sigma(\mu_i + \nu_i) = p + s$. This correspondence is defined by the conditions

$$\mu_i = \lambda_i - i + 1 \quad (\text{when } \lambda_i - i + 1 > 0)$$

$$\nu_i = \lambda'_i - i + 1 \quad (\text{when } \lambda'_i - i + 1 > 0)$$

Note that in this correspondence,

$$\lambda_i \geq i \quad \text{if and only if } \mu_i \geq 1. \tag{60}$$

This correspondence has a simple visual interpretation which we illustrate with the example $\lambda = (4, 4, 3, 1, 1, 1)$, $\mu = (4, 3, 1)$, $\nu = (6, 2, 1)$.

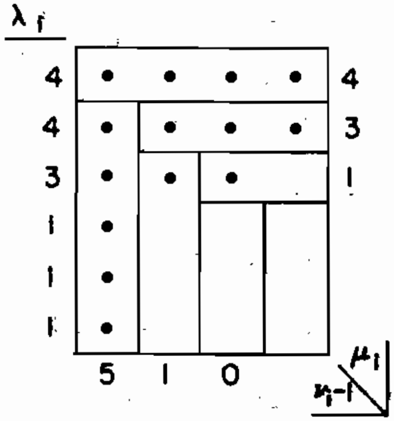

Figure 1. A construction of Frobenius.

Bender and Knuth [18] generalize this construction straightforwardly as follows: If π is a plane partition, then apply the construction of Frobenius to each column to get a pair of column-strict plane partitions π_1 and π_2 of the same shape. For instance, the plane partition

$$\begin{matrix} 4 & 4 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & & \\ 2 & & & \\ 2 & & & \end{matrix}$$

corresponds to the pair

$$\begin{matrix} 4 & 4 & 2 & 1 \\ 3 & 1 & 1 & \\ 2 & & & \end{matrix} \qquad \begin{matrix} 5 & 3 & 2 & 2 \\ 4 & 2 & 1 & \\ 1 & & & \end{matrix}$$

In this correspondence, the number r of rows of π equals the largest part m_2 of π_2; the largest part m of π equals the largest part m_1 of π_1; and by (60) the conjugate trace t of π equals the number of parts $p_1 = p_2$ of π_1 and π_2. Also if π_i is a plane partition of n_i, then π is a plane partition of $n_1 + n_2 - t$.

Thus $T^*_{rmt}(n)$ is equal to the number of pairs π_1, π_2 of column-strict plane partitions of the same shape satisfying:

(i) the largest part of π_1 is $\leq r$
(ii) the largest part of π_2 is $\leq m$
(iii) the number of parts of π_1 or π_2 is t
(iv) the sum of the parts of π_1 and π_2 is $n + t$.

It follows from Knuth's correspondence (Theorem 6.1) that

$$\sum_{n=0}^{\infty}\sum_{t=0}^{\infty} T^*_{rmt}(n)q^t x^n = \prod_{i=1}^{m}\prod_{j=1}^{r}\sum_{a_{ij}=0}^{\infty} q^t x^n,$$

where

$$t = \sum_{i,j} a_{ij}$$

$$n + t = \sum_j \left(j \sum_i a_{ij} \right) + \sum_i \left(i \sum_j a_{ij} \right).$$

The above product thus equals

$$\prod_{i=1}^{m} \prod_{j=1}^{r} \sum_{a_{ij}=0}^{\infty} q^{a_{ij}} x^{(i+j-1)a_{ij}} = \prod_{i=1}^{m} \prod_{j=1}^{r} (1 - qx^{i+j-1})^{-1}. \quad \square$$

Stanley [60] performs an algebraic analysis of the generating function in Theorem 19.3 for the case $m = \infty$. We state two of the more interesting results given there. The first of these results is a stability theorem analogous to Proposition 14.2.

19.4. PROPOSITION. $T_{rt}^{*}(n + t)(= T_{rt}(n + t))$ is the coefficient of x^n in the expansion of

$$\prod_{k=1}^{\infty} (k)^{-\min(k+1,r)},$$

if $t \geq n$. In particular (letting $r \to \infty$), $T_t^{*}(n + t)(= T_t(n + t))$ is the coefficient of x^n in the expansion of

$$\prod_{k=1}^{\infty} (k)^{-(k+1)} = 1 + 2x + 6x^2 + 14x^3 + 33x^4 + 70x^5 + 149x^6 + \cdots,$$

if $t \geq n$. $\square$

The next result is a generalization of the classical formula $\sum_{\lambda \vdash p} (f^{\lambda})^2 = p!$ (Proposition 17.2), obtained by equating coefficients of x^p below.

