# SUPERSOLVABLE LATTICES[1])

R. P. STANLEY

## 1. Introduction

We shall investigate a certain class of finite lattices which we call *supersolvable lattices* (for a reason to be made clear shortly). These lattices $L$ have a number of interesting combinatorial properties connected with the counting of chains in $L$, which can be formulated in terms of Möbius functions. I am grateful to the referee for his helpful suggestions, which have led to more general results with simpler proofs.

1.1. DEFINITION. Let $L$ be a finite lattice and $\Delta$ a maximal chain of $L$. If, for every chain $K$ of $L$, the sublattice generated by $K$ and $\Delta$ is distributive, then we call $\Delta$ an *M-chain* of $L$; and we call $(L, \Delta)$ a *supersolvable lattice* (or *SS-lattice*).

Sometimes, by abuse of notation, we refer to $L$ itself as an $SS$-lattice, the $M$-chain $\Delta$ being tacitly assumed.

A wide variety of examples of $SS$-lattices is given in the next section. In this section, we define two fundamental concepts associated with $SS$-lattices, viz., the *rank-selected Möbius invariant* and the *set of Jordan-Holder permutations.* We shall outline their connection with each other, together with some consequences. Proofs will be given in later sections.

If $L$ is an $SS$-lattice whose $M$-chain $\Delta$ has length $n$ (or cardinality $n+1$), then every maximal chain $K$ of $L$ has length $n$ since all maximal chains of the distributive lattice generated by $\Delta$ and $K$ have the same length. Hence if $\hat{0}$ denotes the bottom element and $\hat{1}$ the top element of $L$, then $L$ has defined on it a unique *rank function* $r: L \to \{0, 1, 2, \ldots n\}$ satisfying $r(\hat{0})=0$, $r(\hat{1})=n$, $r(y)=r(x)+1$ if $y$ *covers* $x$ (i.e., $y>x$ and no $z\in L$ satisfies $y>z>x$). Let $S$ be any subset of the set $\mathbf{n-1}$, where we use the notation

$$\mathbf{k} = \{1, 2, \ldots, k\}.$$

We will also write $S=\{m_1, m_2, \ldots, m_s\}^<$ to signify that $m_1<m_2<\cdots<m_s$. Define $\alpha(S)$ to be the number of chains

$$\hat{0} = y_0 < y_1 < \cdots < y_s < \hat{1}$$

in $L$ such that $r(y_i)=m_i$, $i=1, 2, \ldots, s$. In particular, if $S=\{m\}$, then $\alpha(S)$ is the number

---

[1]) The research was supported by the Air Force Office of Scientific Research, AF 44620-70-C-0079, at M.I.T.

*Presented by R. P. Dilworth. Received June 4, 1971. Accepted for publication in final form May 17, 1972.*

of elements of $L$ of rank $m$; while if $S=\mathbf{n-1}$, then $\alpha(S)$ is the number of maximal chains in $L$. Also, $\alpha(\phi)=1$. Now define

$$\beta(S)=\sum_{T\subseteq S}(-1)^{|S-T|}\alpha(T),$$

so by the Principle of Inclusion-Exclusion,

$$\alpha(S)=\sum_{T\subseteq S}\beta(T).$$

Our main object is to investigate the numbers $\beta(S)$ when $L$ is an $SS$-lattice. It will be seen that these numbers have many remarkable properties. First, we consider an alternative interpretation of $\beta(S)$.

If $S\subseteq\mathbf{n-1}$, define $L(S)$ to be the sub-ordered set of $L$ consisting of $\hat{0}$, $\hat{1}$, and all elements of $L$ whose ranks belong to $S$. Thus, $L(\phi)$ is a two-element chain, and $L(\mathbf{n-1})=L$. It follows from a basic result on Möbius functions due to Philip Hall [11] (see also [15, p. 346]) that

$$\mu_S(\hat{0},\hat{1})=(-1)^{s+1}\beta(S),\tag{1}$$

where $\mu_S$ is the Möbius function of $L(S)$ and $s=|S|$. For this reason, we call $\beta(S)$ the *rank-selected Möbius invariant* of $L$.

Somewhat more generally, if $[x, y]$ is a segment in $L$ of length $m$ (i.e., $r(y)-r(x)=m$) and if $S\subseteq\mathbf{m-1}$, then we denote by $\beta_{xy}(S)$ the rank-selected Möbius invariant of the segment $[x, y]$, considered as a lattice in its own right.

In order to define the second fundamental concept associated with $SS$-lattices, we first review some properties of finite distributive lattices. If $P$ is a partially ordered set, then an *order ideal* of $P$ is a subset $I$ of $P$ such that if $x\in I$ and $y\leqslant x$, then $y\in I$. Recall the structure theorem of Birkhoff [1, p. 59] that every finite distributive lattice $L$ is isomorphic to the set of order ideals of some finite partially ordered set $P$, ordered by inclusion. This correspondence is denoted $L=J(P)$. If $I$ is an element of $L$ of rank $m$, then as an order ideal of $P$, $I$ has cardinality $m$. Every maximal chain $\phi=I_0<I_1<\cdots<I_n=P$ in $L$ corresponds to an *order-compatible permutation* $\sigma=(x_1, x_2,\ldots, x_n)$ of the elements $x_i$ of $P$, i.e., if $x<y$ in $P$, then $x$ appears before $y$ in $\sigma$. This correspondence is determined by the condition

$$x_i\in I_{i+1}-I_i.$$

Now let $\Delta:\phi=I_0<I_1<\cdots<I_n=P$ be any fixed maximal chain in $L$, and let $J$ cover $I$ in $L$ (so $|J-I|=1$). Then there is a unique integer $i\in\mathbf{n}$ such that the prime interval $[I, J]$ is projective to the prime interval $[I_{i-1}, I_i]$. We denote this integer $i$ as $\gamma(I, J)$.

It is easily seen that $i$ is determined by the condition

$$J - I = I_i - I_{i-1}.$$

This fact is essentially the well-known Jordan-Hölder correspondence for finite distributive lattices.

Suppose now that $I<J$ in $L$. Define the subset $\Gamma(I, J)$ of $\mathbf{n}$ by the condition

$$\Gamma(I, J) = \{i \mid x_i \in J - I\}.$$

Hence, $|\Gamma(I, J)| = |J-I|$. If $I=\phi$ (the bottom element of $L$), then $\Gamma(I, J)$ is denoted simply $\Gamma(J)$, so $\Gamma(I, J) = \Gamma(J) - \Gamma(I)$. Note that if $J$ covers $I$, then $\gamma(I, J)$ is the unique element of $\Gamma(I, J)$.

Suppose that

$$K: I = I'_0 < I'_1 < \cdots < I'_m = J$$

is an unrefinable chain between $I$ and $J$. Define the *Jordan-Hölder permutation* (or *J-H permutation*) $\pi_K: \Gamma(I, J) \to \Gamma(I, J)$ associated with $K$ (relative to $\Delta$) by

$$\pi_K = (\gamma(I'_0, I'_1), \gamma(I'_1, I'_2), \ldots, \gamma(I'_{m-1}, I'_m)).$$

Now let $(L, \Delta)$ be an $SS$-lattice, with $x<y$ in $L$. The set $\Gamma(x, y)$ is still defined, viz., $\Gamma(x, y)$ is computed in the distributive lattice generated by $\Delta$ and the chain $x<y$. Similarly, we still have the notion of $\gamma(x, y)$ (when $y$ covers $x$) and of the $J$-$H$ permutation $\pi_K: \Gamma(x, y) \to \Gamma(x, y)$, where $K$ is an unrefinable chain between $x$ and $y$. The set of all $J$-$H$ permutations $\pi_K$, *including repetitions*, as $K$ ranges over all unrefinable chains between $x$ and $y$, is called the $J$-$H$ set of $(L, \Delta; x, y)$ and is denoted $\mathscr{J}_{xy}(L, \Delta)$ (or $\mathscr{J}_{xy}$ for short). If $x=\hat{0}$ and $y=\hat{1}$, then the corresponding $J$-$H$ set is called the $J$-$H$ set of $(L, \Delta)$ and is denoted simply $\mathscr{J}(L, \Delta)$ or just $\mathscr{J}$.

