The Descent Set and Connectivity Set of a Permutation[1]

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version of 16 August 2005

Abstract

The descent set $D(w)$ of a permutation w of $1, 2, \ldots, n$ is a standard and well-studied statistic. We introduce a new statistic, the *connectivity set* $C(w)$, and show that it is a kind of dual object to $D(w)$. The duality is stated in terms of the inverse of a matrix that records the joint distribution of $D(w)$ and $C(w)$. We also give a variation involving permutations of a multiset and a q-analogue that keeps track of the number of inversions of w.

1 A duality between descents and connectivity.

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] = \{1, 2, \ldots, n\}$, and let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. The *descent set* $D(w)$ is defined by

$$D(w) = \{i \,:\, a_i > a_{i+1}\} \subseteq [n-1].$$

The descent set is a well-known and much studied statistic on permutations with many applications, e.g., [6, Exam. 2.24, Thm. 3.12.1][7, §7.23]. Now define the *connectivity set* $C(w)$ by

$$C(w) = \{i \,:\, a_j < a_k \text{ for all } j \le i < k\} \subseteq [n-1]. \tag{1}$$

[1]2000 Mathematics Subject Classification: 05A05
Key words and phrases: descent set, connected permutation, connectivity set
[2]Partially supported by NSF grant #DMS-9988459 and by the Institut Mittag-Leffler.

The connectivity set seems not to have been considered before except for equivalent definitions by Comtet [3, Exer. VI.14] and Callan [1] with no further development. H. Wilf has pointed out to me that the set of splitters of a permutation arising in the algorithm Quicksort [8, §2.2] coincides with the connectivity set. Some notions related to the connectivity set have been investigated. In particular, a permutation w with $C(w) = \emptyset$ is called *connected* or *indecomposable*. If $f(n)$ denotes the number of connected permutations in $\mathfrak{S}_n$, then Comtet [3, Exer. VI.14] showed that

$$\sum_{n\geq 1} f(n)x^n = 1 - \frac{1}{\sum_{n\geq 0} n!x^n},$$

and he also considered the number $\#C(w)$ of components. He also obtained [2][3, Exer. VII.16] the complete asymptotic expansion of $f(n)$. For further references on connected permutations, see Sloane [4]. In this paper we will establish a kind of "duality" between descent sets and connectivity sets.

We write $S = \{i_1, \ldots, i_k\}_<$ to denote that $S = \{i_1, \ldots, i_k\}$ and $i_1 < \cdots < i_k$. Given $S = \{i_1, \ldots, i_k\}_< \subseteq [n-1]$, define

$$\eta(S) = i_1!\,(i_2 - i_1)! \cdots (i_k - i_{k-1})!\,(n - i_k)!.$$

Note that $\eta(S)$ depends not only on S but also on n. The integer n will always be clear from the context. The first indication of a duality between C and D is the following result.

Proposition 1.1. *Let $S \subseteq [n-1]$. Then*

$$\begin{aligned} \#\{w \in \mathfrak{S}_n \,:\, S \subseteq C(w)\} &= \eta(S) \\ \#\{w \in \mathfrak{S}_n \,:\, S \supseteq D(w)\} &= \frac{n!}{\eta(S)}. \end{aligned}$$

Proof. The result for $D(w)$ is well-known, e.g., [6, Prop. 1.3.11]. To obtain a permutation w satisfying $S \supseteq D(w)$, choose an ordered partition $(A_1, \ldots, A_{k+1})$ of $[n]$ with $\#A_j = i_j - i_{j-1}$ (with $i_0 = 0$, $i_{k+1} = n$) in $n!/\eta(S)$ ways, then arrange the elements of A_1 in increasing order, followed by the elements of A_2 in increasing order, etc.