19.5. PROPOSITION. Let x be an indeterminate. Then

$$\sum_{\lambda \vdash p} (f^{\lambda})^2 (x + c_1)(x + c_2) \ldots (x + c_p) = p! x^p,$$

where the c_i's are the contents of λ. $\square$

20. *Asymptotics*

It is natural to ask for an asymptotic estimate of the number of plane partitions of n with various properties. The simplest case occurs when the corresponding generating function is a rational function; in theory the entire asymptotic expansion can be determined by the method of partial fractions. The next proposition gives some results of this nature; we only give the first term of the asymptotic expansion, and we omit the proofs.

20.1. PROPOSITION. (i) Let λ be a fixed partition of p, and let $a_\lambda(n)$, $b_\lambda(n)$, $c_\lambda(n)$ denote respectively the number of ordinary plane partitions, column-strict plane partitions, and row and column-strict plane partitions of n of shape λ. Then

$$a_\lambda(n) \sim b_\lambda(n) \sim c_\lambda(n) \sim f^{\lambda} n^{p-1}/p!(p-1)!$$

(ii) Let p be a fixed positive integer, and let $a^{(p)}(n)$, $b^{(p)}(n)$, $c^{(p)}(n)$ denote respectively the number of ordinary plane partitions, column-strict plane partitions, and row and column-strict plane partitions of n with exactly p parts. Then

$$a^{(p)}(n) \sim b^{(p)}(n) \sim c^{(p)}(n) \sim t_p n^{p-1}/p!(p-1)!,$$

with $t_p = \sum_{\lambda \vdash p} f^\lambda$ (see Proposition 17.3).

(iii) Let t and r be fixed positive integers, and let $T_{rt}^*(n)$, $T_t^*(n)$, $T_{rt}(n)$, $T_t(n)$ be as in Section 19. Then

$$T_{rt}^*(n) = T_{rt}(n) \sim r^t n^{t-1}/t!(t-1)!$$

$$T_t^*(n) = T_t(n) \sim n^{2t-1}/t!(2t-1)! \quad \square$$

It is considerably more difficult, however, to obtain asymptotic results when the appropriate generating function is not rational. Using the generating function of Corollary 18.2, E. M. Wright [63] obtained the asymptotic expansion for the number $a(n)$ of plane partitions of n. His proof is based on the techniques developed by Hardy and Ramanujan [36] (see, e.g., Ayoub [1, Ch. III]) in their analysis of the ordinary partition function $p(n)$. There does not, however, appear to be a plane partition analog of Rademacher's convergent series for $p(n)$ [51], essentially because the generating function for $a(n)$ is not a modular function. The leading term of Wright's expansion is as follows:

20.2. Proposition. Let $a(n)$ be the number of plane partitions of n. Then

$$a(n) \sim (\zeta(3)2^{-11})^{1/36} n^{-25/36} \exp(3 \cdot 2^{-2/3}\zeta(3)^{1/3} n^{2/3} + 2C),$$

where ζ is the Riemann zeta function and

$$C = \int_0^\infty \frac{y \log y \, dy}{e^{2\pi y} - 1}. \quad \square$$

Subsequently Gordon and Houten [32] obtained an asymptotic formula for the number $b_k(n)$ of column-strict plane partitions of n with $\leq k$ columns, and for the total number $b(n)$ of column-strict plane partitions of n, using the generating function of Proposition 16.1 (as $m \to \infty$). We state their result only for $b(n)$.

20.3. Proposition. Let $b(n)$ be the number of column-strict plane partitions of n. Then

$$b(n) \sim 2^{-3/4}(3\pi\zeta(3))^{-1/2} N^{-49/24} \cdot \exp\left(\frac{3}{2}\zeta(3)N^2 + \frac{\pi^2}{24}N - \frac{\pi^4}{3456\zeta(3)} + C\right),$$

where $N = (n/\zeta(3))^{1/3}$, and ζ, C as in Proposition 20.2. $\square$

VI. Conclusion

21. *Open problems*

We summarize most of the open problems concerning plane partitions mentioned previously, and discuss some new ones.

(i) Find a combinatorial proof of Littlewood's identity (31).

(ii) What is the effect of transposing A on Knuth's dual correspondence $A \overset{K^*}{\leftrightarrows} (\pi, \sigma)$ (Theorem 9.1)? Tie in with Littlewood's identity (32).

(iii) Develop further the theory of skew plane partitions (Section 12). In particular, when can the determinant $e_{\lambda/\mu}(x, x^2, \ldots, x^m)$ given by Proposition 12.2 be explicitly evaluated?

(iv) Find a combinatorial proof of Gordon's generating function for column-strict plane partitions with $\leq c$ columns and largest part $\leq m$ (Proposition 16.1).

(v) When $m \to \infty$ in Proposition 16.1, we get

$$\prod_{i=1}^{\infty} \prod_{j=i}^{\infty} \frac{(c+i+j-1)}{(i+j-1)}$$

as the generating function for column-strict plane partitions with $\leq c$ columns. This leads us to ask for a combinatorial interpretation of the coefficient of $q^t x^n$ in the expansion of

$$\prod_{i=1}^{\infty} \prod_{j=i}^{\infty} \frac{(1-qx^{c+i+j-1})}{(1-qx^{i+j-1})}.$$

Even the cases $c = 1, 2$ are not trivial. In the case $c = 1$, t corresponds to the sum of the largest powers of 2 dividing the parts of the plane partition. In the case $c = 2$, it follows from a construction of Sudler [61] that t corresponds to one of the parts n_{11} or n_{12} of the plane partition, but the rule for deciding which one is rather complicated.