Figure 1, for example, shows an $SS$-lattice $(L, \Delta)$ (actually the lattice of subgroups

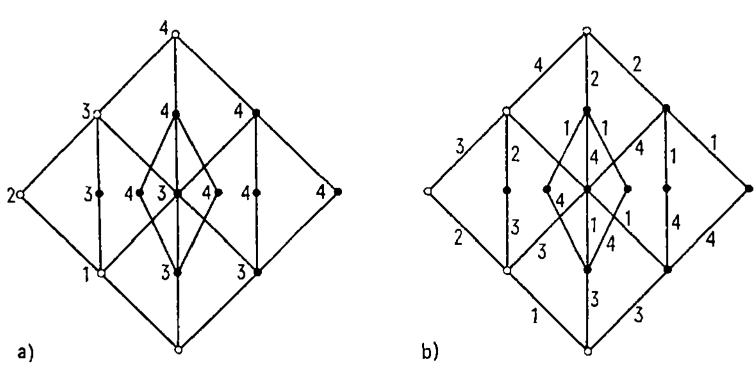

Figure 1.

of an abelian 2-group of type (2,2)), with the $M$-chain $\Delta$ indicated by open dots. In Figure 1(a), each element $x \in L$ is marked by the largest element of $\Gamma(x)$; in Figure 1(b), the numbers $\gamma(x, y)$ are indicated. From Figure 1(b), we read off the $J$-$H$ set $\mathscr{J}(L, \Delta)$:

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 1 | 3 | 2 | 4 |
| 1 | 3 | 2 | 4 |
| 1 | 3 | 4 | 2 |
| 1 | 3 | 4 | 2 |
| 3 | 4 | 1 | 2 |
| 3 | 1 | 2 | 4 |
| 3 | 1 | 4 | 2 |
| 3 | 1 | 4 | 2 |
| 3 | 4 | 1 | 2 |
| 3 | 1 | 2 | 4 |
| 3 | 1 | 4 | 2 |
| 3 | 1 | 4 | 2 |
| 3 | 4 | 1 | 2 |
| 3 | 4 | 1 | 2 |

If $\pi = (i_1, i_2, \ldots, i_m)$ is a permutation of some finite subset of the integers, then a pair $i_j > i_{j+1}$ is called a *descent* of $\pi$, and the set

$$D(\pi) = \{j : i_j > i_{j+1}\}$$

is called the *descent set* of $\pi$. The fundamental result connecting the $J$-$H$ set $\mathscr{J}_{xy}(L, \Delta)$ with the rank-selected Möbius function $\beta_{xy}(S)$ of the segment $[x, y]$ is the following:

1.2 THEOREM. *Let $(L, \Delta)$ be an SS-lattice, and let $[x, y]$ be a segment of $L$ of length $m$. If $S \subseteq \mathbf{m-1}$, then the number of permutations $\pi$ in the J-H set $\mathscr{J}_{xy}(L, \Delta)$ with descent set $D(\pi) = S$ is equal to $\beta_{xy}(S)$*

The proof of Theorem 1.2 is given in Section 3. It was proved in [16, Thm 9.1] when $L$ is a distributive lattice ( under a different terminology). The present paper is a result of extending the lattice-theoretical portions of [16] as far as possible.

By way of illustration, we list the descent sets of the 15 elements of the $J$-$H$ set $\mathscr{J}(L, \Delta)$ of the $SS$-lattice $(L, \Delta)$ in Figure 1.

| $\pi$ | $D(\pi)$ |
|---|---|
| 1 2 3 4 | $\phi$ |
| 1 3 2 4 | 2 |

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 2 | 4 | 2 |
| 1 | 3 | 4 | 2 | 3 |
| 1 | 3 | 4 | 2 | 3 |
| 3 | 4 | 1 | 2 | 2 |
| 3 | 1 | 2 | 4 | 1 |
| 3 | 1 | 4 | 2 | 1,3 |
| 3 | 1 | 4 | 2 | 1,3 |
| 3 | 4 | 1 | 2 | 2 |
| 3 | 1 | 2 | 4 | 1 |
| 3 | 1 | 4 | 2 | 1,3 |
| 3 | 1 | 4 | 2 | 1,3 |
| 3 | 4 | 1 | 2 | 2 |
| 3 | 4 | 1 | 2 | 2 |

Hence $\beta(\phi)=1$, $\beta(1)=2$, $\beta(2)=6$, $\beta(3)=2$, $\beta(1,3)=4$, and all other $\beta(S)=0$.

Using the interpretation (1) of $\beta(S)$, we immediately get two interesting corollaries to Theorem 1.2.

1.3 COROLLARY. *Let $(L, \Delta)$ be an SS-lattice with $x<y$ in $L$. Then there is a unique unrefinable chain*

$$x = x_0' < x_1' < \cdots < x_m' = y$$

*between $x$ and $y$ such that*

$$\gamma(x_0', x_1') < \gamma(x_1', x_2') < \cdots < \gamma(x_{m-1}', x_m').$$

1.4 COROLLARY. *Let $(L, \Delta)$ be an SS-lattice with $x<y$ in $L$, with $r(y)-r(x)=m$. Then the number of unrefinable chains*

$$x = x_0' < x_1' < \cdots < x_m' = y$$

*between $x$ and $y$ satisfying*

$$\gamma(x_0', x_1') > \gamma(x_1', x_2') > \cdots > \gamma(x_{m-1}', x_m')$$

*is equal to $(-1)^m \mu(x, y)$, where $\mu$ is the Möbius function of $L$.* □

Let $\Delta_{xy}$ denote the unique chain between $x$ and $y$ given by Corollary 1.3. It is not hard to see that $\Delta_{xy}$ is an $M$-chain of $[x, y]$ (this will be done in Section 3), so in fact $([x, y], \Delta_{xy})$ is an $SS$-lattice. Hence there is no loss of generality in restricting our attention to $\beta$ rather than to $\beta_{xy}$.

From Theorem 1.2, we immediately see that

$$\beta_{xy}(S) \geqslant 0. \tag{2}$$

Using the Möbius-theoretic interpretation (1) of $\beta$, (2) may be restated as follows:

1.5 COROLLARY. *Let $(L, \Delta)$ be an SS-lattice of rank n, and let $S \subseteq \mathbf{n-1}$. Then the Möbius function $\mu_S$ of the rank-selected subset $L(S)$ of $L$ alternates in sign, i.e., if $[x, y]$ is a segment of $L(S)$ of length k, then*

$$(-1)^k \mu_S(x, y) \geqslant 0. \quad \square$$

Theorem 1.2 is a powerful tool for studying properties of $\beta(S)$, e.g., we trivially have (2). Observe also that the collection of descent sets (including repetitions) of the elements of $\mathscr{J}(L, \Delta)$ depend only on $L$ and not on the $M$-chain $\Delta$, since the same is obviously true of $\beta(S)$. A further result which will be given in Section 3 is:

(a) If $\beta(S)>0$ and $T \subseteq S$, then $\beta(T)>0$ (Proposition 3.3).

*SS*-lattices $L$ which are also upper semimodular enjoy a number of properties not shared by general *SS*-lattices. Some of these are:

(b) When $L$ is an upper semimodular *SS*-lattice (*USS-lattice*, for short), a necessary and sufficient condition for $\beta(S)>0$ is given in terms of the existence of Loewy chains (i.e., chains $y_0 < y_1 < \cdots < y_s$ for which every $y_j$, $1 \leqslant j \leqslant s$, is the join of atoms of the segment $[y_{j-1}, y_j]$) in $L$ (Theorem 5.2).

(c) The *Birkhoff polynomial* (also called the *characteristic polynomial*) of a *USS*-lattice $L$ has positive integral roots related to the structure of $L$ (Theorem 4.1).

(d) For a special class of *USS*-lattices, $\beta(S)$ is divisible by a large power of an integer $q$ (Section 6). This gives a lattice-theoretical generalization of some well-known results in the enumerative theory of $p$-groups.

In addition to the lattice-theoretical results mentioned above, our work also has applications to the combinatorial theory of permutations. The problem of analyzing permutations by their descents has received considerable attention (see, e.g., [9], [16]), and here we introduce a new and more general point of view.

## 2. Examples

The next proposition will make it easy to give a wide variety of examples of *SS*-lattices. First, we recall some lattice-theoretical results. These results were discovered mostly in the 1930s by Garrett Birkhoff, L. R. Wilcox, R. P. Dilworth, and others (see [1]). If $x$ and $y$ belong to a lattice L, we say that $(x, y)$ is a *modular pair* (written $xMy$) if, for all $z \leqslant y$, we have $z \vee (x \wedge y) = (z \vee x) \wedge y$. In general, the relation of being a modular pair is *not* symmetric. In fact, Wilcox [18] showed that the relation of being a modular pair is symmetric if and only if $L$ is upper semimodular. We say that $x$ is a *modular element* of the lattice $L$ if and only if $xMy$ and $yMx$ for all $y \in L$. If every element of $L$ is modular, then $L$ is a *modular lattice*.