Similarly, to obtain a permutation w satisfying $S \subseteq C(w)$, choose a permutation of $[i_1]$ in $i_1!$ ways, followed by a permutation of $[i_1 + 1, i_2] := \{i_1+1, i_1+2, \ldots, i_2\}$ in $(i_2 - i_1)!$ ways, etc. $\square$

Let $S, T \subseteq [n-1]$. Our main interest is in the joint distribution of the statistics C and D, i.e., in the numbers

$$X_{ST} = \#\{w \in \mathfrak{S}_n \,:\, C(w) = \overline{S},\ D(w) = T\},$$

where $\overline{S} = [n-1] - S$. (It will be more notationally convenient to use this definition of X_{ST} rather than having $C(w) = S$.) To this end, define

$$\begin{aligned} Z_{ST} &= \#\{w \in \mathfrak{S}_n \,:\, \overline{S} \subseteq C(w),\ T \subseteq D(w)\} \\ &= \sum_{\substack{S' \supseteq S \\ T' \supseteq T}} X_{S'T'}. \end{aligned} \tag{2}$$

For instance, if $n = 4$, $S = \{2, 3\}$, and $T = \{3\}$, then $Z_{ST} = 3$, corresponding to the permutations 1243, 1342, 1432, while $X_{ST} = 1$, corresponding to 1342. Tables of X_{ST} for $n = 3$ and $n = 4$ are given in Figure 1, and for $n = 5$ in Figure 2.

Theorem 1.2. *We have*

$$Z_{ST} = \begin{cases} \eta(\overline{S})/\eta(\overline{T}), & \text{if } S \supseteq T; \\ 0, & \text{otherwise,} \end{cases}$$

Proof. Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$. If $i \in C(w)$ then $a_i < a_{i+1}$, so $i \notin D(w)$. Hence $Z_{ST} = 0$ if $S \not\supseteq T$.

Assume therefore that $S \supseteq T$. Let $C(w) = \{c_1, \ldots, c_j\}_<$ with $c_0 = 0$ and $c_{j+1} = n$. Fix $0 \le h \le j$, and let

$$[c_h, c_{h+1}] \cap \overline{T} = \{c_h = i_1, i_2, \ldots, i_k = c_{h+1}\}_<.$$

If $w = a_1 \cdots a_n$ with $\overline{S} \subseteq C(w)$ and $T \subseteq D(w)$, then the number of choices for $a_{c_h}+1, a_{c_h}+2, \ldots, a_{c_{h+1}}$ is just the multinomial coefficient

$$\binom{c_{h+1} - c_h}{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}} := \frac{(c_{h+1} - c_h)!}{(i_2 - i_1)!\,(i_3 - i_2)! \cdots (i_k - i_{k-1})!}.$$

$S\backslash T$	$\emptyset$	1	2	12
$\emptyset$	1			
1	0	1		
2	0	0	1	
12	0	1	1	1

$S\backslash T$	$\emptyset$	1	2	3	12	13	23	123
$\emptyset$	1							
1	0	1						
2	0	0	1					
3	0	0	0	1				
12	0	1	1	0	1			
13	0	0	0	0	0	1		
23	0	0	1	1	0	0	1	
123	0	1	2	1	2	4	2	1

Figure 1: Table of X_{ST} for $n = 3$ and $n = 4$

Taking the product over all $0 \leq h \leq j$ yields $\eta(\overline{S})/\eta(\overline{T})$. $\square$

Theorem 1.2 can be restated matrix-theoretically. Let $M = (M_{ST})$ be the matrix whose rows and columns are indexed by subsets $S, T \subseteq [n-1]$ (taken in some order), with

$$M_{ST} = \begin{cases} 1, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D = (D_{ST})$ be the diagonal matrix with $D_{SS} = \eta(\overline{S})$. Let $Z = (Z_{ST})$, i.e., the matrix whose (S, T)-entry is Z_{ST} as defined in (2). Then it is straightforward to check that Theorem 1.2 can be restated as follows:

$$Z = DMD^{-1}. \tag{3}$$

Similarly, let $X = (X_{ST})$. Then it is immediate from equations (2) and (3) that

$$MXM = Z. \tag{4}$$

The main result of this section (Theorem 1.4 below) computes the inverse of the matrices X, Z, and a matrix $Y = (Y_{ST})$ intermediate

