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# Theory and Application of Plane Partitions: Part 1

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# I. Introduction

## 1. *Definitions*

A *partition* $\lambda$ of a non-negative integer $n$ can be regarded as a decreasing sequence of positive integers,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \tag{1}$$

satisfying $\Sigma\lambda_i = n$. We say that $\lambda$ has $r$ *parts.* Because of the linear nature of the array (1), we also refer to $\lambda$ as a *linear partition* of $n$. Similarly a partition of $n$ into distinct parts may be regarded as a strictly decreasing array of positive integers,

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0, \tag{2}$$

satisfying $\Sigma\lambda_i = n$. Such a partition is called a *strict partition* of $n$.

We denote partitions in three ways:

(i) $\lambda \vdash n$ signifies that $\lambda$ is a partition of $n$ (a notation due to Philip Hall [35]);
(ii) $\lambda = (\lambda_1, \lambda_2, \ldots)$ signifies that the parts of $\lambda$ are $\lambda_1 \geq \lambda_2 \geq \ldots$,
(iii) $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$ signifies that exactly $r_i$ parts of $\lambda$ are equal to $i$.

It is natural to extend these concepts to more general arrays of integers. A general theory along these lines has been developed by Stanley [59], but we will be concerned here with the special case known as *plane partitions.* The theory of plane partitions forms one of the most beautiful branches of combinatorial theory, with applications to such diverse topics as ballot problems, symmetric functions, and the representation theory of the symmetric group. This paper is devoted to giving a survey, not intended to be completely comprehensive, of the theory of plane partitions, including a selection of proofs large enough to impart the flavor of the subject.

A *plane partition* $\pi$ of $n$ is an array of non-negative integers,

$$\begin{array}{cccc} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{array} \tag{3}$$

for which $\Sigma n_{ij} = n$ and the rows and columns are in decreasing order:

$$n_{ij} \geq n_{(i+1)j}, \qquad n_{ij} \geq n_{i(j+1)}, \qquad \text{for all } i, j \geq 1.$$

The non-zero entries $n_{ij} > 0$ are called the *parts* of $\pi$. If there are $\lambda_i$ parts in the $i$th row of $\pi$, so that for some $r$,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0,$$

then we call the partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ of the integer $p = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ the *shape* of $\pi$, denoted by $\lambda$. We also say that $\pi$ has $r$ *rows* and $p$ *parts.* Similarly if $\lambda_i'$ is the number of parts in the $i$th column of $\pi$, then for some $c$,

$$\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_c' > \lambda_{c+1} = 0.$$

The partition $\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_c'$ of $p$ is the *conjugate partition* to $\lambda$ [6, Ch. 19.2], denoted $\lambda'$, and we say that $\pi$ has $c$ *columns.*

If the non-zero entries of $\pi$ are strictly decreasing in each row, we say that $\pi$ is *row-strict*, *Column-strict* is similarly defined. If $\pi$ is both row-strict and column-strict, we say that $\pi$ is *row and column-strict*.

## II. Symmetric functions

### 2. *The four basic symmetric functions*

The wide variety of results known about plane partitions can be unified greatly by appealing to the theory of symmetric functions. We use a method involving elementary linear algebra, due to Philip Hall [35]. Let $A_n$ denote the set of all homogeneous symmetric functions of degree $n$ in the infinitely many indeterminates $x_1, x_2, \ldots$, with coefficients in the field $Q$ of rational numbers. We regard elements of $A_n$ merely as formal expressions. $A_n$ has the structure of a vector space over $Q$. We can also make the $A_n$'s into a *graded algebra*,

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots,$$

by defining multiplication to be ordinary power series multiplication. We are interested in studying various bases for the vector space $A_n$.

If $\lambda \vdash n$, define

$$k_\lambda = \Sigma x_1^{\lambda_1} x_2^{\lambda_2} \ldots, \tag{4}$$

where the summation sign indicates that we are to form all *distinct* monomials in the $x_i$'s with exponents $\lambda_1, \lambda_2, \ldots$ (in some order). The $k_\lambda$'s are known as the *monomial symmetric functions*. It is easily seen that the $k_\lambda$'s form a basis for $A_n$ as $\lambda$ runs over all partitions of $n$. Thus $A_n$ has dimension $p(n)$, the number of partitions of $n$. For an introduction to the function $p(n)$, see Hardy and Wright [6, Ch. 19].

If we wish to specialize certain values of $x_i$, we indicate this by notation such as

$$k_{1,1}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$$

$$k_n(x, x^2, x^3, \ldots) = x^n + x^{2n} + x^{3n} + \cdots = x^n/(1 - x^n)$$

(here $k_n$ denotes $k_\lambda$ where $\lambda = (n, 0, 0, \ldots) = \langle n^1 \rangle$). We also use $x$ and $y$ to denote the vectors $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$, so $k_\lambda(x) = \Sigma x_1^{\lambda_1} x_2^{\lambda_2} \ldots$ and $k_\lambda(y) = \Sigma y_1^{\lambda_1} y_2^{\lambda_2} \ldots$. The $x_i$'s and $y_i$'s are to be regarded as independent indeterminates.

Any basis which can be obtained from $k_\lambda$ *via* a matrix with integral coefficients and determinant $\pm 1$ is called an *integral basis*. We now consider two important integral bases. Define

$$h_n = \sum_{\lambda \vdash n} k_\lambda,$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots. \tag{5}$$

The $h_n$'s are the *complete homogeneous symmetric functions*. Also define

$$a_n = k_{\langle 1^n \rangle} = \Sigma x_1 x_2 \ldots x_n,$$

$$a_\lambda = a_{\lambda_1} a_{\lambda_2} \ldots. \tag{6}$$

The $a_n$'s are the *elementary symmetric functions*. It is easily seen that $h_\lambda$ and $a_\lambda$ are bases for $A_n$. Because of (5) and (6), we say that the bases $h_\lambda$ and $a_\lambda$ are *multiplicative*. The fact that $a_\lambda$ is a basis for $A_n$ is equivalent to the basic theorem that every symmetric function can be uniquely expressed as a polynomial in the elementary symmetric functions. A simple induction argument shows that we can express the $k_\lambda$'s as integral linear combinations of the $h_\lambda$'s or $a_\lambda$'s, so $h_\lambda$ and $a_\lambda$ are integral bases.

In analogy to $a_\lambda$ we define a fourth basis $s_\lambda$ by

$$s_n = k_n = \Sigma x_1^n$$
$$s_\lambda = s_{\lambda_1} s_{\lambda_2} \dots \tag{7}$$

The $s_n$'s are the *power sum symmetric functions*. Although $s_\lambda$ is a multiplicative basis, it is not integral, e.g., $k_{1,1} = \frac{1}{2}(s_1^2 - s_2)$. Soon we will determine the determinant of the transformation $k_\lambda \to s_\lambda$.

### 3. *Relations among the symmetric functions*

Some basic relations among the symmetric functions can be expressed in terms of linear transformations among the various bases. Define linear transformations (or matrices) $\phi$ and $\theta$ by

$$\phi : k_\lambda \to h_\lambda$$
$$\theta : a_\lambda \to h_\lambda. \tag{8}$$

Note that since $a_\lambda$ and $h_\lambda$ are multiplicative, $\theta$ preserves multiplication and is therefore an automorphism of the algebra $A$.