2.1 PROPOSITION. *Let L be a finite lattice and $\Delta$ a maximal chain of L such that every element of $\Delta$ is modular. Then $\Delta$ is an M-chain of L.*

*Proof.* The proof is essentially the same as Birkhoff's proof of the less general result that a modular lattice generated by two chains is distributive (see [1, pp. 65–66]). We will merely point out why Birkhoff's proof applies to our more general result.

The only point of Birkhoff's proof which invokes modularity is in establishing the identities

$$(a_1 \wedge b_1) \vee \cdots \vee (a_r \wedge b_r) = a_1 \wedge (b_1 \vee a_2) \wedge \cdots \wedge (b_{r-1} \vee a_r) \wedge b_r; \qquad (3)$$
$$(b_1 \vee a_1) \wedge \cdots \wedge (b_r \vee a_r) = b_1 \vee (a_1 \wedge b_2) \vee \cdots \vee (a_{r-1} \wedge b_r) \vee a_r, \qquad (4)$$

when $a_i \geqslant a_{i+1}$ and $b_i \leqslant b_{i+1}$ in a modular lattice. We show that these identities still hold, however, if one only assumes that the $a_i$'s are modular, from which the proof of Proposition 2.1 follows.

We first prove (4) by induction on $r$. Set $A = (b_1 \vee a_1) \wedge \cdots \wedge (b_r \vee a_r)$. Now $b_1 \leqslant (b_2 \vee a_2) \wedge \cdots \wedge (b_r \vee a_r)$. Hence since $a_1 M (b_2 \vee a_2) \wedge \cdots \wedge (b_r \vee a_r)$, we have

$$A = b_1 \vee [a_1 \wedge (b_2 \vee a_2) \wedge \cdots \wedge (b_r \vee a_r)].$$

By induction,

$$A = b_1 \vee [a_1 \wedge \{b_2 \vee (a_2 \wedge b_3) \vee \cdots \vee (a_{r-1} \wedge b_r) \vee a_r\}].$$

Now $a_r \leqslant a_1$. Hence since $b_2 \vee (a_2 \wedge b_3) \vee \cdots \vee (a_{r-1} \wedge b_r)\, M a_1$, we have

$$A = b_1 \vee [a_1 \wedge \{b_2 \vee (a_2 \wedge b_3) \vee \cdots \vee (a_{r-1} \wedge b_r)\}] \vee a_r.$$

Now $(a_2 \wedge b_3) \vee \cdots \vee (a_{r-1} \wedge b_r) \leqslant a_1$. Hence since $b_2 M a_1$, we have

$$A = b_1 \vee (a_1 \wedge b_2) \vee (a_2 \wedge b_3) \vee \cdots \vee (a_{r-1} \wedge b_r) \vee a_r.$$

This proves (4).

We cannot just dualize (4) to prove (3), since the property of being a modular element is not self-dual. Instead, setting $B = (a_1 \wedge b_1) \vee \cdots \vee (a_r \wedge b_r)$, we have $(a_2 \vee b_2) \vee \cdots \vee (a_r \wedge b_r) \leqslant a_1$ and $b_1 M a_1$, so

$$B = a_1 \wedge [b_1 \vee (a_2 \wedge b_2) \vee \cdots \vee (a_r \wedge b_r)].$$

Now $b_1 \vee (a_2 \wedge b_2) \vee \cdots \vee (a_{r-1} \wedge b_{r-1}) \leqslant b_r$ and $a_r M b_r$, so

$$B = a_1 \wedge [b_1 \vee (a_2 \wedge b_2) \vee \cdots \vee (a_{r-1} \wedge b_{r-1}) \vee a_r] \wedge b_r.$$

By (4),

$$B = a_1 \wedge (b_1 \vee a_2) \wedge (b_2 \vee a_3) \wedge \cdots \wedge (b_{r-1} \vee a_r) \wedge b_r.$$

This proves (3), and with it, the proposition. □

The converse to Proposition 2.1 is false in general. For instance, take $L$ to be the lattice of subsets of $\{a, b, c\}$ ordered by inclusion except that the relation $\{a\} \subset \{a, b\}$ is excluded. Let $\Delta$ be the chain $\phi < \{c\} < \{b, c\} < \{a, b, c\}$. Then $\Delta$ is an $M$-chain, but $\{a\}\ M\ \{b, c\}$ is false. We have, however, the following partial converse.

2.2 PROPOSITION. *Let $(L, \Delta)$ be an SS-lattice. If $x \in \Delta$ and $y \in L$, then $xMy$.*

*Proof.* Let $z \leqslant y$. Since the sublattice generated by $\Delta$, $y$, and $z$ is distributive, *a fortiori* so is the sublattice generated by $x$, $y$, and $z$. Hence, $xMy$. □

The converse to Proposition 2.2 is also false: take $L$ to be the 5-element non-modular lattice and $\Delta$ to be the maximal chain of length 3.

Since the relation $xMy$ is symmetric in an upper semimodular lattice, we deduce from Propositions 2.1 and 2.2, the following corollary.

2.3 COROLLARY. *Let $L$ be a finite upper semimodular lattice, and let $\Delta$ be a maximal chain of $L$. Then $\Delta$ is an $M$-chain if and only if every element of $\Delta$ is modular.*

We are now in a position to give numerous examples of $SS$-lattices.

2.4 *Example.* If $L$ is a finite modular lattice and $\Delta$ is any maximal chain of $L$, then trivially $(L, \Delta)$ is an $SS$-lattice.

2.5 *Example.* Let $G$ be a supersolvable finite group and $L(G)$ its lattice of subgroups. Now every normal subgroup of any group $G$ is a modular element of its lattice of subgroups [1, p. 172]. Hence since $G$ is supersolvable, $L(G)$ contains a maximal chain of normal subgroups (corresponding to a chief series of $G$). Hence $L(G)$ is an $SS$-lattice, and every chief series of $G$ is an $M$-chain (there may be other $M$-chains). Since it is the supersolvability of $G$ which implies the existence of an M-chain in $L(G)$, this explains our terminology 'supersolvable lattice'.

Observe that from Corollary 1.5 (with $S = \mathbf{n} - \mathbf{1}$), we deduce the following interesting result: the Möbius function of the lattice of subgroups of a finite supersolvable group alternates in sign.

2.6 *Example.* Let $\Pi_n$ denote the lattice of partitions of an $n$-set $S$ [1, p. 95]. It is not difficult to see that a partition $\pi$ of $S$ is a modular element of $\Pi_n$ if and only if at most one block of $\pi$ has more than one element. From this it follows that $\Pi_n$ is an $SS$-lattice with exactly $n!/2$ $M$-chains ($n > 1$). These $M$-chains are permuted among themselves transitively by the automorphisms of $\Pi_n$.

The next two examples generalize the previous example.

2.7 *Example.* A finite graph whose lattice of contractions [15, Section 9] is an

$SS$-lattice will be called an *SS-graph*. The next proposition (whose proof we omit) gives a characterization of $SS$-graphs. The case of complete graphs corresponds to the lattices $\Pi_n$ of the previous example.

2.8 PROPOSITION. *Let $G$ be a finite graph. Then $G$ is an SS-graph if and only if the vertices of $G$ can be labeled as $v_1, v_2, \ldots, v_n$ such that whenever $1 \leqslant i < j < k \leqslant n$ and $v_k$ is connected by an edge to both $v_i$ and $v_j$, then $v_i$ and $v_j$ are connected by an edge. If $G$ is doubly-connected, then the number of $M$-chains in the lattice of contractions of $G$ is equal to exactly half the number of such labelings (the labeling $v_1, v_2, \ldots, v_n$ being paired with the labeling obtained by interchanging $v_1$ and $v_2$).* □

$SS$-graphs have been considered previously in different contexts under the name *triangulated graphs* or *rigid circuit graphs*. They are defined to be graphs for which every cycle of length at least four contains a chord.

2.9. *Example.* Let $V$ be a projective space of rank $n$ over $GF(q)$. Let $L$ be the lattice of flats of the geometry (in the sense of Crapo-Rota [5]) determined by all the vectors in $V$ with one or two nonzero entries. Then $L$ is a geometric lattice of rank $n$. These interesting lattices were discovered by Dowling [8], who derived their basic properties (see also Doubilet-Rota-Stanley [7, Section 5 (c)]). In particular, when $q=2$, we get the partition lattices $\Pi_{n+1}$. For any $q$, the boolean algebra generated by the vectors in $V$ which contain one nonzero entry consists of modular elements of $L$. (There will be additional modular elements only if $q=2$). Hence $L$ is an $SS$-lattice, with $n!$ $M$-chains when $q>2$.

