Some Congruence Properties of Symmetric Group Character Values

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We follow symmetric function notation and terminology from [4, Ch. 7]. Let $\lambda \vdash n$, and let $f^\lambda$ denote the number of standard Young tableaux of shape $\lambda$. Equivalently, $f^\lambda$ is the dimension of the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by $\lambda$. Let $\ell$ be a prime, and let

$$n = \alpha_0 + \alpha_1 \ell + \alpha_2 \ell^2 + \cdots,$$

with $0 \leq \alpha_i < \ell$, the base $\ell$ expansion of $n$. Let

$$P(x) = \prod_{n \geq 1} (1 - x^n)^{-1}.$$

If $G(x)$ is a power series, then $[x^\alpha]G(x)$ denotes the coefficient of $x^\alpha$ in $G(x)$. Finally, write $m_\ell(n)$ for the number of partitions $\lambda \vdash n$ for which $f^\lambda$ is relatively prime to $\ell$. I. G. Macdonald [1] showed that

$$m_\ell(n) = \prod_{r \geq 0} [x^{\alpha_r}] P(x)^{\ell^r}. \quad (1)$$

In particular, if each $\alpha_r = 0$ or 1, so $n = \ell^{k_1} + \ell^{k_2} + \cdots$ with $k_1 < k_2 < \cdots$, then

$$m_\ell(n) = \ell^{k_1+k_2+\cdots}. \quad (2)$$

Equation (2) had earlier been conjectured by J. McKay for $\ell = 2$, inspiring Macdonald to write his paper.

In this note we give a simpler approach to equation (1) based on symmetric functions, allowing us to extend the result to some other irreducible character values of $\mathfrak{S}_n$. 
Lemma 1. Let $\lambda \vdash n$. The number of ways to add a border strip of size $m > n$ to $\lambda$ is $m$.

Proof. Straightforward. □

First we do the special case (2).

Proof of equation (2). If $f, g$ are symmetric functions over $\mathbb{Z}$, then write $f \equiv g \pmod{\ell}$ to mean that every coefficient of $f - g$ is divisible by $\ell$. Thus $p^n_j \equiv p_{j \ell^r} \pmod{\ell}$, so

$$p^n_1 = p_1^{\ell k_1 + \ell k_2 + \ldots} \equiv p_{\ell k_1} p_{\ell k_2} \cdots \pmod{\ell}$$

By the Murphysan-Nakayama rule,

$$p_{\ell k_1} p_{\ell k_2} \cdots = \sum_B \text{sgn}(B) s_{\text{sh}(B)},$$

where $B$ is obtained by beginning with a hook $B_1$ of size $\ell k_1$, then adjoining a border strip $B_2$ of size $\ell k_2$, etc. Here $\text{sgn}(B) = \pm 1$ and $\text{sh}(B)$ is the shape of $B$. By Lemma 1, there are $\ell k_1$ choices for $B_1$, then $\ell k_2$ choices for $B_2$, etc., so $N = \ell k_1 + k_2 + \ldots$ choices in all. It is easy to see that all the shapes obtained in this way are distinct. Hence $p_{\ell k_1} p_{\ell k_2} \cdots$ is a linear combination of $N$ Schur functions, each with sign $\pm 1$. Now $p^n_1 = \sum_{\lambda \vdash n} f^\lambda s_{\lambda}$, so taking $p^n_1$ modulo $\ell$ completes the proof. □

Proof of equation (1). Now we obtain

$$p^n_1 \equiv p_1^{\alpha_0} p_\ell^{\alpha_1} p_{\ell^2}^{\alpha_2} \cdots \pmod{\ell}.$$ 