The basic properties of $\phi$ and $\theta$ are:

3.1. PROPOSITION. $\phi$ is symmetric.

3.2. PROPOSITION. $\theta^2 = 1$.

3.3. PROPOSITION. The $s_\lambda$'s are eigenvectors for $\theta$; indeed,

$$\theta s_\lambda = (-1)^{n-r} s_\lambda, \qquad \text{if } \lambda \vdash n, \qquad \lambda = (\lambda_1, \dots, \lambda_r).$$

The key to proving these relations lies in observing that we have the generating functions

$$\prod_{i=1}^{\infty} (1 - x_i t)^{-1} = \sum_{n=0}^{\infty} h_n t^n \tag{9}$$

$$\prod_{i=1}^{\infty} (1 + x_i t) = \sum_{n=0}^{\infty} a_n t^n. \tag{10}$$

It follows from (9) that

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \prod_{i=1}^{\infty} \sum_{m=0}^{\infty} h_m(y) x_i^m = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} k_\lambda(x) h_\lambda(y),$$

which we abbreviate to

$$\prod (1 - x_i y_j)^{-1} = \sum_\lambda k_\lambda(x) h_\lambda(y). \tag{11}$$

Now suppose

$$h_\lambda(y) = \sum_\mu \phi_{\lambda\mu} k_\mu(y), \qquad k_\lambda(x) = \sum_\nu \psi_{\lambda\nu} h_\nu(x),$$

where $\psi = \phi^{-1}$. Then from (11) we get

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \sum_{\mu,\nu} \left( \sum_{\lambda} \phi_{\lambda\mu} \psi_{\lambda\nu} \right) k_\mu(y) h_\nu(x). \tag{12}$$

Since the left-hand side of (11) is symmetric in the $x_i$'s and $y_j$'s, we also have

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \sum_{\lambda} k_\lambda(y) h_\lambda(x). \tag{13}$$

Since the distinct products $k_\mu(y)h_\nu(x)$ are linearly independent, we see from comparing (12) and (13) that

$$\sum_{\lambda} \phi_{\lambda\mu} \psi_{\lambda\nu} = \delta_{\mu\nu} \qquad (\delta = \text{Kronecker delta}).$$

Since $\sum_\lambda \phi_{\mu\lambda} \psi_{\lambda\nu} = \delta_{\mu\nu}$, we have $\phi_{\lambda\mu} = \phi_{\mu\lambda}$, so $\phi$ is symmetric.

Similarly we prove $\theta^2 = 1$ using the relation

$$\left[ \sum_{n=0}^{\infty} a_n (-t)^n \right] \left[ \sum_{n=0}^{\infty} h_n t^n \right] = 1. \tag{14}$$

The details are omitted. The identity arising from (14), viz.,

$$\sum_{r=0}^{n} (-1)^r a_r h_{n-r} = \delta_{0n},$$

is thus equivalent to $\theta^2 = 1$.

To prove Proposition 3.3, we need to find a generating function for the $s_n$'s. We have

$$\begin{aligned} \log \sum_{n=0}^{\infty} h_n t^n &= \log \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \\ &= \sum_{i=1}^{\infty} \log(1 - x_i t)^{-1} \\ &= \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x_i^m t^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m} s_m t^m. \end{aligned} \tag{15}$$

Similarly,

$$\log \sum_{n=0}^{\infty} a_n t^n = \sum_{m=1}^{\infty} \frac{1}{m} s_m (-t)^m. \tag{16}$$

Applying $\theta$ to (15) and comparing with (16), we see that $\theta s_n = (-1)^n s_n$, so Proposition 3.3 follows from the multiplicative properties of $\theta$ and $s_\lambda$. □

## 4. *An inner product*

We now impose additional structure on the vector space $A_n$ by defining an inner product. Any linear transformation $\omega: A_n \to A_n$ defines an inner product by the rule

$$(k_\lambda, \omega k_\mu) = \delta_{\lambda\mu}. \tag{17}$$

This inner product will be non-degenerate if $\omega$ is non-singular; symmetric if $\omega$ is symmetric; positive definite if $\omega$ is positive definite, etc. We take $\omega = \phi$, so we now have an inner product on $A_n$ given by

$$(k_\lambda, h_\mu) = \delta_{\lambda\mu}. \tag{18}$$

It follows from Proposition 3.1 that

$$(f, g) = (g, f), \tag{19}$$

for all $f, g \in A_n$.

Recall that two bases $b_\lambda$ and $c_\lambda$ are *dual* if

$$(b_\lambda, c_\mu) = \delta_{\lambda\mu}$$

for all $\lambda, \mu \vdash n$. Thus $h_\lambda$ and $k_\lambda$ are dual. A basis $b_\lambda$ is *self-dual* (or *orthonormal*) if

$$(b_\lambda, b_\mu) = \delta_{\lambda\mu}$$

for all $\lambda, \mu \vdash n$.

4.1. LEMMA. *The bases $b_\lambda, c_\lambda$ are dual if and only if*

$$\sum_\lambda b_\lambda(x)c_\lambda(y) = \prod_{i,j=1}^{\infty} (1 - x_iy_j)^{-1}.$$

*Proof*: Define linear transformations $\omega$ and $\xi$ by $\omega: b_\lambda \to h_\lambda, \xi: c_\lambda \to k_\lambda$. The statement that $b_\lambda$ and $c_\lambda$ are dual is equivalent to $\omega\xi^* = 1$ (* denotes transpose). Equivalently, $\omega^*\xi = 1$, or

$$\sum_\lambda \omega_{\lambda\nu}\xi_{\lambda\mu} = \delta_{\nu\mu}. \tag{20}$$

Thus

$$\begin{aligned}\prod(1 - x_iy_j)^{-1} &= \sum_\lambda h_\lambda(x)k_\lambda(y)\\ &= \sum_\lambda \left(\sum_\nu \omega_{\lambda\nu}b_\nu(x)\right)\left(\sum_\mu \xi_{\lambda\mu}c_\mu(y)\right)\\ &= \sum_{\mu,\nu}\left(\sum_\lambda \omega_{\lambda\nu}\xi_{\lambda\mu}\right) b_\nu(x)c_\mu(y).\end{aligned}$$

Since the functions $b_\nu(x)c_\mu(y)$ are all linearly independent, the proof follows from (20). □

4.2. PROPOSITION. The $s_\lambda$'s are an orthogonal basis for $A_n$. Specifically,

$$(s_\lambda, s_\mu) = 0, \qquad \text{if } \lambda \neq \mu,$$

$$(s_\lambda, s_\lambda) = 1^{r_1}r_1!2^{r_2}r_2!\ldots,$$

where $\lambda = \langle 1^{r_1}2^{r_2}\ldots\rangle$.