2.10 *Example.* Let $a_1, a_2, \ldots, a_n$ be any sequence of positive integers with $a_1=1$. Let $G$ be them geometry consisting of $n$ independent points $v_1, v_2, \ldots, v_n$, with an additional $a_i-1$ points inserted on the line $v_1v_i$, $i>1$. Then the lattice $L$ of flats of $G$ is an $SS$-lattice. $L$ possesses an $M$-chain $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$ such that the number of points contained in $x_i$ but not in $x_{i-1}$ is $a_i$ $(i=1, 2, \ldots, n)$. The significance of this remark will become clear in Example 4.7.

2.11 *Example.* Let $\mathfrak{N}$ denote the set of all partial orderings $P$ on the set $\{1, 2, \ldots, n\}$ satisfying $i<j$ in $P \Rightarrow i<j$ as integers. Define $P \leqslant Q$ in $\mathfrak{N}$ if $i \leqslant j$ in $P \Rightarrow i \leqslant j$ in $Q$. Dean and Keller [6] showed that this order relation makes $\mathfrak{N}$ into a lower semimodular lattice of rank $N=\binom{n}{2}$. Let $\Delta$ be the maximal chain $\hat{0}=P_0<P_1<\cdots<P_N=\hat{1}$ of $\mathfrak{N}$ where $P_i$ is generated by the first $i$ terms of the sequence $1<n, 1<n-1, 1<n-2, \ldots, 1<2, 2<n, 2<n-1, \ldots, 2<3, 3<n, 3<n-1, \ldots, 3<4, \ldots, n-1<n$. Then $\Delta$ is an $M$-chain of $\mathfrak{N}$, so $(\mathfrak{N}, \Delta)$ is an $SS$-lattice.

In all the above examples except for some cases of Example 1.3, either the lattice $L$ or its dual $L^*$ is upper semimodular. (It is clear that if $(L, \Delta)$ is an $SS$-lattice, then so is $(L^*, \Delta^*)$.) Some special properties of supersolvable semimodular lattices will be discussed in Sections 4–6, and additional properties will be given in a later paper.

## 3. Properties of *SS*-lattices

We begin by proving Theorem 1.2, viz., that the number of permutations $\pi$ in the *J-H* set $\mathscr{J}_{xy} = \mathscr{J}_{xy}(L, \Delta)$ (where $[x, y]$ is a segment of length $m$ in an *SS*-lattice $(L, \Delta)$) with descent set $D(\pi) = S \subseteq \mathbf{m-1}$ is equal to $\beta_{xy}(S)$. The proof is based on a simple lemma which is in fact a special case of Corollary 1.3

3.1 LEMMA. *Let $(L, \Delta)$ be a distributive SS-lattice, and let $I < J$ in L. Then there is a unique unrefinable chain*

$$I = J_0 < J_1 < \cdots < J_m = J$$

*in L between I and J such that*

$$\gamma(J_0, J_1) < \gamma(J_1, J_2) < \cdots < \gamma(J_{m-1}, J_m). \tag{5}$$

*Proof.* Let $L = J(P)$. Suppose $\Delta$ is given by $\phi = I_0 < I_1 < \cdots < I_n = P$. Label the elements of $P$ as $x_1, x_2, \ldots, x_m$ so that $x_i \in I_i - I_{i-1}$. Suppose $J - I = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$, where $i_1 < i_2 < \cdots < i_m$. Then condition (5) requires that $J_k = I \cup \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. Moreover, with this definition of $J_k$, we indeed have that $J_0 < J_1 < \cdots < J_m$ is an unrefinable chain between $I$ and $J$, so the proof is complete. $\square$

*Proof of Theorem* 1.2. Let $\delta_{xy}(S)$ be the number of permutations $\pi \in \mathscr{J}_{xy}$ with descent set $S$. Thus we need to show that $\beta_{xy}(S) = \delta_{xy}(S)$ for all $S \subseteq \mathbf{m-1}$.

Let $K$ be a maximal chain of the segment $[x, y]$,

$$K: x = y_0 < y_1 < \cdots < y_m = y, \tag{6}$$

with associated *J-H* permutation $\pi_K$. If $S = \{m_1, m_2, \ldots, m_s\}_< \subseteq \mathbf{m-1}$, denote by $K_S$ the chain

$$K_S: y_{m_1} < y_{m_2} < \cdots < y_{m_s}. \tag{7}$$

We shall show that the correspondence $K \to K_S$ is a bijection between maximal chains $K$ of $[x, y]$ satisfying $D(\pi_K) \subseteq S$ and chains $K_S$ of $L$ satisfying

$$\{r(z) - r(x) \mid z \in K_S\} = S. \tag{8}$$

This will show that $\sum_{T \subseteq S} \delta_{xy}(T)$ is equal to the number of chains satisfying (8), i.e.,

$$\sum_{T \subseteq S} \delta_{xy}(T) = \alpha_{xy}(S). \tag{9}$$

But (9) uniquely determines the $\delta_{xy}(S)$'s and is also the recursion satisfied by the $\beta_{xy}(S)$'s. Hence it will follow that $\delta_{xy}(S)=\beta_{xy}(S)$.

We need to show that given a chain $K'$ satisfying $\{r(z)-r(x) \mid z\in K'\}=S$, then there is a unique maximal chain $K$ of $[x, y]$ satisfying (i) $D(\pi_K)\subseteq S$, and (ii) $K_S=K'$. Condition (ii) is equivalent to: $K'$ is a subchain of $K$. Thus if $K'$ is given by (7) and $K$ by (6), then condition (i) is equivalent to the conditions

$$\begin{gathered}
\gamma(y_0, y_1) < \gamma(y_1, y_2) < \cdots < \gamma(y_{m_1-1}, y_{m_1}),\\
\gamma(y_{m_1}, y_{m_1+1}) < \gamma(y_{m_1+1}, y_{m_1+2}) < \cdots < \gamma(y_{m_2-1}, y_{m_2}),\\
\vdots\\
\gamma(y_{m_s}, y_{m_s+1}) < \gamma(y_{m_s+1}, y_{m_s+2}) < \cdots < \gamma(y_{m-1}, y_m).
\end{gathered} \tag{10}$$

Let $L'$ be the distributive lattice generated by $K'\cup\{x, y\}$ and $\Delta$. Then by Lemma 3.1, there is a unique maximal chain $K$ of $L'$ satisfying (i) and (ii). $K$ is also a maximal chain of $[x, y]$ since every maximal chain of $L'$ has the same length as $\Delta$. If $K_1$ were another maximal chain of $L$ satisfying (i) and (ii), then since $K'\subseteq K_1$ the distributive lattice generated by $K_1\cup\{x, y\}$ and $\Delta$ would contain the two chains $K$ and $K_1$ satisfying (i) and (ii). This contradicts Lemma 3.1, so the theorem is proved. $\square$

We now clarify the relationship between the *J-H* sets $\mathscr{J}_{xy}(L, \Delta)$ (where $x\leqslant y$ in $L$) and $\mathscr{J}(L, \Delta)$.

3.2 PROPOSITION. *Let $(L, \Delta)$ be an SS-lattice, and let $x\leqslant y$ in $L$. Denote by $L'$ the segment $[x, y]$, considered as a lattice in its own right. Let $\Delta'=\Delta_{xy}$ be the unique maximal chain in $L'$ given by Corollary* 1.3. *Then*

(i) $(L', \Delta')$ is an *SS-lattice.*

(ii) *Suppose $\Delta$ is given by $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$ and $\Delta'$ by $x=y_0<y_1<\cdots<y_m=y$. Let $\gamma$ refer to $(L,\Delta)$ and $\gamma'$ to $(L',\Delta')$. Suppose $y'$ covers $x'$, where $x'$ and $y'$ lie in $L'$. If $i=\gamma(x', y')$ and $j=\gamma'(x', y')$, then $i$ and $j$ satisfy*

$$\begin{aligned}
i &= \gamma(x_{i-1}, x_i) = \gamma(y_{j-1}, y_j)\\
j &= \gamma'(y_{j-1}, y_j).
\end{aligned}$$

(iii) *If $y'$ covers $x'$ and $y''$ covers $x''$, where $x', y', x'', y''$ all lie in $L'$, then $\gamma(x', y')>\gamma(x'', y'')$ if and only if $\gamma'(x', y')>\gamma'(x'', y'')$.*

*Proof.* (i) Let $K'$ be a chain in $L'$. We need to prove that $\Delta'$ and $K'$ generate a distributive lattice. Now $K''=K'\cup\{x, y\}$ is also a chain, so $\Delta$ and $K''$ generate a distributive lattice $D$. But $\Delta'$ is contained in the lattice generated by $\Delta$ and $\{x, y\}$. Hence $\Delta'\subseteq D$, so the lattice generated by $\Delta'$ and $K'$ is distributive.