By Lemma 1 it follows that

$$m_\ell(n) = \prod_{r \geq 0} m_\ell(\alpha_r \ell^r).$$

If we expand $p^n_{\ell^r}$ in terms of Schur functions, the shapes $\lambda$ that appear will be those partitions of $\alpha_r \ell^r$ with empty $\ell$-core. Let $\mu_1, \ldots, \mu_\alpha$ be the $\ell^r$-quotient of $\lambda$. Let $c_i = |\mu_i|$. Then by standard properties of cores and quotients [2, Exam. I.1.8, p. 12, and Exam. I.5.2(b), p. 75],

$$\langle p^\alpha_{\ell^r}, s_\lambda \rangle = \pm \left( \sum_{\mu_1, \mu_2, \ldots} \mu_1 \mu_2 \cdots \right) f^{\mu_1} f^{\mu_2} \cdots.$$
Because $\alpha_r < p$, it follows easily that
\[ \langle p^{\alpha_r}_r, s_\lambda \rangle \not\equiv 0 \pmod{p}. \]
Hence $m_\ell(\alpha_r, \ell^r)$ is equal to the number of partitions of $\alpha_r \ell^r$ with empty $\ell^r$-core. By [4, Exer. 7.59(e)] this number is $[x^{\alpha_r}] P(x)^{\ell^r}$, and the proof follows.

Numerous generalizations suggest themselves.

- Can one determine for each $0 \leq i < \ell$ the number of $\lambda \vdash n$ for which $f^\lambda \equiv i \pmod{\ell}$?
- Rather than using $p^n_1 = \sum_{\lambda \vdash n} f^\lambda s_\lambda$, use
  \[ p^n_j = \sum_{\lambda \vdash j^n} \chi^\lambda(\langle j^n \rangle) s_\lambda, \]
  where $\langle j^n \rangle$ denotes the partition with $n$ parts equal to $j$. For instance, taking $j = \ell^k$ gives:

**Proposition 2.** Let $\lambda \vdash \ell^k n$. The number of character values $\chi^\lambda(\ell^k, \ell^k, \ldots)$ (n terms equal to $\ell^k$) that are not divisible by $\ell$ is equal to the number of $\mu \vdash \ell^k n$ for which $f^\mu$ is not divisible by $\ell$ (given by equation (1)).

What about other values of $j$, i.e., $j \neq \ell^k$?

- Use $h_j$ instead of $p_j$. Use [4, Exer. 7.61] to expand $h_j^{\ell^r} \equiv h_j(x_1^{\ell^r}, x_2^{\ell^r}, \ldots)$ in terms of Schur functions. This will give Kostka number congruences. For instance, let $g(n)$ denote the number of odd Kostka numbers $K_{\lambda, (n+1)}$, $\lambda \vdash 2n$. Since $h_2(x_1^n, x_2^n, \ldots) = h_2[p_n]$ (plethysm) is a linear combination of $\binom{n+1}{2}$ Schur functions with coefficients $\pm 1$, we get $g(2^r) = \binom{2^r+1}{2}$.

We apparently have
\[
\begin{align*}
g(2^r + 1) &= \binom{2^r + 1}{2} \\
g(2^r + 2) &= 3 \binom{2^r + 1}{2} \\
g(2^r + 3) &= 5 \binom{2^r + 1}{2}.
\end{align*}
\]
What about $g(2^r - 1)$? the values of $g(n)$ for $1 \leq n \leq 15$ are 1, 3, 5, 10, 10, 30, 50, 36, 36, 108, 180, 312, 312, 840, 1368 (I think).
• What about $g^\lambda$ (shifted SYT) instead of $f^\lambda$? And projective characters of $\mathfrak{S}_n$ instead of ordinary ones? A relevant exercise might be [2, Exam. I.1.9, p. 14].

• What about differential posets? I.e., replace $f^\lambda$ for $\lambda \vdash n$ with the number $e(x)$ of saturated chains from $\bar{0}$ to an element $x$ of rank $n$. The Fibonacci differential poset in particular may be interesting. In this case $e(x)$ is the dimension of an irreducible representation of the Okada algebra $\mathcal{O}_n$ [3], so we can also ask about congruence properties of character values of $\mathcal{O}_n$.

References