*Remark*: We denote the number $n!/(s_\lambda, s_\lambda)$ by $c^\lambda$ (the usual notation is $h_\lambda$, which has obvious disadvantages here). The number $c^\lambda$ is equal to the number of elements in the symmetric group $S_n$ of degree $n$ in the conjugacy class corresponding to the partition $\lambda$[8, 5.2; 1]. Clearly then $\sum_{\lambda \vdash n} c^\lambda = n!$

*Proof*: In the same way that (15) was derived, we get

$$\log \Pi(1 - x_i y_j)^{-1} = \sum_{m=1}^{\infty} \frac{1}{m} s_m(x) s_m(y). \tag{21}$$

It is well-known (see Riordan [11, p. 68]) that

$$\exp\left(t_1 + \frac{t_2}{2} + \frac{t_3}{3} + \dots\right) = \sum_{\lambda} (t_1^{r_1} t_2^{r_2} t_3^{r_3} \dots)/1^{r_1} r_1 ! 2^{r_2} r_2 ! \dots .$$

where $\lambda = \langle 1^{r_1} 2^{r_2} 3^{r_3} \dots \rangle$. Since the $s_\lambda(x)$'s and $s_\lambda(y)$'s are multiplicative, we get from (21),

$$\Pi(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)/1^{r_1} r_1 ! 2^{r_2} r_2 ! \dots$$

It follows from Lemma 4.1 that the basis $s_\lambda/\sqrt{1^{r_1} r_1 ! 2^{r_2} r_2 ! \dots}$ is self-dual, and the proof follows. □

Since the bases $h_\lambda$, $k_\lambda$ are dual and $s_\lambda$ is orthogonal, it follows that the determinant of the transformation $k_\lambda \to s_\lambda$ is given by $\prod_{\lambda \vdash n} (s_\lambda, s_\lambda)^{1/2} = \prod_{\lambda \vdash n} (1^{r_1} r_1 ! 2^{r_2} r_2 ! \dots)^{1/2}$. It is not difficult to see that this product is equal to either of $\prod_{\lambda \vdash n} (r_1 ! r_2 ! \dots)$ or $\prod_{\lambda \vdash n} (1^{r_1} 2^{r_2} \dots)$. This last product is simply the product of all the parts of all the partitions of $n$. (One can also evaluate this determinant by showing that with a suitable ordering of the $\lambda$'s, the matrix defined by $k_\lambda \to s_\lambda$ is in triangular form with the terms $r_1 ! r_2 ! \dots$ on the main diagonal.)

Since there exists an orthogonal basis for $A_n$ (over the field $Q$), viz., $s_\lambda$, such that $(s_\lambda, s_\lambda) > 0$, we immediately have:

4.3 Corollary. *$\phi$ is positive definite, i.e., $(f, f) \geq 0$, with equality if and only if $f = 0$, for all $f \in A_n$.*

Indeed, if $f = \Sigma \alpha_\lambda s_\lambda$, then $(f, f) = \Sigma \alpha_\lambda^2 (1^{r_1} r_1 ! 2^{r_2} r_2 ! \dots)$.

4.4 Corollary. *$\theta$ is an isometry, i.e., $(f, g) = (\theta f, \theta g)$.*

*Proof*: $s_\lambda$ is an orthogonal basis, and $\theta s_\lambda = \pm s_\lambda$ by Proposition 3.3. □

## III. Schur functions

### 5. *The combinatorial definition*

We now consider a fifth basis $e_\lambda$ (also denoted $\{\lambda\}$) for the space $A_n$. The functions $e_\lambda$ are known as the *Schur functions* (or $S$-functions) and have many remarkable properties. The term "Schur function" is due to Littlewood–Richardson [44], who give in this paper a systematic account of their properties. Littlewood and Richardson named them in honor of the pioneering work of Schur's doctoral dissertation [56].

We will give six basic expressions for the Schur functions, viz., the classical definition in terms of a generalized Vandermonde determinant, the expansion of $e_\lambda$ in terms of the four bases $k_\lambda, h_\lambda, a_\lambda, s_\lambda$, and a characterization in terms of the inner product (18). These six expressions will tie together the theory of symmetric functions, plane partitions, and the representation theory of the symmetric group. (For some further combinatorial ramifications of Schur functions, see Read [52]). It is interesting to realize that the Schur functions were considered (under a different terminology, e.g., *bialternants*) long before the theory of plane partitions or group representation theory was born. The classical results on Schur functions can be found in Muir [10], in the chapters on "Alternants".

In order to keep sight of the theory of plane partitions, we will adopt as our basic definition of Schur functions one involving plane partitions. If $\pi$ is a plane partition, define

$$M(\pi) = x_1^{a_1} x_2^{a_2} \dots, \tag{22}$$

where $a_i$ parts of $\pi$ are equal to $i$. Thus $M(\pi)$ is a monomial whose degree is equal to the number of parts of $\pi$.

5.1. DEFINITION. Let $\lambda$ be a partition of $n$. Define the *Schur function* associated with $\lambda$, denoted $e_\lambda$ or $\{\lambda\}$, to be the formal expression

$$e_\lambda = \sum_\pi M(\pi),$$

where the sum is over all column-strict plane partitions $\pi$ of shape $\lambda$.

Thus $e_\lambda$ is a homogeneous function of degree $n$ in the $x_i$'s. Our next object is to prove the remarkable fact that the $e_\lambda$'s are symmetric functions.

## 6. *The correspondence of Knuth*

Our goal in this section is to prove that the $e_\lambda$'s are symmetric functions. This fact follows easily from a combinatorial construction due to Knuth [39] (Knuth also discusses this construction in [7, § 5.2.4]) which generalizes a construction due to Robinson [53, no. 1, § 5] (given in a rather vague form) and Schensted [55]. Knuth's theorem is the following.

6.1. THEOREM. *There exists a one-to-one correspondence, denoted* $A \xrightarrow{K} (\pi, \sigma)$, *between matrices* $A = (a_{ij})$ *of non-negative integers* $(i, j \geq 1)$ *with finitely many non-zero entries, and ordered pairs* $(\pi, \sigma)$ *of column-strict plane partitions of the same shape. In this correspondence,*

$$i \text{ occurs in } \pi \text{ exactly } \sum_j a_{ij} \text{ times}$$

$$j \text{ occurs in } \sigma \text{ exactly } \sum_i a_{ij} \text{ times.}$$

*Proof*: We will describe the correspondence $A \xrightarrow{K} (\pi, \sigma)$, leaving the reader to verify the desired properties (the most crucial being invertibility). Complete details are given by Knuth [39].

Regard $A$ as a "generalized permutation"

$$\begin{matrix} i_1 & i_2 & i_3 & \dots & i_m \\ j_1 & j_2 & j_3 & \dots & j_m \end{matrix} \tag{23}$$

where (i) $i_1 \geq i_2 \geq \dots \geq i_m$, (ii) if $i_r = i_s$ and $r \leq s$, then $j_r \geq j_s$, and (iii) for each pair $(i, j)$, there are exactly $a_{ij}$ values of $r$ for which $(i_r, j_r) = (i, j)$. It is easily seen that each matrix $A$ determines a unique such array (23), and conversely.

We now build up $\sigma$ out of the $i_r$'s and $\pi$ out of the $j_r$'s inductively as follows. Define $\sigma_1$ to be the one element array $i_1$ and $\pi_1$ to be the array $j_1$. Suppose now $\sigma_r$ and $\pi_r$ are defined. These will be column-strict plane partitions of the same shape whose parts consist of $i_1, i_2, \dots, i_r$ and $j_1, j_2, \dots, j_r$, respectively. We now "insert" $j_{r+1}$ into $\pi_r$ by the following procedure. We put $j_{r+1}$ in the first row of $\pi_r$

in the space immediately following the right-most occurrence of an element $\geq j_{r+1}$. (If there is no such element put $j_{r+1}$ at the beginning of the row.) If some element $k$ already occupies this space, then $k$ is "bumped" down to the second row, where it is inserted in the same manner as $j_{r+1}$, possibly bumping another element to the third row. This bumping process is continued until some element is finally inserted at the end of a row without replacing another element. This gives the array $\pi_{r+1}$. To obtain $\sigma_{r+1}$, insert $i_{r+1}$ into $\sigma_r$ so that the array obtained has the same shape as $\pi_{r+1}$.