(ii) That $i=\gamma(x_{i-1}, x_i)$ and $j=\gamma'(y_{j-1}, y_j)$ is immediate from the definition of $\gamma$ and $\gamma'$. Thus the prime intervals $[x', y']$ and $[y_{j-1}, y_j]$ are projective in $L'$; hence they are projective in $L$. Since projectivity is transitive, $[y_{j-1}, y_j]$ and $[x_{i-1}, x_i]$ are projective. Thus $\gamma(y_{j-1}, y_j)=i$.

(iii) Follows immediately from (ii). $\square$

The significance of Proposition 3.2(iii) is that it allows one to compute the *J-H* set $\mathscr{J}(L', \Delta')$ once the 'relative' *J-H* set $\mathscr{J}_{xy}(L, \Delta)$ is known. For instance, a permutation (4, 8, 2, 6, 3) in $\mathscr{J}_{xy}(L, \Delta)$ corresponds to a permutation (3, 5, 1, 4, 2) in $\mathscr{J}(L', \Delta')$.

We turn to an interesting property of the numbers $\beta(S)$.

3.3 PROPOSITION. *Let $(L, \Delta)$ be an SS-lattice of rank n, and let $S \subseteq \mathbf{n-1}$. If $\beta(S)>0$ and $T \subseteq S$, then $\beta(T)>0$.*

*Proof.* Suppose $S=\{m_1, m_2, \ldots, m_s\}_<$ and $\beta(S)>0$. Then there is a maximal chain

$$K: \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$$

such that $D(\pi_K)=S$, where $D(\pi_K)$ is the descent set of the *J-H* permutation $\pi_K$. For any $i$ satisfying $1 \leqslant i \leqslant s$, we have, by Corollary 1.3, a (unique) unrefinable chain

$$y_{m_{i-1}} = y'_0 < y'_1 < \cdots < y'_r = y_{m_{i+1}}$$

between $y_{m_{i-1}}$ and $y_{m_{i+1}}$ (with the convention $m_0=0$, $m_{s+1}=n$) satisfying

$$\gamma(y'_0, y'_1) < \gamma(y'_1, y'_2) < \cdots < \gamma(y'_{r-1}, y'_r).$$

Hence the chain $K'$ given by

$$\hat{0} = y_0 < y_1 < \cdots < y_{m_{i-1}} < y'_1 < y'_2 < \cdots < y'_r < y_{m_{i+1}+1} < y_{m_{i+1}+2} < \cdots < y_n = \hat{1}$$

satisfies $D(\pi_K')=S-\{m_i\}$, so $\beta(S-\{m_i\})>0$. By a trivial inductive argument, $\beta(T)>0$ for any $T \subseteq S$. $\square$

The numbers $\beta(S)$ provide a means of giving '$q$-analogues' and '$\Pi$-analogues' of certain well-known combinatorial numbers. For instance, let $L$ be a direct product of chains of lengths $n_1, n_2, \ldots, n_r$, so that $L$ is a distributive lattice of rank $n = n_1+n_2+\cdots+n_r$. If $S=\{m_1, m_2, \ldots, m_s\}_< \subseteq \mathbf{n-1}$, then MacMahon [13, Sections 167–168] has studied the relation between the numbers $\beta(S)$ and the theory of distribution of objects. He uses the notation $N(\mu)_\lambda = \beta(S)$, where $\lambda$ is the partition of $n$ into parts $n_1, n_2, \ldots, n_s$, and where $S=\{\mu_1, \mu_1+\mu_2, \mu_1+\mu_2+\mu_3, \ldots, \mu_1+\mu_2+\cdots+\mu_{r-1}\}$. The $p$-analog of the lattice $L$ (at least when $p$ is prime) is the lattice of subgroups of an abelian $p$-group of type $(n_1, n_2, \ldots, n_r)$. Hence we get a $p$-analog of MacMahon's invariants $N(\mu)_\lambda$. Some properties of these numbers will be discussed in a subsequent paper.

As a further example, if $L$ is a boolean algebra of rank $n$ and $0 \leqslant s \leqslant n-1$, then $A_{ns} = \sum \beta(S)$ is an Eulerian number [14], where the sum is over all subsets of $\mathbf{n-1}$ of cardinality $n-s$ (see [16, Section 13]) Hence if $L$ is the lattice of subspaces of a projective geometry of dimension $n$ over $GF(q)$, we get a $q$-analog of the Eulerian numbers. These $q$-Eulerian numbers differ from those of Carlitz [3]. In fact, Carlitz's $q$-Eulerian numbers $A_{ns}(q)$ can be defined as

$$A_{ns}(q) = \sum_{S} \beta(S)\, q^{\Sigma S},$$

where $L$ is a boolean algebra of rank $n$, $S$ ranges over all subsets of $\mathbf{n-1}$ of cardinality $n-s$, and $\sum S$ is the sum of the elements of $S$.

If we take $L=\Pi_{n+1}$ (cf. Example 2.6), then $\sum \beta(S)$, where $S$ ranges over all subsets of $\mathbf{n-1}$ of cardinality $n-s$, is a '$\Pi$-Eulerian number'. These numbers seem never to have been considered before and may be of further interest.

## 4. The Birkhoff polynomial

In the remaining three sections of this paper, we will be largely concerned with upper-semimodular *SS*-lattices, or *USS*-lattices for short. Let $L$ be a finite lattice with a rank function $r$, and let $n$ be the rank of $L$. The *Birkhoff polynomial* $p(\lambda)$ (also called the *characteristic polynomial* [5] or *Poincaré polynomial* [4]) of $L$ is defined by

$$p(\lambda) = \sum_{x \in L} \mu(\hat{0}, x)\, \lambda^{n-r(x)},$$

where $\mu$ denotes the Möbius function of $L$. This concept is due to G. D. Birkhoff [2], though usually it is only defined for more restrictive classes of lattices.

We shall show that the Birkhoff polynomial $p(\lambda)$ of a *USS*-lattice $L$ of rank $n$ has nonnegative integral roots $a_1=1, a_2, \ldots, a_n$ connected with the structure of $L$. In particular, $\mu(\hat{0}, \hat{1})=(-1)^n a_1 a_2 \ldots a_n$. This fact can be proved in a purely lattice-theoretic way by applying the 'factorization theorem' of Stanley [17] (in fact, the following theorem led to the discovery of the factorization theorem), but in the supersolvable case we can gain more insight into the structure of $p(\lambda)$ by employing a different approach. Specifically, the coefficients of $p(\lambda)$ are symmetric functions in the $a_i$'s, and we shall attach a combinatorial meaning to each term of these symmetric functions.

4.1 THEOREM. *Let $(L, \Delta)$ be a USS-lattice of rank $n$ with $\Delta$ given by $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$, and let $a_i$ be the number of atoms $x$ of $L$ satisfying $\gamma(\hat{0}, x)=i$ (i.e., $x\leqslant x_i$ but $x\nleqslant x_{i-1}$). Then*

$$p(\lambda) = (\lambda - a_1)(\lambda - a_2)\ldots(\lambda - a_n).$$

*In particular*, $\mu(\hat{0}, \hat{1})=(-1)^n a_1 a_2 \ldots a_n$.

*Proof.* Let $x\in L$ with $r(x)=m$. By Corollary 1.4, $(-1)^m \mu(\hat{0}, x)$ is equal to the number of chains $\hat{0}=y_0<y_1<\cdots<y_m=x$ such that $i_1>i_2>\cdots>i_m$, where $i_j=\gamma(y_{j-1}, y_j)$. Hence the coefficient of $\lambda^{n-m}$ in $p(\lambda)$ is equal to $(-1)^m$ times the total number of chains $\hat{0}=y_0<y_1<\cdots<y_m$ with $i_1>i_2>\cdots>i_m$ (where $i_j=\gamma(y_{j-1}, y_j)$ as before).

Fix a sequence $n>i_1>i_2>\cdots>i_m\geqslant 1$. We prove by induction on $m$ that:

(A) The number of chains $\hat{0}=y_0<y_1<\cdots<y_m$ satisfying $r(y_m)=m$ and $i_j=\gamma(y_{j-1}, y_j)$ $(j=1, 2,\dots, m)$ is $a_{i_1} a_{i_2}\dots a_{i_m}$.