$S \backslash T$	$\emptyset$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
$\emptyset$	1															
1	0	1														
2	0	0	1													
3	0	0	0	1												
4	0	0	0	0	1											
12	0	1	1	0	0	1										
13	0	0	0	0	0	0	1									
14	0	0	0	0	0	0	0	1								
23	0	0	1	1	0	0	0	0	1							
24	0	0	0	0	0	0	0	0	0	1						
34	0	0	0	1	1	0	0	0	0	0	1					
123	0	1	2	1	0	2	4	0	2	0	0	1				
124	0	0	0	0	0	0	0	1	0	1	0	0	1			
134	0	0	0	0	0	0	1	1	0	0	0	0	0	1		
234	0	0	1	2	1	0	0	0	2	4	2	0	0	0	1	
1234	0	1	3	3	1	3	10	8	6	10	3	3	8	8	3	1

Figure 2: Table of X_{ST} for $n = 5$

between X and Z. Namely, define

$$Y_{ST} = \#\{w \in \mathfrak{S}_n : \overline{S} \subseteq C(w),\ T = D(w)\}. \tag{5}$$

It is immediate from the definition of matrix multiplication and (4) that the matrix Y satisfies

$$Y = MX = ZM^{-1}. \tag{6}$$

In view of equations (3), (4) and (6) the computation of Z^{-1}, Y^{-1}, and X^{-1} will reduce to computing M^{-1}, which is a simple and well-known result. For any invertible matrix $N = (N_{ST})$, write N_{ST}^{-1} for the (S,T)-entry of N^{-1}.

Lemma 1.3. *We have*

$$M_{ST}^{-1} = (-1)^{\#S+\#T} M_{ST}. \tag{7}$$

Proof. Let f, g be functions from subsets of $[n]$ to $\mathbb{R}$ (say) related by

$$f(S) = \sum_{T \subseteq S} g(T). \tag{8}$$

Equation (7) is then equivalent to the inversion formula

$$g(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} f(T). \tag{9}$$

This is a standard combinatorial result with many proofs, e.g., [6, Thm. 2.1.1, Exam. 3.8.3]. $\square$

Theorem 1.4. *The matrices Z, Y, X have the following inverses:*

$$\begin{aligned} Z_{ST}^{-1} &= (-1)^{\#S+\#T} Z_{ST} && (10)\\ Y_{ST}^{-1} &= (-1)^{\#S+\#T} \#\{w \in \mathfrak{S}_n : \overline{S} = C(w),\ T \subseteq D(w)\} && (11)\\ X_{ST}^{-1} &= (-1)^{\#S+\#T} X_{ST}. && (12) \end{aligned}$$

Proof. By equations (3), (4), and (6) we have

$$Z^{-1} = DM^{-1}D^{-1},\ Y^{-1} = MDM^{-1}D^{-1},\ X^{-1} = MDM^{-1}D^{-1}M.$$

Equation (10) is then an immediate consequence of Lemma 1.3 and the definition of matrix multiplication.

Since $Y^{-1} = MZ^{-1}$ we have for fixed $S \supseteq U$ that

$$\begin{aligned} Y^{-1}_{SU} &= \sum_{T\,:\,S \supseteq T \supseteq U} (-1)^{\#T+\#U} Z_{TU} \\ &= \sum_{T\,:\,S \supseteq T \supseteq U} (-1)^{\#T+\#U} \#\{w \in \mathfrak{S}_n \,:\, \overline{T} \subseteq C(w),\ U \subseteq D(w)\} \\ &= \sum_{\overline{T}\,:\,\overline{U} \subseteq \overline{T} \subseteq \overline{S}} (-1)^{\#T+\#U} \#\{w \in \mathfrak{S}_n \,:\, \overline{T} \subseteq C(w),\ U \subseteq D(w)\}. \end{aligned}$$

Equation (11) is now an immediate consequence of the Principle of Inclusion-Exclusion (or of the equivalence of equations (8) and (9)). Equation (12) is proved analogously to (11) using $X^{-1} = Y^{-1}M$. $\square$

NOTE. The matrix M represents the zeta function of the boolean algebra $\mathcal{B}_n$ [6, §3.6]. Hence Lemma 1.3 can be regarded as the determination of the Möbius function of $\mathcal{B}_n$ [6, Exam. 3.8.3]. All our results can easily be formulated in terms of the incidence algebra of $\mathcal{B}_n$.