This process is continued until the array (23) is exhausted, resulting in $\pi = \pi_m$, $\sigma = \sigma_m$. This is the desired correspondence. $\square$

*Example*: Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The array (23) is given by

$$\begin{array}{cccccc} 3 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 1. \end{array}$$

The plane partitions $\pi_1, \ldots, \pi_6 = \pi$ and $\sigma_1, \ldots, \sigma_6 = \sigma$ are as follows:

$$\begin{array}{lcl} \underline{\pi_i} & & \underline{\sigma_i} \\ 1 & & 3 \\[1em] \begin{array}{l} 2 \\ 1 \end{array} & & \begin{array}{l} 3 \\ 2 \end{array} \\[1em] \begin{array}{ll} 2 & 2 \\ 1 & \end{array} & & \begin{array}{ll} 3 & 2 \\ 2 & \end{array} \\[1em] \begin{array}{ll} 3 & 2 \\ 2 & \\ 1 & \end{array} & & \begin{array}{ll} 3 & 2 \\ 2 & \\ 1 & \end{array} \\[1em] \begin{array}{ll} 3 & 3 \\ 2 & 2 \\ 1 & \end{array} & & \begin{array}{ll} 3 & 2 \\ 2 & 1 \\ 1 & \end{array} \\[1em] \begin{array}{lll} 3 & 3 & 1 \\ 2 & 2 & \\ 1 & & \end{array} & & \begin{array}{lll} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{array} \end{array}$$

6.2. THEOREM (Littlewood [8, p. 191]). *$e_\lambda$ is a symmetric function.*

*Proof*: (Bender and Knuth [18]). Let $(\mu_1, \mu_2, \ldots)$ be a fixed vector of non-negative integers, with finitely many non-zero entries. Consider the generating function $\Sigma x_1^{\nu_1} x_2^{\nu_2} \ldots$, where the sum is over all matrices $A = (a_{ij})$ of non-negative integers with row sums $\sum_i a_{ij} = \mu_j$, and where $\nu_1, \nu_2, \ldots$ are the column sums $\sum_i a_{ji} = \nu_j$. It is easily seen that this generating function is given by

$$\Sigma x_1^{\nu_1} x_2^{\nu_2} = h_{\mu_1} h_{\mu_2} \ldots = h_\mu, \tag{24}$$

where $\mu$ is the partition with parts $\mu_i$.

By Theorem 6.1 and (24), we also have $h_\mu = \Sigma x_1^{\nu_1} x_2^{\nu_2} \dots$, where the sum now is over all ordered pairs $(\pi, \sigma)$ of column-strict plane partitions of the same shape such that $\pi$ contains $\mu_i$ parts equal to $i$ and $\sigma$ contains $\nu_i$ parts equal to $i$. Hence if we define $K_{\lambda\mu}$ to be the number of column-strict plane partitions of shape $\lambda$ and $\mu_i$ parts equal to $i$, there follows

$$h_\mu = \sum_\lambda K_{\lambda\mu} e_\lambda. \tag{25}$$

Since the $h_\mu$'s are linearly independent, the matrix $(K_{\lambda\mu})$, where $\lambda, \mu \vdash n$, can be inverted to express $e_\lambda$ in terms of the $h_\lambda$'s, so $e_\lambda$ is a symmetric function. □

7. *Kosta's theorem and the orthonormality of the Schur functions*

We have seen in (25) that

$$h_\mu = \sum_\lambda K_{\lambda\mu} e_\lambda, \tag{26}$$

where $K_{\lambda\mu}$ is the number of column-strict plane partitions of shape $\lambda$ and any fixed set of parts occurring with multiplicities $\mu_1, \mu_2, \dots$ On the other hand, it follows from Definition 5.1 (once it is known that $e_\lambda$ is symmetric) that

$$e_\lambda = \sum_\mu K_{\lambda\mu} k_\mu. \tag{27}$$

The appearance of the same coefficients $K_{\lambda\mu}$ in (26) and (27) is *Kosta's theorem* [41] (see also Muir [10, 4, pp. 145–146], Littlewood [8, 6.4; 6]). In [41] Kosta constructs tables of the coefficients $K_{\lambda\mu}$ and of the inverse matrix $H_{\lambda\mu}$. He extends this table in [42], where he also gives a more unified account of his work. The following restatement of Kosta's theorem is due to Philip Hall [35].

7.1. THEOREM. *For $\lambda \vdash n$, the $e_\lambda$'s form an orthonormal integral basis for $A_n$.*

*Proof*: (26) expresses the integral basis $h_\mu$ as an integral combination of $e_\lambda$'s, while (27) shows that $e_\lambda$ is an integral combination of the integral basis $k_\mu$. Hence $e_\lambda$ is an integral basis. Moreover (26) and (27) state that the linear transformations $h_\lambda \to e_\lambda$ and $k_\lambda \to e_\lambda$ are inverse transposes of one another, which is precisely the condition for orthonormality of the $e_\lambda$'s. □

Philip Hall [35] points out that Theorem 7.1 characterizes the Schur functions up to sign and order, since any two orthonormal integral bases for $A_n$ can be transformed into one another by an integral orthogonal matrix, which must therefore be a signed permutation matrix. From the standpoint of linear algebra, there exists an integral orthonormal basis for $A_n$ if and only if there exists an *integral* matrix $\omega$ such that

$$\omega\phi\omega^* = 1.$$

We then say that $\phi$ is *integrally equivalent* (or *Z-equivalent*) *to the identity*. It is an important unsolved problem to determine in general which integral matrices are integrally equivalent to the identity. [3, § 73].

Note that unlike the orthogonal basis $s_\lambda$, the basis $e_\lambda$ is *not* multiplicative. In fact, Farahat [27] has shown that each $e_\lambda$ is *irreducible*.

As an immediate consequence of Lemma 4.1 we have:

7.2. COROLLARY. (Littlewood [8, p. 103], Knuth [39]). *We have*

$$\sum_\lambda e_\lambda(x) e_\lambda(y) = \Pi(1 - x_i y_j)^{-1}. \quad \square$$

This corrollary can also be proved directly from Knuth's theorem (Theorem 6.1), as follows. By Theorem 6.1,

$$\sum_{\lambda} e_\lambda(x)e_\lambda(y) = \sum_{A} \prod_{i,j=1}^{\infty} (x_i y_j)^{a_{ij}} \tag{28}$$

where the sum is over all matrices $A = (a_{ij})$ of non-negative integers, with finitely many non-zero entries. But the right-hand side of (28) is equal to

$$\prod_{i,j=1}^{\infty} \sum_{a_{ij}=0}^{\infty} (x_i y_j)^{a_{ij}} = \prod (1 - x_i y_j)^{-1}.$$

## 8. *Further properties of Knuth's correspondence*

We will discuss some further properties of Knuth's correspondence $A \xrightarrow{K} (\pi, \sigma)$, some of which lead to interesting identities involving Schur functions and are of great importance to the enumeration of plane partitions. In general, the proofs will be omitted, but references to them will be given.

8.1. PROPOSITION. (Knuth [39]). If $A \xrightarrow{K} (\sigma, \pi)$, then $A^* \xrightarrow{K} (\pi, \sigma)$, where $A^*$ denotes the transpose of $A$. □

8.2. COROLLARY. (Knuth [39]). *There exists a one-to-one correspondence between symmetric matrices $A = (a_{ij})$ of non-negative integers $(i, j \geq 1)$ with finitely many non-zero entries, and column-strict plane partitions $\pi$. In this correspondence, $i$ occurs in $\pi$ exactly $\sum_j a_{ij}$ times.*

*Proof*: If $A$ is symmetric and $A \xrightarrow{K} (\sigma, \pi)$, then by Proposition 8.1, $\sigma = \pi$. Thus $A \to \pi$ achieves the desired correspondence. □

The next corollary was first obtained by Littlewood [8, p. 238] by group-theoretic means (the essence of his proof actually appears on pp. 92–94), while the combinatorial method we are pursuing is due to Bender and Knuth [18].