Assertion (A) is clearly true for $m=1$, by definition of $a_i$. Assume true for $m-1$. We thus have $a_{i_1} a_{i_2}\dots a_{i_{m-1}}$ chains $\hat{0}=y_0<y_1<\cdots<y_{m-1}$ with $i_j=\gamma(y_{j-1}, y_j)$ $(1\leqslant j\leqslant m-1)$, and it suffices to prove that there are precisely $a_{i_m}$ elements $y_m$ covering $y_{m-1}$ such that $i_m=\gamma(y_{m-1}, y_m)$. Suppose $y_m$ covers $y_{m-1}$ and $i_m=\gamma(y_{m-1}, y_m)$. Then $i_m$ is the least element of the set $\Gamma(y_m)$. Applying Corollary 1.3 to the case $x=\hat{0}$, $y=y_m$, it follows that there is a unique atom $x'\in L$ satisfying $x'\leqslant y_m$ and $\gamma(\hat{0}, x')=i_m$. Conversely, given any atom $x'$ satisfying $\gamma(\hat{0}, x')=i_m$, then we can take $y_m=x'\vee y_{m-1}$. (This is where the assumption of upper-semimodularity is needed.) Hence there is a one-to-one correspondence between the $y_m$'s and the atoms $x'$ of $L$ satisfying $\gamma(\hat{0}, x')=i_m$. Since there are $a_{i_m}$ such atoms, the proof of (A) follows by induction.

Thus the coefficient of $\lambda^{n-m}$ in $p(\lambda)$ is equal to $(-1)^m\sum a_{i_1} a_{i_2}\dots a_{i_m}$, the sum being over all sequences $n\geqslant i_1>i_2>\cdots>i_m\geqslant 1$, so $p(\lambda)=(\lambda-a_1)(\lambda-a_2)\dots(\lambda-a_n)$. □

Note that Theorem 4.1 shows that no matter what $M$-chain $\Delta$ we choose for $L$, the set of $a_i$'s is uniquely determined; on the other hand, easy examples show that their order can vary (though trivially, $a_1=1$).

We now give various examples which illustrate Theorem 4.1.

4.2 *Example.* If $L$ is a boolean algebra of rank $n$, then each $a_i=1$ so $p(\lambda)=(\lambda-1)^n$.

4.3 *Example.* If $L$ is a projective geometry of rank $n$ over $GF(q)$, then $a_i=q^i$ so $p(\lambda)=(\lambda-1)(\lambda-q)\dots(\lambda-q^{n-1})$.

4.4 *Example.* Let $L=\Pi_{n+1}$ with the $M$-chain of Example 2.6. Then $a_i=i$ so $p(\lambda)=(\lambda-1)(\lambda-2)\dots(\lambda-n)$.

4.5 *Example.* Let $L$ be Dowling's lattice of rank $n$ over $GF(q)$ (cf. Example 2.9). Let $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$ be the $M$-chain in $L$ such that $x_i$ contains all the vectors $v$ generating $L$ whose nonzero entries appear among the first coordinates of $v$. Thus $x_i$ contains $i$ vectors with one nonzero entry and $\binom{i}{2}(q-1)$ vectors with two nonzero entries. Hence

$$\begin{aligned} a_i &= \left[i+\binom{i}{2}(q-1)\right]-\left[i-1+\binom{i-1}{2}(q-1)\right] \\ &= 1+(i-1)(q-1). \end{aligned}$$

Therefore $p(\lambda)=(\lambda-1)(\lambda-q)(\lambda-2q+1)(\lambda-3q+2)\cdots(\lambda-(n-1)q+n-2)$, a result proved by Dowling by other means.

4.6 *Example.* Let $G$ be an $SS$-graph, as defined in Example 2.7. Let $v_1, v_2, \ldots, v_n$ be a labeling of the vertices of $G$ in accordance with Proposition 2.8. Suppose $v_i$ is connected to exactly $a_{i-1}$ vertices $v_j$ with $j<i$. Then by Proposition 2.8, these vertices form a clique (complete subgraph), from which it follows that the chromatic polynomial of $G$ is given by $\lambda(\lambda-a_1)(\lambda-a_2)\ldots(\lambda-a_n)$. By [15, Section 9], the Birkhoff polynomial $p(\lambda)$ of the lattice of contractions of $G$ is $(\lambda-a_1)(\lambda-a_2)\ldots(\lambda-a_n)$. Loosely speaking, an $SS$-graph is a graph whose chromatic polynomial can be 'trivially' calculated. Note that a polynomial $p(\lambda)$ is the Birkhoff polynomial of the lattice of contractions of some $SS$-graph if and only if the roots of $p(\lambda)$ are positive integers such that if $a>1$ is a root, then $a-1$ is a root.

4.7 *Example.* In Example 2.10, a geometric $SS$-lattice was constructed with an $M$-chain $\hat{0}=x_0<x_1<\cdots<x_n=\hat{1}$, such that exactly $a_i$ atoms $x$ satisfy $\gamma(\hat{0}, x)=i$, for any set $a_1, a_2, \ldots, a_n$ of positive integers with $a_1=1$. Hence $p(\lambda)=(\lambda-a_1)(\lambda-a_2)\ldots(\lambda-a_n)$. It follows that a polynomial $p(\lambda)$ is the Birkhoff polynomial of a geometric $SS$-lattice if and only if the roots of $p(\lambda)$ are positive integers, including the integer 1. Unfortunately, there exist finite geometric lattices which are *not* $SS$-lattices but whose Birkhoff polynomials have only positive integral roots. For instance, take a 5-point line and two others points in the same plane. The Birkhoff polynomial of the resulting geometric lattice $L$ is $(\lambda-1)(\lambda-3)^2$, but $L$ is not supersolvable.

## 5. Loewy chains

Following Zassenhaus [19, p. 215] (who, however, only considers modular lattices), we define a *Loewy chain* in a lattice $L$ of finite length as a chain

$$\hat{0}=y_0<y_1<\cdots<y_r=\hat{1}$$

such that each $y_i$ is the join of the atoms of the segment $[y_{i-1}, y_i]$. If $L$ is upper-semimodular, this is equivalent to saying that each segment $[y_{i-1}, y_i]$ is complemented. This does not mean that each segment $[y_{i-1}, y_i]$ is relatively complemented, or that every element of $[y_{i-1}, y_i]$ is a join of atoms of $[y_{i-1}, y_i]$ (in other words, $[y_{i-1}, y_i]$ need not be a geometric lattice). Recall, however, that every complemented modular lattice is relatively complemented [1, p. 16].

A sufficient condition for a chain of an $SS$-lattice to be a Loewy chain is provided by the next lemma.

5.1 LEMMA. *Let $(L, \Delta)$ be an SS-lattice, and let $K$ be a maximal chain* *in L,*

$$K : \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}.$$

*Let* $0 < m_1 < m_2 < \cdots < m_r = n$. *Then the subchain*

$$\hat{0} = y_0 < y_{m_1} < y_{m_2} < \cdots < y_{m_r} = \hat{1}$$

*is a Loewy chain if*

$$\mathbf{n-1} - D(\pi_K) \subseteq \{m_1, m_2, \ldots, m_{r-1}\}.$$

*Proof.* We need to prove that if, for $1 \leqslant j < k \leqslant n$, we have $\gamma(y_j, y_{j+1}) > \gamma(y_{j+1}, y_{j+2}) > \cdots > \gamma(y_{k-1}, y_k)$, then $y_k$ is the join of atoms of the segment $[y_j, y_k]$.

*First method.* If $\gamma(y_j, y_{j+1}) > \cdots > \gamma(y_{k-1}, y_k)$, then by Corollary 1.4, $\mu(y_j, y_k) \neq 0$. By a theorem of P. Hall [11, Thm. 2.3] (see also, Rota [15, p. 349]), $y_k$ is the join of atoms of $[y_j, y_k]$.

*Second method.* Let $\Delta'$ be the $M$-chain of $[y_j, y_k]$ induced by $\Delta$ (via Proposition 3.2), say, $\Delta' : y_j = x'_0 < x'_1 < \cdots < x'_m = y_k$ $(m = k - j)$. Let $L' = J(P)$ be the distributive lattice generated by $\Delta'$ and the chain $y_j < y_{j+1} < \cdots < y_k$. Regarding each $x'_i$ and $y_i$ as an order ideal of $P$, and assuming $\gamma(y_j, y_{j+1}) > \cdots > \gamma(y_{k-1}, y_k)$, we see that $x_{i+1} - x_i = y_{m-i} - y_{m-i-1}$. It follows that $P$ is an antichain, so $L'$ is a boolean algebra. Thus $y_k$ is the join of atoms of $L'$. Since every atom of $L'$ is an atom of $[y_j, y_k]$, the proof follows. $\square$

We now come to a fundamental theorem telling us when $\beta(S) > 0$ in a *USS*-lattice.