NOTE. The matrix Y arose from the theory of quasisymmetric functions in response to a question from Louis Billera and Vic Reiner and was the original motivation for this paper, as we now explain. See for example [7, §7.19] for an introduction to quasisymmetric functions. We will not use quasisymmetric functions elsewhere in this paper.

Let $\mathrm{Comp}(n)$ denote the set of all compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, i.e, $\alpha_i \geq 1$ and $\sum \alpha_i = n$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathrm{Comp}(n)$, and let $\mathfrak{S}_\alpha$ denote the subgroup of $\mathfrak{S}_n$ consisting of all permutations $w = a_1 \cdots a_n$ such that $\{1, \ldots, \alpha_1\} = \{a_1, \ldots, a_{\alpha_1}\}$, $\{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\} = \{a_{\alpha_1+1}, \ldots, a_{\alpha_1+\alpha_2}\}$, etc. Thus $\mathfrak{S}_\alpha \cong \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}$ and $\#\mathfrak{S}_\alpha = \eta(S)$, where $S = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. If $w \in \mathfrak{S}_n$ and $D(w) = \{i_1, \ldots, i_k\}_<$, then define the *descent composition* $\mathrm{co}(w)$ by

$$\mathrm{co}(w) = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k) \in \mathrm{Comp}(n).$$

Let L_α denote the fundamental quasisymmetric function indexed by α [7, (7.89)], and define

$$R_\alpha = \sum_{w \in \mathfrak{S}_\alpha} L_{\mathrm{co}(w)}. \tag{13}$$

Given $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathrm{Comp}(n)$, let $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. Note that $w \in \mathfrak{S}_\alpha$ if and only if $S_\alpha \subseteq C(w)$. Hence equation (13) can be rewritten as

$$R_\alpha = \sum_\beta Y_{\overline{S_\alpha} S_\beta} L_\beta,$$

with $Y_{\overline{S_\alpha} S_\beta}$ as in (5). It follows from (5) that the transition matrix between the bases L_α and R_α is lower unitriangular (with respect to a suitable ordering of the rows and columns). Thus the set $\{R_\alpha : \alpha \in \mathrm{Comp}(n)\}$ is a $\mathbb{Z}$-basis for the additive group of all homogeneous quasisymmetric functions over $\mathbb{Z}$ of degree n. Moreover, the problem of expressing the L_β's as linear combinations of the R_α's is equivalent to inverting the matrix $Y = (Y_{ST})$.

The question of Billera and Reiner mentioned above is the following. Let P be a finite poset, and define the quasisymmetric function

$$K_P = \sum_f x^f,$$

where f ranges over all order-preserving maps $f : P \to \{1, 2, \ldots\}$ and $x^f = \prod_{t \in P} x_{f(t)}$ (see [7, (7.92)]). Billera and Reiner asked whether the quasisymmetric functions K_P generate (as a $\mathbb{Z}$-algebra) or even span (as an additive abelian group) the space of all quasisymmetric functions. Let $\boldsymbol{m}$ denote an m-element antichain. The *ordinal sum* $P \oplus Q$ of two posets P, Q with disjoint elements is the poset on the union of their elements satisfying $s \leq t$ if either (1) $s, t \in P$ and $s \leq t$ in P, (2) $s, t \in Q$ and $s \leq t$ in Q, or (3) $s \in P$ and $t \in Q$. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathrm{Comp}(n)$ then let $P_\alpha = \boldsymbol{\alpha_1} \oplus \cdots \oplus \boldsymbol{\alpha_k}$. It is easy to see that $K_{P_\alpha} = R_\alpha$, so the K_{P_α}'s form a $\mathbb{Z}$-basis for the homogeneous quasisymmetric functions of degree n, thereby answering the question of Billera and Reiner.