(vi) Simplify when possible MacMahon's determinant generating function for plane partitions of shape λ and largest part $\leq m$ (see Section 18).

(vii) Is there a simple expression for the generating function $\Sigma\Sigma a(p, n) q^p x^n$, where $a(p, n)$ is the number of plane partitions of n with exactly p parts?

(viii) (Gordon) Find a simple combinatorial interpretation of the coefficient $s(n)$ of x^n in the expansion of $\prod_{k=1}^{\infty} (1 + x^k)^k$. It follows from Corollary 9.2, upon substituting $x_i = x^i$, $y_i = x^{i-1}$, that $s(n)$ is equal to the number of pairs (π, σ) of column-strict plane partitions of conjugate shape, the combined sum of whose parts is $n + p$, where p is the number of parts of π or σ. A simpler interpretation, however, is desired.

(ix) Gordon [33] investigates the generating functions $D_r(x)$ for row and column-strict plane partitions with $\leq r$ rows. He is able only to express $D_r(x)$ as a sum of determinants, except in the case $r = 2$. In this case he obtains

$$D_2(x) = \frac{\left(1 - (1-x) \sum_{n=1}^{\infty} x^{(2n-1)(n+1)}\right)}{(1)(2)(3)\ldots}.$$

Can similar expressions be obtained for $r > 2$?

(x) Let $E(n)$ denote the expected length of the longest increasing subsequence of a random permutation of $1, 2, \ldots, n$. The asymptotic rate of growth of $E(n)$ is unknown. It follows from Schensted's theorem (Proposition 17.5) that

$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 (f^\lambda)^2.$$

Using this fact, Baer and Brock [67] have compiled extensive tables which suggest that possibly $E(n) \sim 2\sqrt{n}$, but no proof is known.

(xi) Develop a "q-analog" of the theory of plane partitions. In this q-theory, the role of the symmetric group should be replaced with the general linear group over the field $GF(q)$. For some combinatorial aspects of the finite general linear groups, see Green [34], Klein [38], and the references cited there.

(xii) *Multi-dimensional partitions.* Develop a theory of multi-dimensional partitions. For instance, a *3-dimensional partition* (or *solid partition*) is an array n_{ijk} of non-negative integers decreasing in all three directions. MacMahon [9, Section 424], conjectured that the generating function for r-dimensional partitions is

$$\prod_{k=1}^{\infty} (k)^{-\binom{k+r-2}{r-1}},$$

but this was shown by Atkin et al. [17], and later by E. M. Wright [65], to be false. (Nanda [49], [50] erroneously assumes this conjecture to be true for $r = 3$.) Bender feels that the key to understanding solid partitions lies in finding an analog to Theorem 6.2 (symmetry of e_λ).

Very few positive results are known concerning multi-dimensional partitions. E. M. Wright [64] has found a complicated generating function for a special class of solid partitions. G. Kreweras [43, eqn. (85)] has found the number of "Young tableaux" of solid partitions with a certain kind of shape. Knuth [71] proves a theorem upon which he bases an algorithm for counting solid partitions. Knuth's theorem was also proved in a more general form by Stanley [59, p. 55]. Finally we mention the *reciprocity theorem* of Stanley [59, Ch. II, Thm. 6.5] for multi-dimensional partitions, which is a consequence of his more general Theorem 6.2. Gordon [33] also gives a proof of this theorem in the case $r = 2$. The terminology of the theorem is self-evident.

21.1. THEOREM. *Let* $\{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, r\}$ *and let* $\{j_1, j_2, \ldots, j_{r-s}\} = \{1, 2, \ldots, r\} - \{i_1, i_2, \ldots, i_s\}$. *Let* $A(x)$ *be the generating function for r-dimensional partitions strict in directions* $i_1, i_2, \ldots, i_s$ *and of a fixed shape* π, (π *is an* r-1*-dimensional partition) and let* $B(x)$ *be the generating function for r-dimensional partitions strict in the complementary directions* $j_1, j_2, \ldots, j_{r-s}$ *and also of shape* π. *Then* $A(x)$ *and* $B(x)$ *are rational functions of x related by*

$$B(x) = (-1)^p x^p A(1/x),$$

where p is the number of parts of π. □

References

The following bibliography is not intended to be complete, though most purely combinatorial results on plane partitions are included. For further references to Schur functions and their connection with group theory, see the bibliographies in Littlewood [8] and Robinson [12]. For further references to ballot problems, see the first chapter of Takács [14] or the survey paper of Barton and Mallows [68].

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Massachusetts Institute of Technology

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