8.3. COROLLARY.

$$\sum_{\lambda} e_\lambda = \prod (1 - x_i)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1}.$$

*Proof*: By Corollary 8.2,

$$\sum_{\lambda} e_\lambda = \sum_{A} \prod_{i,j=1}^{\infty} x_i^{a_{ij}}, \tag{29}$$

where the sum is over all symmetric matrices $A = (a_{ij})$ of non-negative integers with finitely many non-zero entries. But the right-hand side of (29) is equal to

$$\begin{aligned}\sum_{A}\left(\prod_{i=j} x_i^{a_{ij}}\right)\left(\prod_{i<j} x_i^{a_{ij}} x_j^{a_{ji}}\right) &= \sum_{A}\left(\prod_{i} x_i^{a_{ii}}\right)\left(\prod_{i<j} (x_i x_j)^{a_{ij}}\right)\\ &= \left(\prod_{i} \sum_{a_{ii}=0}^{\infty} x_i^{a_{ii}}\right)\left(\prod_{i<j} \sum_{a_{ij}=0}^{\infty} (x_i x_j)^{a_{ij}}\right)\\ &= \prod_{i} (1 - x_i)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1}. \ \square\end{aligned}$$

8.4. PROPOSITION. (Schutzenberger [57], Knuth [39]). If $A$ is symmetric and $A \xrightarrow{K} (\sigma, \sigma)$, then the number of columns of $\sigma$ of odd length is equal to the trace of $A$. □

As a corollary, we can modify the proof of Corollary 8.3 by summing over symmetric matrices $A$ of trace 0 to obtain the expansion

$$\sum_{\mu} e_{\mu} = \prod_{i<j} (1 - x_i x_j)^{-1}, \tag{30}$$

where $\mu$ ranges over all partitions whose shape has no columns of odd length. The expansion (30) was first obtained by Littlewood [8, p. 238]. He also obtains the conjugate result

$$\sum_{\nu} e_{\nu} = \prod_{i} (1 - x_i^2)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1}, \tag{31}$$

where $\nu$ ranges over all partitions whose shape has no *rows* of odd length. It would be interesting to relate (31) to some property of Knuth's correspondence.

The next proposition gives an interpretation for the number of rows and number of columns of $\pi$ (or $\sigma$) if $A \xrightarrow{K} (\pi, \sigma)$, in terms of the "generalized permutation" (23) to which $A$ corresponds. Some applications of this result will be given in Section 17.

8.5. PROPOSITION. (Schensted [55], Knuth [39]). If $A \xrightarrow{K} (\pi, \sigma)$ and $A$ corresponds to the "generalized permutation" (23), then the number of rows of $\pi$ (or $\sigma$) is equal to the length of the longest strictly increasing subsequence of the sequence $j_1$, $j_2, \ldots, j_m$, while the number of columns of $\pi$ (or $\sigma$) is equal to the length of the longest decreasing (not necessarily strictly) subsequence of $j_1, j_2, \ldots, j_m$. □

For example, in the example following Theorem 6.1 the sequence $j_1, j_2, \ldots, j_m$ is given by 1, 2, 2, 3, 3, 1. The longest strictly increasing subsequence is 1, 2, 3, so $\pi$ has three rows. The longest decreasing subsequence is 2, 2, 1 or 3, 3, 1, so $\pi$ has three columns.

## 9. *The dual correspondence*

By modifying the "bumping process" of Knuth's correspondence $A \xrightarrow{K} (\pi, \sigma)$, we obtain another correspondence, called by Knuth the "dual correspondence," which has important applications to Schur functions and plane partitions.

9.1. THEOREM (Knuth [39]). *There exists a one-to-one correspondence, denoted* $A \xrightarrow{K^*} (\pi, \sigma)$, *between* 0–1 *matrices* $A = (a_{ij})$ $(i, j \geq 1)$ *with finitely many* 1's, *and ordered pairs* $(\pi, \sigma)$ *of column-strict plane partitions of* conjugate *shape. In this correspondence,*

$$i \text{ occurs in } \pi \text{ exactly } \sum_j a_{ij} \text{ times}$$

$$j \text{ occurs in } \sigma \text{ exactly } \sum_i a_{ij} \text{ times.}$$

*Proof*: The correspondence $K^*$ is defined identically to $K$, except that rather than inserting an element $j$ in the space immediately following the right-most occurrence of an element $\geq j$, we insert $j$ following the right-most occurrence of an element $> j$. (If there is no such element, put $j$ at the beginning of the row.) At the end we have a pair $(\pi^*, \sigma)$ of plane partitions of the same shape such that $\pi^*$ is row-strict and $\sigma$ is column-strict, so we take $\pi$ to be the transpose of $\pi^*$. □

*Example*: Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The array (23) is given by

$$\begin{matrix} 3 & 3 & 2 & 1 & 1 \\ 3 & 1 & 3 & 2 & 1 \end{matrix}$$

The plane partitions $\pi_1^*, \ldots, \pi_5^* = \pi^*$ and $\sigma_1, \ldots, \sigma_5 = \sigma$ are as follows:

| $\pi_i^*$ | $\sigma_i$ |
|---|---|
| $\begin{matrix} 3 \end{matrix}$ | $\begin{matrix} 3 \end{matrix}$ |
| $\begin{matrix} 3 & 1 \end{matrix}$ | $\begin{matrix} 3 & 3 \end{matrix}$ |
| $\begin{matrix} 3 & 1 \\ 3 & \end{matrix}$ | $\begin{matrix} 3 & 3 \\ 2 & \end{matrix}$ |
| $\begin{matrix} 3 & 2 \\ 3 & 1 \end{matrix}$ | $\begin{matrix} 3 & 3 \\ 2 & 1 \end{matrix}$ |
| $\begin{matrix} 3 & 2 & 1 \\ 3 & 1 & \end{matrix}$ | $\begin{matrix} 3 & 3 & 1 \\ 2 & 1 & \end{matrix}$ |

so $\pi$ is

$$\begin{matrix} 3 & 3 \\ 2 & 1 \\ 1 & \end{matrix}$$

In exactly the same way that Corollary 7.2 was derived from (28), we obtain:

9.2. COROLLARY. (Littlewood [8, p. 103], Knuth [39]). *We have*

$$\sum_{\lambda} e_\lambda(x) e_{\lambda'}(y) = \prod (1 + x_i y_j). \quad \square$$

From Corollary 9.2 we can deduce the effect of the linear transformation $\theta$ on the $e_\lambda$'s. Note that since $e_\lambda$ is an orthonormal basis and $\theta$ an isometry, the basis $\theta e_\lambda$ must also be orthonormal. The next result shows in fact that $\theta$ merely permutes the $e_\lambda$'s.