2 Multisets and inversions.

In this section we consider two further aspects of the connectivity set: (1) an extension to permutations of a multiset and (2) a q-analogue of Theorem 1.4 when the number of inversions of w is taken into account.

Let $T = \{i_1, \dots, i_k\}_< \subseteq [n-1]$. Define the multiset

$$N_T = \{1^{i_1}, 2^{i_2 - i_1}, \dots, (k+1)^{n-i_k}\}.$$

Let $\mathfrak{S}_{N_T}$ denote the set of all permutations of N_T, so $\#\mathfrak{S}_{N_T} = n!/\eta(T)$; and let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_{N_T}$. In analogy with equation (1) define

$$C(w) = \{i \,:\, a_j < a_k \text{ for all } j \leq i < k\}.$$

(Note that we could have instead required only $a_j \leq a_k$ rather than $a_j < a_k$. We will not consider this alternative definition here.)

Proposition 2.1. *Let $S, T \subseteq [n-1]$. Then*

$$\begin{aligned}
\#\{w \in \mathfrak{S}_{N_T} \,:\, C(w) = S\} &= (XM)_{\overline{S}\,\overline{T}} \\
&= \sum_{U \,:\, U \supseteq \overline{T}} X_{\overline{S}U} \\
&= \#\{w \in \mathfrak{S}_n \,:\, C(w) = S,\ D(w) \supseteq \overline{T}\}.
\end{aligned}$$

Proof. The equality of the three expressions on the right-hand side is clear, so we need only show that

$$\#\{w \in \mathfrak{S}_{N_T} \,:\, C(w) = S\} = \#\{w \in \mathfrak{S}_n \,:\, C(w) = S,\ D(w) \supseteq \overline{T}\}. \tag{14}$$

Let $T = \{i_1, \dots, i_k\}_< \subseteq [n-1]$. Given $w \in \mathfrak{S}_n$ with $C(w) = S$ and $D(w) \supseteq \overline{T}$, in w^{-1} replace $1, 2, \dots, i_1$ with 1's, replace $i_1 + 1, \dots, i_2$ with 2's, etc. It is easy to check that this yields a bijection between the sets appearing on the two sides of (14). $\square$

Let us now consider q-analogues $Z(q)$, $Y(q)$, $X(q)$ of the matrices Z, Y, X. The q-analogue will keep track of the number $\mathrm{inv}(w)$ of inversions of $w = a_1 \cdots a_n \in \mathfrak{S}_n$, where we define

$$\mathrm{inv}(w) = \#\{(i,j) \,:\, i < j,\ a_i > a_j\}.$$

Thus define

$$X(q)_{ST} = \sum_{\substack{w \in \mathfrak{S}_n \\ C(w)=\overline{S},\ D(w)=T}} q^{\mathrm{inv}(w)},$$

and similarly for $Z(q)_{ST}$ and $Y(q)_{ST}$. We will obtain q-analogues of Theorems 1.2 and 1.4 with completely analogous proofs.

Write $\boldsymbol{(j)} = 1 + q + \cdots + q^{j-1}$ and $\boldsymbol{(j)!} = \boldsymbol{(1)(2)\cdots(j)}$, the standard q-analogues of j and $j!$. Let $S = \{i_1, \ldots, i_k\}_< \subseteq [n-1]$, and define

$$\eta(S,q) = \boldsymbol{i_1!\,(i_2 - i_1)! \cdots (i_k - i_{k-1})!\,(n - i_k)!}.$$

Let $T \subseteq [n-1]$, and let $\overline{T} = \{i_1, \ldots, i_k\}_<$. Define

$$z(T) = \binom{i_1}{2} + \binom{i_2 - i_1}{2} + \cdots + \binom{n - i_k}{2}.$$

Note that $z(T)$ is the least number of inversions of a permutation $w \in \mathfrak{S}_n$ with $T \subseteq D(w)$.