5.2 THEOREM. *Let* $(L, \Delta)$ *be a USS-lattice of rank* $n$, *and let* $S = \{m_1, m_2, \ldots, m_s\}_< \subseteq \mathbf{n-1}$. *There exists a chain* $C$,

$$C : \hat{0} = y_0 < y_1 < \cdots < y_s < y_{s+1} = \hat{1},$$

*satisfying the two conditions*

(i) $r(y_i) = m_i$, $1 \leqslant i \leqslant s$;

(ii) *each* $y_i$ $(1 \leqslant i \leqslant s+1)$ *is the join of atoms of the segment* $[y_{i-1}, y_i]$, *i.e., C is a Loewy chain, if and only if* $\beta(\mathbf{n-1} - S) > 0$.

*Proof.* If $\beta(\mathbf{n-1} - S) > 0$, then there exists a maximal chain $K$ such that $D(\pi_K) = \mathbf{n-1} - S$. By Lemma 5.1, the subchain $\hat{0} = y_0 < y_1 < \cdots < y_s < y_{s+1} = \hat{1}$ of $K$ satisfying $r(y_i) = m_i$ $(1 \leqslant i \leqslant s)$ is a Loewy chain.

Conversely, suppose we have a chain $C : \hat{0} = y_0 < y_1 < \cdots < y_s < y_{s+1} = \hat{1}$, satisfying (i) and (ii). Let $k$ satisfy $1 \leqslant k \leqslant s+1$, and define $x = y_{k-1}$, $y = y_k$. Suppose $r(y) - r(x) = m$. We will establish the existence of a maximal chain $K_k$ in $[x, y]$,

$$K_k : x = z_0 < z_1 < \cdots < z_m = y,$$

such that

$$\gamma(z_0, z_1) > \gamma(z_1, z_2) > \cdots > \gamma(z_{m-1}, z_m).$$

The proof is based on the following lemma:

(A) *If* $[u, v]$ *is a segment of an SS-lattice such that* $v$ *is the join of atoms of* $[u, v]$, *then there is an atom* $w$ *of* $[u, v]$ *such that* $\gamma(u, w)$ *is the largest element of* $\Gamma(u, v)$.

*Proof of* (A). Let $v'$ be the element of the induced $M$-chain $\Delta_{uv}$ which is covered by $v$. Since $v$ is a join of atoms, some atom $w$ of $[u, v]$ satisfies $w \nleq v'$. Hence $\gamma(u, w)$ is the largest element of $\Gamma(u, v)$, proving (A).

Now, since $y$ is the join of atoms of $[x, y]$, by (A), we have an atom $z_1$ such that $\gamma(z_0, z_1)$ is the largest element of $\Gamma(x, y)$. Since $L$ is upper-semimodular, for any $z \in [x, y]$, we have that $y$ is the join of atoms of $[z, x]$. (This is where the assumption of upper-semimodularity is needed.) In particular, $y$ is the join of atoms of $[z_1, y]$. Thus, by (A), there is an atom $z_2$ of $[z_1, y]$ such that $\gamma(z_1, z_2)$ is the largest element of $\Gamma(z_1, y)$. Continuing in this way, we get a maximal chain $K_k: x = z_0 < z_1 < \cdots < z_m = y$ such that $\gamma(z_{i-1}, z_i)$ is the largest element of $\Gamma(z_{i-1}, y)$, so

$$\gamma(z_0, z_1) > \gamma(z_1, z_2) > \cdots > \gamma(z_{m-1}, z_m).$$

The union of these chains $K_k$ $(1 \leqslant k \leqslant s+1)$ is then a maximal chain $K$ of $L$ (containing $C$), satisfying $\mathbf{n}-\mathbf{1}-D(\pi_K) \subseteq S$. Then $\mathbf{n}-\mathbf{1}-S \subseteq D(\pi_K)$, so by Proposition 3.3, $\beta(\mathbf{n}-\mathbf{1}-S) > 0$. $\square$

*Conjecture*. Equation (2), Proposition 3.3, and Theorem 5.2 are valid for any finite upper-semimodular lattice $L$.

## 6. *q-USS*-lattices

Recall that the *degree* of a projective geometry $V$ is one less than the number of points on a line. In particular, if $V$ is coordinatized by the field $GF(q)$, then deg $V = q$. If $L$ is the lattice of subspaces of a projective geometry of degree $q$, then the integer $q$ enters into many of the combinatorial properties of $L$. Our object in this section is to define a general class of upper-semimodular lattices $L$ 'based on' the integer $q$, and, in the $SS$-case, to show how $q$ enters into the global combinatorial properties of $L$.

6.1 DEFINITION. Let $q$ be a fixed positive integer. A *q-lattice* is a lattice $L$ of finite length with the property that every segment $[x, y]$ of $L$ for which $y$ is the join of atoms of $[x, y]$ is isomorphic to the lattice of subspaces of a projective geometry of degree $q$ (or to a boolean algebra if $q = 1$).

Observe that every $q$-lattice is necessarily upper-semimodular, since if $y$ and $z$

cover $x$, then the segment $[x, y \vee z]$ is a projective geometry or boolean algebra, so $y \vee z$ covers $y$ and $z$.

6.2 *Example.* Let $L^*(G)$ denote the dual of the lattice of subgroups of a finite $p$-group $G$. Since $G$ is supersolvable, we know from Example 2.5 that $L^*(G)$ is an $SS$-lattice. In addition, $L^*(G)$ is a $p$-lattice. This fact is essentially well known, but for the sake of completeness we shall sketch a proof. If $G$ is any finite $p$-group and if $\Phi(G)$ is the Frattini subgroup of $G$ (intersection of all maximal subgroups), then the quotient group $G/\Phi(G)$ is an elementary abelian $p$-group [10, Thm. 12.2.1]. Hence, if $H \in L^*(G)$ and $H'$ denotes the join of all elements in $L^*(G)$ covering $H$, then $H' = \Phi(H)$, and the segment $[H, H']$ is isomorphic to the dual of the lattice of subgroups of the elementary abelian $p$-group $H/H'$. Since this lattice is a projective geometry over $GF(p)$, $L^*(G)$ is a $p$-lattice. We therefore say that $L^*(G)$ is a *p-SS-lattice.*

6.3 *Example.* Let $\mathfrak{N}$ denote the lattice of natural partial orders on the set $\mathbf{n}$, as defined in Example 2.11. It is easily seen that the dual $\mathfrak{N}^*$ is a 1-$SS$-lattice. Indeed, if $[P, Q]$ is a segment of $\mathfrak{N}^*$ such that $y$ is a join of atoms of $[P, Q]$, then $Q$ is obtained from $P$ by removing a set $S$ of relations of the form $i<j$ where $j$ covers $i$ in $P$; and the general element of $[P, Q]$ is obtained by removing an arbitrary subset of the relations in S. Hence $[P, Q]$ is a boolean algebra.

The basic result on $q$-$SS$-lattices is the following.

6.4 LEMMA. Let $(L, \Delta)$ be a $q$-$SS$-lattice of rank $n$; let $\sigma$ be a permutation of $\{1, 2, \dots, n\}$ with $\mathbf{n}-\mathbf{1}-D(\sigma)=\{j_1, j_2, \dots, j_t\}_<$ (where $D(\sigma)$ is the descent set of $\sigma$); and let $j_0=0, j_t=n$. Then the number of maximal chains $K$ of $L$ satisfying $\sigma=\pi_K$ is equal to $q^k M$, where

$$k = \sum_{r=1}^{t} \binom{j_r - j_{r-1}}{2}, \tag{11}$$

and where $M$ is the number of Loewy chains

$$C: \hat{0} = y_0 < y_1 < \cdots < y_t = \hat{1} \tag{12}$$

such that $r(y_i)=j_i$ and $\Gamma(y_i)=\{\sigma(1), \sigma(2), \dots, \sigma(j_i)\}$, $0 \leqslant i \leqslant t$.

*Proof.* If $K$ is a maximal chain of $L$, satisfying $\pi_K=\sigma$, then by Lemma 5.1, the subchain $C$ of $K$ consisting of all $x \in K$ such that $r(x)=j_i$ $(0 \leqslant i \leqslant t)$ is a Loewy chain, while, by definition of $\pi_K$ and $\Gamma$, we have $\Gamma(y_i)=\{\sigma(1), \sigma(2), \dots, \sigma(j_i)\}$. Hence it suffices to prove that if we have a Loewy chain (12) with $r(y_i)=j_i$ and $\Gamma(y_i)=\{\sigma(1), \sigma(2), \dots, \sigma(j_i)\}$, then the number of refinements of $C$ to a maximal chain $K$, satisfying $\pi_K=\sigma$, is equal to $q^k$, where $k$ is given by (11).