9.3. COROLLARY. (P. Hall [35]) $\theta e_\lambda = e_{\lambda'}$.

*Proof*: Regard $\theta$ as acting on symmetric functions in the variables $y_i$ only, so symmetric functions in the variables $x_i$ are left invariant by $\theta$. Then

$$\begin{aligned} \Sigma e_\lambda(x)(\theta e_\lambda(y)) &= \theta \Sigma e_\lambda(x) e_\lambda(y) \\ &= \theta \Sigma k_\lambda(x) h_\lambda(y), \qquad \text{by Lemma 4.1} \\ &= \Sigma k_\lambda(x)(\theta h_\lambda(y)) \\ &= \Sigma k_\lambda(x) a_\lambda(y) \\ &\phantom{=} \Sigma e_\lambda(x) e_{\lambda'}(y), \qquad \text{by Corollary 9.2.} \end{aligned}$$

Hence $\theta e_\lambda = e_{\lambda'}$. $\square$

Although we have attributed the above corollary to Philip Hall, he actually only restated a classical result, due to Naegelbasch [47] and Kosta [40], that if $e_\lambda = \sum_\mu H_{\lambda\mu} h_\mu$, then $e_{\lambda'} = \sum_\mu H_{\lambda\mu} a_\mu$ (see also Muir [10, 3, pp. 144–148, 154–156]). Naegelbasch and Kosta prove this result using the classical definition of the Schur functions, to be discussed in the next section. A method for computing the matrix $H_{\lambda\mu}$ (the inverse matrix to $K_{\lambda\mu}$), called the *Jacobi–Trudi identity*, will be given in Section 11.

In view of Proposition 8.1, which states that $A^* \xrightarrow{K} (\pi, \sigma)$ if $A \xrightarrow{K} (\sigma, \pi)$, it is natural to consider the effect of transposing $A$ on the correspondence $A \xrightarrow{K^*} (\pi, \sigma)$. Unfortunately no result analogous to Proposition 8.1 is known, and it is also unknown what the range of $K^*$ is when $A$ is symmetric. There are three formulas of Littlewood [8, p. 238, nos. (11.9; 1), (11.9; 3), (11.9; 5)] which suggest some result along these lines exists. For instance, one of Littlewood's results states that

$$\prod (1 - x_i) \prod_{i<j} (1 - x_i x_j) = \sum_\lambda \alpha_\lambda e_\lambda, \tag{32}$$

where the coefficient $\alpha_\lambda$ is given by

$$\alpha_\lambda = \begin{cases} 0, & \text{if } \lambda \neq \lambda' \\ (-1)^{(n+r)/2}, & \text{if } \lambda \text{ is a self-conjugate partition of } n \text{ of rank } r. \end{cases}$$

Here the *rank* of a partition is the number of elements on the main diagonal of its shape (or the size of its "Durfee square" [6, p. 281]). No combinatorial proof of (32) is known. The identity (32) may be regarded as the "inverse" of Corollary 8.3, and thus as the column-strict plane partition analog of Euler's formula [6, Thm. 353] for inverting the ordinary partition function $p(n)$,

$$1 \Big/ \sum_{n=0}^{\infty} p(n) x^n = \sum_{n=-\infty}^{\infty} (-1)^n x^{(1/2)n(3n+1)}.$$

## 10. *The classical definition of the Schur functions*

We are now in a position to give the classical definition of the Schur functions, apparently due to Jacobi [37], though the terminology "Schur functions" did not come until Schur [56] tied them in with the characters of the symmetric group (see Section 12). We will then give a remarkably simple proof of the equivalence of the classical expression with our Definition 5.1.

Recall the definition of the Vandermonde determinant $\Delta(x_1, x_2, \ldots, x_n)$,

$$\Delta(x_1, x_2, \ldots, x_n) = |x_s^{n-t}| \qquad (s, t = 1, \ldots, n), \tag{33}$$

where $|\beta_{st}|$ stands for the determinant of the matrix $(\beta_{st})$. It is well-known that

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j). \tag{34}$$

The function $\Delta(x_1, \ldots, x_n)$ is *alternating*, since interchanging any two variables changes the sign of the function. It is natural to consider determinants analogous to (33) with the exponents $n - t$ replaced by an arbitrary partition $\lambda$ of $n$, viz., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ (further terms are superfluous since $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$

if $\lambda \vdash n$). Since such a determinant will equal 0 unless all the $\lambda_i$'s are distinct, we are led to consider the determinant

$$|x_s^{\lambda_t+n-t}| \qquad (s, t = 1, \ldots, n) \tag{35}$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. Here the exponents $\lambda_t + n - t$ are all distinct. The determinant (35) is again an alternating function of $x_1, \ldots, x_n$. Moreover, setting $x_i = x_j$ $(i \neq j)$ results in 0, so (35) is divisible by $\Delta(x_1, \ldots, x_n)$. Consider the function

$$|x_s^{\lambda_t+n-t}|/|x_s^{n-t}|. \tag{36}$$

Since (36) is the quotient of two alternating functions, it is a symmetric function of $x_1, \ldots, x_n$ and is clearly of degree $\lambda_1 + \cdots + \lambda_n = n$. Thus (36) can be "extended" to a unique symmetric function in $A_n$ which "agrees" with (36) in those terms involving just $x_1, \ldots, x_n$. We now come to the surprising result that this symmetric function is just the Schur function $e_\lambda$.

10.1. THEOREM. $e_\lambda(x_1, \ldots, x_n) = |x_s^{\lambda_t+n-t}|/|x_s^{n-t}|$.

*Proof*: In [8, p. 68], Littlewood gives an argument which proves the theorem for the coefficients of $x_1x_2 \ldots x_n$ (he proves the entire theorem by other means). We give a straightforward generalization of his argument. Essentially the same argument was given by Bender and Knuth [18].

According to (26), $h_\mu = \sum_\lambda K_{\lambda\mu} e_\lambda$. Applying $\theta$ to both sides, we get from Corollary 9.3 that

$$a_\mu = \sum_\lambda K_{\lambda\mu} e_{\lambda'}.$$

Since the matrix $(K_{\lambda\mu})$ is invertible, it suffices to prove

$$a_\mu = \sum_\lambda K_{\lambda\mu} |x_s^{\lambda_t'+n-t}|/|x_s^{n-t}|.$$

Equivalently, we need to show that the coefficient of $x_1^{\lambda_1'+n-1}x_2^{\lambda_2'+n-2} \ldots x_n^{\lambda_n'}$ in the expansion of $a_\mu|x_s^{n-t}|$ is equal to $K_{\lambda\mu}$.

Consider the process of multiplying $|x_s^{n-t}|$ with $a_\mu = a_{\mu_1}a_{\mu_2} \ldots$, by multiplying by each $a_{\mu_1}, a_{\mu_2}, \ldots$ in succession. Since $|x_s^{n-t}|$ is alternating and each $a_{\mu_i}$ is symmetric, each partial product $|x_s^{n-t}|a_{\mu_1}a_{\mu_2} \ldots a_{\mu_k}$ is alternating. Hence the coefficient of any term $x_1^{i_1} \ldots x_n^{i_n}$ of $|x_s^{n-t}|a_{\mu_1}a_{\mu_2} \ldots a_{\mu_k}$ is zero unless the $i_j$'s are all distinct. On the other hand, each term of the symmetric function $a_j$ is of the form $x_{m_1}x_{m_2} \ldots x_{m_j}$, $m_1 < m_2 < \cdots < m_j$, and when this is multiplied by a term $x_1^{i_1} \ldots x_n^{i_n}$ with distinct exponents, either the order of the exponents is preserved or else two exponents become equal.

