Dear Chern,

it was a great pleasure to receive your postcard from Nankai written jointly with Michael.

You ask about Catalan numbers. The $n$-th Catalan number $C_n$ is given by

$$C_n = \binom{2n}{n} / (n+1).$$

Thus for $n = 0, 1, 2, 3, \ldots$ we have

$$C_n : 1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

There is the characteristic function

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{2x} \left( 1 - \sqrt{1 - 4x} \right)$$
Let $X_n$ be the manifold of all lines of the complex projective space $P_{n+1}$.

$$\dim X_n = 2n$$

Over $X_n$ we have the tautological $\mathbb{C}^2$-vector bundle obtained by using that $X_n$ equals the Grassmannian of 2-dim. complex linear subspaces of $\mathbb{C}^{n+2}$. Using compact groups

$$X_n = \frac{U(n+2)}{(U(2) \times U(n))}$$

The Chern classes $c_1, c_2$ of the (dual) tautological bundle are according to one of your definitions dual to certain subvarieties of $X_n$ (of complex codimension 1, 2):

$c_1$ : Variety of all lines intersecting a fixed $P_{n-1} \subset P_{n+1}$

$c_2$ : $X_{n-1} \subset X_n$

Schubert (Math. Annalen 1885) already determines $c_1^{2n} [X_n]$. It is the number of lines intersecting all of 2n given subspaces of codimension 2 in $P_{n+1}$ in general position.
We have

\[(2) \quad c_1^{2n} [X_n] = C_n\]

and can determine all Chern numbers

\[(3) \quad c_1 c_2^{2n} [X_n^{2n}] = C_{2n} \quad \text{for} \quad 2r + 2s = 2n.\]

In particular the matrix of intersection (for the signature) is a matrix of Catalan numbers which has determinant 1 and is equivalent over \(\mathbb{Z}\) to the standard diagonal matrix (all 1's in the diagonal).

Of course \((3)\) does not give the Chern numbers of the tangent bundle of \(X_n\). But these, in principal, can be expressed by using \((3)\).

The formulas of A. Borel and myself express the Chern classes of the tangent bundle of \(X_n\) in terms of \(c_1, c_2\). For example, \((n+2)c_1\) is the first Chern class of the tangent bundle of \(X_n\).

We can embed

\[(4) \quad X_n \hookrightarrow \mathbb{P}(n+2) - 1\]

by the Plücker coordinates. Then \(c_1\) is dual to the hyperplane section \(H_0\). By \((2)\) the Catalan number is the degree \(C_n\).
of the embedding \((4)\).


For example consider the variety of all lines in \(X_n\) which intersect a given \(P_{n-2} \subset P_{n+1}\). This variety has codimension 2.

It follows from Schubert that it is dual to

\[ e_1^2 - c_2 \]

Therefore the numbers

\[(5) \quad C_2(n) = (e_1^2 - c_2)^n \left[ X_n \right] \]

are interesting.

They occur in Schubert, \(C_2(n)\) is the number of lines intersecting all of \(n\) given projective subspaces of codimension 3 in \(P_{n+1}\) in general position.

By (2) and (3) and (5)

\[ C_2(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_k \]

For \(n = 0, 1, 2, 3, \ldots\) we have

\[ C_2(n) = 1, 0, 1, 4, 3, 6, 15, 36, 91, 232, 603, \ldots \]

Up to \(n = 9\) these numbers are in Schubert.
I looked into Sloane's impressive list of integral sequences and found the sequence \( C_2(n) \) under number M2587. The given references show that \( C_2(n) \) has several combinatorial interpretations (the Catalan numbers have dozens of combinatorial meanings, see the books by Stanley). I showed \( C_2(n) \) to Don Zagier. He proved immediately that indeed \( C_2(n) \) is M2587, and

\[
(6) \quad \sum_{n=0}^{\infty} C_2(n) x^n = \frac{1}{2x} \left( 1 - \sqrt{1 - \frac{1-3x}{1+x}} \right).
\]

Formula (6) for M2587 occurs in the literature. But I did not find anywhere that M2587 are the Chern numbers

\[(C_1 - C_2)^n \left[ x_n \right].\]

Schubert calculus of lines is very amusing. I showed other things to Don Zagier and he developed a very interesting machinery. I could write many pages. But let me stop here. I wish you the best for your health. Inge just returned from hospital after a knee operation. Both of us send you our best wishes.

Fritz

continued pages 6, 7
To Chern:

Apparently I am unable to stop. First let me mention that the Catalan numbers satisfy

\[ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \]

whereas the \( C_{2n}^{(h)} \) satisfy

\[ C_{2n}^{(h+1)} = \sum_{i=0}^{n} C_{2i}^{(h)} C_{2n-2i}^{(h)} + (-1)^i \]

(Don Zagier).

Secondly let me mention the following fact which is proved using the relation between representation theory (Herman Weyl) and my Riemann-Roch formulas observed by Borel and myself during our Princeton time 1952-54.

Consider the embedding (4) and let \( H \) be a hyperplane section of \( X_n \) dual to \( i \).

The Hilbert polynomial

\[ \chi(X_n, rH) = \dim H^0(X_n, rH) \]

for \( r > -(h+2) \)

("postulation formula") is given by

\[ -(h+2)H \text{ is the canonical divisor of } X_n \]
\[(7) \quad \chi(X_N, r) = \frac{(r+1)(r+2)^2 \cdots (r+n)^2 (r+n+1)}{1 \cdot 2^2 \cdots n^2 \cdot (n+1)}.\]

It is a polynomial of degree \(2n\) which vanishes for \(r = -1, -2, \ldots, -(n+1)\), as it must by the Kodaira vanishing theorem. By Riemann-Roch, the coefficient of \(r^{2n}\) \((2n = \dim C_{X_N})\) equals

\[\frac{H^{2n}[X_N]}{(2n)!} = \frac{1}{(n+1)! \cdot n!} \quad (7)\]

Hence

\[H^{2n}[X_N] = \frac{(2n)!}{(n+1)! \cdot n!} = C_n\]

Hence we obtain (2) by Riemann-Roch.

Once again,

Best wishes,

Fritz

* For \(r = 0\) it has the value 1.