Assume we have such a Loewy chain $C$. Since $L$ is a $q$-lattice, each segment

$[y_{r-1}, y_r]$ $(1 \leqslant r \leqslant t)$ is a projective geometry of degree $q$. Hence $\mu(y_{r-1}, y_r) = (-1)^b q^{k_r}$, where $b = j_r - j_{r-1}$ and $k_r = \binom{j_r - j_{r-1}}{2}$. Now, by Corollary 1.4, the number of maximal chains $y_{r-1} = z_0 < z_1 < \cdots < z_b = y_r$ of the segment $[y_{r-1}, y_r]$ such that

$$\gamma(z_0, z_1) > \gamma(z_1, z_2) > \cdots > \gamma(z_{b-1}, z_b)$$

is just $(-1)^b \mu(y_{r-1}, y_r) = q^{k_r}$. Hence the total number of refinements of $C$ to a maximal chain $K$, satisfying $\pi_K = \sigma$, is equal to $q^{k_1} q^{k_2} \ldots q^{k_t} = q^k$, and the proof follows. □

6.5 COROLLARY. *Let $(L, \Delta)$ be a q-SS-lattice of rank n, and let $S \subseteq \mathbf{n-1}$. Set $\mathbf{n-1} - S = \{j_1, j_2, \ldots, j_{t-1}\}_<$, with $j_0 = 0, j_t = n$. Then $\beta(S)$ is divisible by $q^k$, where k is given by* (11). □

If $n$ is a positive integer and $0 \leqslant s \leqslant n-1$, define

$$Q(n, s) = \frac{1}{2}\left[\frac{n}{n-s}\right]\left(n + s - (n-s)\left[\frac{n}{n-s}\right]\right)$$

(brackets denote the integer part).

6.6 THEOREM. *Let $(L, \Delta)$ be a q-SS-lattice of rank n, and let $S \subseteq \mathbf{n-1}$ with $|S| = s$. Then $\beta(S)$ is divisible by $q^{Q(n,s)}$. This result is best possible in the sense that, given n and $0 \leqslant s \leqslant n-1$, there exists a q-SS-lattice of rank n and a set $S \subseteq \mathbf{n-1}$ of cardinality s such that $\beta(S) = q^{Q(n,s)}$.*

*Proof.* Let $K$ be a maximal chain of $L$ with $D(\pi_K) = S$, and let $\mathbf{n-1} - S = \{j_1, j_2, \ldots, j_{t-1}\}_<$, so $s + t = n$. It follows from Corollary 6.5 that $\beta(S)$ is divisible by $q^m$, where

$$m = \min \sum_{r=1}^{t} \binom{j_r - j_{r-1}}{2},$$

the minimum being taken over all sequences $0 = j_0 < j_1 < \cdots < j_t = n$. We shall show that this minimum is equal to $Q(n, s)$.

If $0 \leqslant a < b$, then it is easily verified that

$$\binom{a}{2} + \binom{b}{2} \geqslant \binom{a+1}{2} + \binom{b-1}{2}.$$

Hence the sum (11) is minimized when the differences $j_1 - j_0, j_2 - j_1, \ldots, j_t - {}_{t-1}$ are as nearly equal as possible. Given $n$ and $t \geqslant 1$, it is easy to show that there is a unique partition $\lambda = \lambda(n, t)$ of $n$ into $t$ parts such that each part is one of two consecutive integers. In this partition $\lambda$, the part $[n/t]+1$ appears $n - t[n/t]$ times and the part $[n/t]$ appears $t(1+[n/t]) - n$ times. Hence the sum (11) is minimized when $n - t[n/t]$

of the differences $j_r - j_{r-1}$ are equal to $[n/t]+1$ and $t(1+[n/t])-n$ of them are equal to $[n/t]$. Setting $C=[n/t]$ and using $t=n-s$, we get

$$\begin{aligned} m &= (n-tC)\binom{C+1}{2} + (t(1+C)-n)\binom{C}{2} \\ &= \tfrac{1}{2}C(2n-t-tC) \\ &= Q(n,s). \end{aligned}$$

To prove that this result is best possible given $n$ and $s$, choose $0=j_0<j_1<\cdots<$ $<j_t=n$ so as to minimize the sum (11) (so this minimum value is $Q(n,s)$). For $1\leqslant r\leqslant t$, let $V_r$ be a projective geometry over $GF(q)$ of rank $j_r - j_{r-1}$. Let $L$ be obtained by 'stacking' the $V_r$'s, i.e., by identifying the top element of $V_{r-1}$ with the bottom element of $V_r$. Then $L$ is a $q$-$SS$-lattice (in fact, a $q$-modular lattice), and it is clear (e.g., from Theorem 1.2) that if $S=\mathbf{n-1}-\{j_1,j_2,\ldots,j_{t-1}\}$, then $|S|=s$ and $\beta(S)=$ $=q^{Q(n,s)}$. □

For fixed $s$, the function $Q(n,s)$ exhibits the following behavior:

$$\begin{aligned} \binom{s+1}{2} &= Q(s+1,s) > Q(s+2,s) > \cdots > Q(2s-1,s) > Q(2s,s) \\ &= Q(2s+1,s) = \cdots = s. \end{aligned}$$

Thus the sequence $Q(n,s)$ (for fixed $s$) becomes 'stable' at the value $s$ when $n\geqslant 2s$.

6.7 COROLLARY. *Let $L$ be a $q$-SS-lattice of rank $n$, and let $0\leqslant m\leqslant n$. The number of elements of $L$ of rank $m$ is congruent to* $1\,(\mathrm{mod}\,q)$.

*Proof.* The number of such elements is $\alpha(m)=\beta(m)+1$. Now $Q(n,1)=1$, so $\beta(m)$ is divisible by $q$. □

As a special case of Corollary 6.7, we get the well-known result of P. Hall [12] that the number of subgroups of a given order in a finite $p$-group is congruent to $1\,(\mathrm{mod}\,p)$. Thus Theorem 6.6 gives a lattice-theoretical generalization of this result, which, even in the case of the lattice of subgroups of $p$-groups, appears to be new. We ask whether there are similar lattice-theoretic arguments for proving stronger results in the enumerative theory of $p$-groups, e.g., Kulakoff's theorem [19, p. 153].

*Conjecture.* Corollary 6.5 and Theorem 6.6 are valid for any finite $q$-lattice $L$.

REFERENCES

[1] Garrett Birkhoff, *Lattice Theory* (third ed.), American Mathematical Society, Providence, R.I. (1967).

[2] G. D. Birkhoff, *A determinant formula for the number of ways of coloring a map,* Annals of Math. *14* (1912), 42–46.

[3] L. Carlitz, *q-Bernoulli and Eulerian numbers,*) Trans. Amer. Math. Soc. *76* (1954), 332–350.
[4] H. H. Crapo, *The Tutte polynomial,* Aequationes Math. *3* (1969), 211–229.
[5] H. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries,* M.I.T. Press (1970).
[6] R. A. Dean and G. Keller, *Natural partial orders,* Canad. J. Math. *20* (1968), 535–554.
[7] P. Doubilet, G.-C. Rota, and R. P. Stanley, *On the foundations of combinatorial theory: The idea of generating function,* Sixth Berkeley Symposium on Mathematical Statistics and Probability (to appear).
[8] T. A. Dowling, *A g-analog of the partition lattice,* Proc. International Symposium on Combinatorial Mathematics and Its Applications, Colorado State University, Fort Collins, Colorado, September 9–11 (1971) (to appear).
[9] D. Foata and M. P. Schützenberger, *Théorie géométrique des polynômes Eulériens,* Lecture Notes in Math. *138,* Springer-Verlag, Berlin (1970).
[10] Marshall Hall, Jr., *The Theory of Groups,* Macmillan, New York (1959).
[11] P. Hall, *The Eulerian functions of a group,* Quart. J. Math. *7* (1936), 134–151.
[12] P. Hall, *A contribution to the theory of groups of prime-power orders,* Proc. London Math. Soc. (2) *36* (1933), 29–95.
[13] P. A. MacMahon, *Combinatory Analysis,* Vols. 1–2, Cambridge (1916), reprinted by Chelsea (1960).
[14] J. Riordan, *An Introduction to Combinatorial Analysis,* John Wiley & Sons, New York (1958).
[15] G.-C. Rota, *On the foundations of combinatorial theory, I. Theory of Möbius functions,* Z. Wahrscheinlichkeitstheorie *2* (1964), 340–368.
[16] R. P. Stanley, *Ordered structures and partitions,* Memoirs of the Amer. Math. Soc. *119* (1972).
[17] R. P. Stanley, *Modular elements of geometric lattices,* Algebra Universalis *1* (1971), 214–217.
[18] L. R. Wilcox, *Modularity in the theory of lattices,* Annals of Math. (2) *40* (1939), 490–505.
[19] H. J. Zassenhaus, *The Theory of Groups* (second ed.), Chelsea, New York (1958).

*University of California*
*Berkeley, California*
*U.S.A.*