It follows that if we have a term $x_1^{i_1} \ldots x_n^{i_n}$ of $|x_s^{n-t}|a_{\mu_1}a_{\mu_2} \ldots a_{\mu_k}$ with a non-zero coefficient, then it was obtained from terms $x_1^{j_1} \ldots x_n^{j_n}$ of $|x_s^{n-t}|a_{\mu_1}a_{\mu_2} \ldots a_{\mu_{k-1}}$ by multiplying by appropriate terms of $a_{\mu_k}$, such that *the relative order of the numbers* $i_1, \ldots, i_n$ *is the same as that of* $j_1, \ldots, j_n$. In particular, the only non-zero contributions to the coefficient of $x_1^{\lambda_1'+n-1}x_2^{\lambda_2'+n-2} \ldots x_n^{\lambda_n'}$ in $a_\mu|x_s^{n-t}|$ are obtained by considering the process of multiplying $a_\mu$ by the term $x_1^{n-1}x_2^{n-2} \ldots x_{n-1}$ of $|x_s^{n-t}|$.

Hence the coefficient of $x_1^{\lambda_1'+n-1}x_2^{\lambda_2'+n-2} \ldots x_n^{\lambda_n'}$ in $a_\mu|x_s^{n-t}|$ is equal to the number of ways of "building" the term $x_1^{\lambda_1'+n-1} \ldots x_n^{\lambda_n'}$ from the term $x_1^{n-1}x_2^{n-2} \ldots x_{n-1}$ of $|x_s^{n-t}|$ by successively multiplying by some term of $a_{\mu_1}$, then of $a_{\mu_2}, \ldots$, in such a way that at no stage are any two exponents equal. Given such a choice of terms from each $a_{\mu_j}$, define a column-strict plane partition $\pi$ as follows: the term $j$ appears in the $k$th column of $\pi$ if and only if the variable $x_k$ appears in the term chosen from $a_j$. It is easily seen that $\pi$ is a uniquely defined column-strict plane partition of shape $\lambda$ and $\mu_j$ parts equal to $j$. Moreover, any such $\pi$ corresponds to a choice of terms from each $a_{\mu_j}$. Hence the coefficient in question is equal to the number of such $\pi$, which by definition is just $K_{\lambda\mu}$. □

Note that this proof implicitly includes a proof of the fact that the $e_\lambda$'s, as defined by Definition 5.1, are symmetric functions, since the order in which we multiply by the $a_{\mu_j}$'s can be arbitrary.

## 11. *The Jacobi–Trudi identity*

We now turn to the problem of expressing the $e_\lambda$'s in terms of the $h_\mu$'s; or equivalently, of inverting the matrix $K_{\lambda\mu}$. This result was first obtained by Jacobi [37] in 1841 and later simplified by his student Trudi [62] in 1864. Subsequently a combinatorial proof was given by Bender and Knuth [18], but we will give Trudi's proof, based on the classical expression Theorem 10.1.

11.1. THEOREM. *Let $\lambda$ be a partition into $r$ non-zero parts*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0.$$

*Then*

$$e_\lambda = |h_{\lambda_s - s + t}| \qquad (s, t = 1, 2, \ldots, r)$$

(*with the convention $h_0 = 1, h_{-m} = 0$ if $m > 0$*).

Since $h_\lambda$ is a multiplicative basis, each term of the above determinant is of the form $\pm h_\mu$, and we get the expansion of $e_\lambda$ in terms of $h_\mu$.

*Example*: Take $r = 1, \lambda_1 = n$. Then

$$e_n = |h_n| = h_n,$$

a result which is evident from the definition of $e_n$. Similarly if $r = 2, \lambda_1 = a, \lambda_2 = b$, then

$$e_{a,b} = \begin{vmatrix} h_a & h_{a+1} \\ h_{b-1} & h_b \end{vmatrix} = h_a h_b - h_{a+1} h_{b-1}.$$

*Proof of Theorem* 11.1: The following identities are readily verified:

$$h_m(x_1, x_2, \ldots, x_n, y_1) - h_m(x_1, x_2, \ldots, x_n, y_2) = (y_1 - y_2) h_{m-1}(x_1, x_2, \ldots, x_n, y_1, y_2) \tag{37}$$

$$h_m(x_1, x_2, \ldots, x_n, x_{n+1}) = h_m(x_1, x_2, \ldots, x_n) + x_{n+1} h_{m-1}(x_1, x_2, \ldots, x_n, x_{n+1}). \tag{38}$$

The idea of the proof is to factor the product $\prod_{i<j}(x_i - x_j)$ out of the determinant $|x_s^{\lambda_t+n-t}|$. Thus in $|x_s^{\lambda_t+n-t}|$, subtract the first row ($s = 1$) from every subsequent row ($s > 1$), and remove the factor $(x_s - x_1)$ from the $s$th row ($s > 1$). Thus the $(s, t)$ entry for $s > 1$ becomes

$$(x_s^{\lambda_t+n-t} - x_1^{\lambda_t+n-t})/(x_s - x_1) = h_{\lambda_t+n-t-1}(x_1, x_s).$$

Now subtract the second row ($s = 2$) from every subsequent row ($s > 2$), and remove the factor $(x_s - x_2)$ from the $s$th row ($s > 2$). Thus the $(s, t)$ entry for $s > 2$ becomes

$$(h_{\lambda_t+n-t-1}(x_1, x_s) - h_{\lambda_t+n-t-1}(x_1, x_2))/(x_s - x_2).$$

By (37) this is equal to $h_{\lambda_t+n-t-2}(x_1, x_2, x_s)$. Continuing in this way, we finally obtain

$$|x_s^{\lambda_t+n-t}| = \prod_{i<j} (x_j - x_i)|h_{\lambda_t+n-t-s+1}(x_1, x_2, \ldots, x_s)|. \tag{39}$$

In the determinant of (39), reverse the order of the rows and interchange rows with columns, giving

$$|x_s^{\lambda_t+n-t}| = \prod_{i<j} (x_i - x_j)|h_{\lambda_s-s+t}(x_1, \ldots, x_{n-t+1})|. \tag{40}$$

Equation (40) is an alternative form of the Jacobi–Trudi identity, differing from the desired result in that the $h_i$'s are not in the full set of variables $x_1, \ldots, x_n$.

Now add $x_2$ times the $n - 1$st column ($t = n - 1$) to the last column ($t = n$), then $x_3$ times the column $t = n - 2$ to the column $t = n - 1$, continuing to $x_n$ times the first column ($t = 1$) to the second column. By (38), (40) becomes

$$e_\lambda(x_1, \ldots, x_n) = |h_{\lambda_s-s+t}(x_1, \ldots, x_{n-t+2})|, \tag{41}$$

with the understanding that any variables $x_i$ with $i > n$ are to be ignored.

Now add $x_3$ times the column $t = n - 1$ to the column $t = n$, then $x_4$ times the column $t = n - 2$ to the column $t = n - 1$, up to $x_n$ times the column $t = 2$ to the column $t = 3$. Once again the number of variables appearing in each $h_i$ is increased by 1 (unless this number is already $n$). Continuing in this way, we finally obtain the $n \times n$ determinant $|h_{\lambda_s-s+t}|_1^n$, which has the form

$$\left|\begin{array}{c|cccc} |h_{\lambda_s-s+t}|_1^r & & * & & \\ \hline & 1 & & & * \\ & & 1 & & \\ 0 & & & 1 & \\ & 0 & & & 1 \end{array}\right| = |h_{\lambda_s-s+t}|_1^r. \quad \square$$

As an immediate corollary of the Jacobi–Trudi identity and Corollary 9.3, we get the classical form of the Naegelbasch–Kosta theorem, of which Corollary 9.3 is the "combinatorial" form.