Theorem 2.2. *We have*

$$Z(q)_{ST} = \begin{cases} q^{z(T)}\eta(\overline{S},q)/\eta(\overline{T},q), & \text{if } \overline{S} \cap T = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Preserve the notation from the proof of Theorem 1.2. If (s,t) is an inversion of w (i.e., $s < t$ and $a_s > a_t$) then for some $0 \le h \le j$ we have $c_h + 1 \le s < t \le c_{h+1}$. It is a standard fact of enumerative combinatorics (e.g., [5, (21)][6, Prop. 1.3.17]) that if $U = \{u_1, \ldots, u_r\}_< \subseteq [m-1]$ then

$$\begin{aligned} \sum_{\substack{v \in \mathfrak{S}_m \\ D(v) \subseteq U}} q^{\mathrm{inv}(v)} &= \binom{\boldsymbol{m}}{\boldsymbol{u_1, u_2 - u_1, \ldots, m - u_r}} \\ &:= \frac{\boldsymbol{(m)!}}{\boldsymbol{(u_1)!\,(u_2 - u_1)! \cdots (m - u_r)!}}, \end{aligned}$$

a q-multinomial coefficient. From this it follows easily that if $\overline{U} = \{y_1, \ldots, y_s\}_<$ then

$$\sum_{\substack{v \in \mathfrak{S}_m \\ D(v) \supseteq U}} q^{\mathrm{inv}(v)} = q^{z(T)} \binom{\boldsymbol{m}}{\boldsymbol{y_1, y_2 - y_1, \ldots, m - y_s}}.$$

Hence we can parallel the proof of Theorem 1.2, except instead of merely counting the number of choices for the sequence $u = (a_{c_h}, a_{c_h}+1, \ldots, a_{c_{h+1}})$ we can weight this choice by $q^{\mathrm{inv}(u)}$. Then

$$\sum_u q^{\mathrm{inv}(u)} = q^{\binom{i_2-i_1}{2}+\cdots+\binom{i_k-i_{k-1}}{2}} \binom{\boldsymbol{c_{h+1}-c_h}}{\boldsymbol{i_2-i_1, i_3-i_2, \ldots, i_k-i_{k-1}}},$$

summed over all choices $u = (a_{c_h}, a_{c_h}+1, \ldots, a_{c_{h+1}})$. Taking the product over all $0 \le h \le j$ yields $q^{z(T)}\eta(\overline{S}, q)/\eta(\overline{T}, q)$. $\square$

Theorem 2.3. *The matrices $Z(q), Y(q), X(q)$ have the following inverses:*

$$\begin{aligned}
Z(q)^{-1}_{ST} &= (-1)^{\#S+\#T} Z(1/q)_{ST} \\
Y(q)^{-1}_{ST} &= (-1)^{\#S+\#T} \sum_{\substack{w \in \mathfrak{S}_n \\ \overline{S}=C(w),\ T \subseteq D(w)}} q^{-\mathrm{inv}(w)} \\
X(q)^{-1}_{ST} &= (-1)^{\#S+\#T} X(1/q)_{ST}.
\end{aligned}$$

Proof. Let $D(q) = (D(q)_{ST})$ be the diagonal matrix with $D(q)_{SS} = \eta(\overline{S}, q)$. Let $Q(q)$ be the diagonal matrix with $Q(q)_{SS} = q^{z(S)}$. Exactly as for (3), (4) and (6) we obtain

$$\begin{aligned}
Z(q) &= D(q)MD(q)^{-1}Q(q) \\
MX(q)M &= Z(q) \\
Y(q) &= MX(q) = Z(q)M^{-1}.
\end{aligned}$$

The proof now is identical to that of Theorem 1.4. $\square$

Let us note that Proposition 2.1 also has a straightforward q-analogue; we omit the details.

ACKNOWLEDGMENT. I am grateful to the referee for several suggestions that have enhanced the readability of this paper.

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