11.2. Corollary. *Let $\lambda$ be a partition with largest part $\lambda_1 = q$. Then*

$$e_\lambda = |a_{\lambda'_s-s+t}| \qquad (s, t = 1, 2, \ldots, q). \quad \square$$

### 12. *Skew plane partitions and the multiplication of Schur functions*

The concept of plane partitions can be generalized to "skew plane partitions", leading to a new class of symmetric functions related to taking a product of Schur functions, and to a generalization of the Jacobi–Trudi identity. Let $\lambda$ and $\mu$ be partitions such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$
$$\mu_1 \geq \mu_2 \geq \cdots \mu_r \geq 0$$
$$\mu_i \leq \lambda_i \qquad \text{for } i = 1, 2, \ldots, r. \tag{42}$$

Define a *skew plane partition, of shape* $\lambda/\mu$, to be a plane partition of shape $\lambda$ from which the shape $\mu$ has been "removed." For instance, if $\lambda = (5, 5, 3, 1)$ and $\mu = (2, 1, 1, 0)$ then the array

$$\begin{array}{ccccc} \cdot & \cdot & 3 & 3 & 1 \\ \cdot & 3 & 2 & 2 & 1 \\ \cdot & 3 & 1 & & \\ 4 & & & & \end{array} \tag{43}$$

is a skew plane partition of 23 of shape $\lambda/\mu$. The obvious definition is made for "column-strict skew plane partition," etc. In analogy to our original combinatorial definition of $e_\lambda$ (Definition 5.1), if $\pi$ is a skew plane partition, define

$$M(\pi) = x_1^{a_1} x_2^{a_2} \ldots,$$

where $a_i$ parts of $\pi$ are equal to $i$. Thus for the array $\pi$ of (43), $M(\pi) = x_1^3 x_2^2 x_3^4 x_4$.

12.1. Definition. Let $\lambda$ and $\mu$ be partitions satisfying (42). Define

$$e_{\lambda/\mu} = \sum_{\pi} M(\pi),$$

where the sum is over all column-strict skew plane partitions $\pi$ of shape $\lambda/\mu$.

We state without proof the basic result on the functions $e_{\lambda/\mu}$, generalizing the Jacobi–Trudi identity.

12.2. Proposition. $e_{\lambda/\mu} = |h_{\lambda_s - \mu_t - s + t}|$. □

In the form given by Proposition 12.2, the functions $e_{\lambda/\mu}$ were investigated by Naegelbasch [47] and Aitken [15], [16]. The connection with plane partitions was first pointed out by Littlewood [8, p. 109] (see also Robinson [12], § 2.5]). Since the right-hand side of Proposition 12.2 is a symmetric function, we have the following generalization of Theorem 6.2.

12.3 Corollary. *$e_{\lambda/\mu}$ is a symmetric function.* □

It is thus natural to ask how $e_{\lambda/\mu}$ may be expressed in terms of the $e_\nu$'s. The next proposition gives such an expression, apparently due to Littlewood [8, p. 110]; its proof is omitted.

12.4. Proposition. If $e_{\lambda/\mu} = \sum_\nu g_{\nu\mu\lambda} e_\nu$, then $g_{\nu\mu\lambda}$ is the coefficient of $e_\lambda$ in the product $e_\nu e_\mu$, i.e., $g_{\nu\mu\lambda} = (e_{\lambda/\mu}, e_\nu) = (e_\lambda . e_\mu e_\nu)$. □

Proposition 12.4 is the basic result on the ordinary multiplication $e_\nu e_\mu$ of Schur functions (see Littlewood [8, pp. 91–98]), and shows how multiplying Schur functions is related to building up plane partitions. Other methods of multiplying Schur functions, in particular the *plethyism* $e_\mu \otimes e_\nu$, have been considered. We will not go into them here, but instead refer the reader to the bibliographies in Littlewood [8] and Robinson [12]. For an interesting relation between the numbers $g_{\nu\mu\lambda}$ and the structure of finite abelian $p$-groups, see P. Hall [35] and Klein [38].

By applying the operator $\theta$ to Proposition 12.2 and Proposition 12.4, we get an expression for $e_{\lambda/\mu}$ in terms of the $a_\nu$'s directly generalizing Corollary 11.2. This result was first proved by Aitken [15], [16] by other means.

12.5. COROLLARY (Aitken). $e_{\lambda/\mu} = |a_{\lambda'_s - \mu'_t - s + t}|$. □

## 13. *Frobenius' formula for the characters of the symmetric group*

We have succeeded in expressing the $e_\lambda$'s in terms of the symmetric functions $k_\mu$, $h_\mu$, and $a_\mu$. The remaining basis to be considered is $s_\mu$. Let us write

$$s_\lambda = \sum_{\mu \vdash n} \chi_\lambda^\mu e_\mu, \tag{44}$$

so $\chi$ is the matrix transforming $e_\lambda$ to $s_\lambda$, and $\chi_\lambda^\mu = (e_\mu, s_\lambda)$. The significance of the coefficients $\chi_\lambda^\mu$, first obtained by Frobenius [29], is perhaps the most profound result known about Schur functions.

13.1. THEOREM (Frobenius) *The matrix $(\chi_\lambda^\mu)$ is the character table of the symmetric group $S_n$. Specifically, $\chi_n^\mu$ is the character $\chi^\mu$ corresponding to the partition $\mu$ evaluated at the conjugacy class of $S_n$ corresponding to the partition $\lambda$.* □

We will not prove this theorem here, since we are assuming no group-theoretic background on the part of the reader, and since this result will not be needed for our enumeration of plane partitions (Parts IV and V). A straightforward account is given by Littlewood [8, § 5.2]. Further results on the representation theory of the symmetric group may be found in Littlewood [8], Robinson [12], and the references given there. In particular, Young [66] was the first person to recognize the connection between plane partitions and the symmetric group. An account of Young's highly significant work is given by Rutherford [13].

A number of properties of the matrix $\chi_\lambda^\mu$ can be deduced without recourse to group theory (i.e., without using Theorem 13.1). These results normally are regarded as special cases of theorems in group representation theory. We prove two such results here.

13.2 PROPOSITION. $(\chi_\lambda^\mu)$ is a column-orthogonal matrix, indeed,

$$\sum_\lambda \chi_\lambda^\mu \chi_\lambda^\nu = \delta_{\mu\nu} \cdot 1^{r_1} r_1! 2^{r_2} r_2! \ldots,$$

where $\mu = \langle 1^{r_1} 2^{r_2} \ldots \rangle$.

*Proof*: Since $e_\mu$ is an orthonormal basis and $s_\mu$ is an orthogonal basis,

$$\sum_\lambda \chi_\lambda^\mu \chi_\lambda^\nu = (s_\mu, s_\nu),$$

which was evaluated in Proposition 4.2. □

13.3. PROPOSITION. $|\chi_\lambda^\mu| = \prod_{\lambda \vdash n} (1^{r_1} 2^{r_2} \ldots)$, where $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$.

*Proof*: By the previous proposition, $|\chi_\lambda^\mu| = \prod_\nu (s_\nu, s_\nu)^{1/2} = \prod_{\lambda \vdash n} (1^{r_1} 2^{r_2} \ldots)$ (see Section 4). □

Since the matrix $\chi_\lambda^\mu$ is orthogonal, its inverse is simply $\chi_\mu^\lambda/(s_\mu, s_\mu)$. In other words,

$$e_\lambda = \sum_{\mu \vdash n} \chi_\mu^\lambda s_\mu/(s_\mu, s_\mu). \tag{45}$$

(to be continued)

## References

The following bibliography is not intended to be complete, though most purely combinatorial results on plane partitions are included. For further references to Schur functions and their connection with group theory, see the bibliographies in Littlewood [8] and Robinson [12]. For further references to ballot problems, see the first chapter of Takács [14].


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