EC2 SUPPLEMENT: PAPERBACK EDITION OF 2001

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version of 7 November 2009

Here I will maintain supplementary material for *Enumerative Combinatorics*, volume 2 (paperback edition of 2001). This will include errata, updated references, and new material. I will be continually updating this supplement.

NOTE. References to math.CO refer to the combinatorics section of the xxx Mathematics Archive at xxx.lanl.gov/archive/math. A front end site for math.CO is front.math.ucdavis.edu/math.CO.

- p. 6, line 10. Change situtations to situations.
- p. 11, line 3. Change $E_c(n)$ to $E_c(x)$.
- p. 18, line 3. Change $(n)_2$ to $n(n-2)$.
- p. 20, line 9. Change $Z(\mathfrak{S}_n)$ to $\tilde{Z}(\mathfrak{S}_n)$.
- p. 24, line 4 (after figure). Change $\lim_{n\to\infty}$ to $\lim_{k\to\infty}$.
- p. 25, line 5. Change $\subseteq$ to $\in$.
- p. 33, line 5–. Change $\mathrm{ord}(\tau_k)$ to $\mathrm{ord}(\tau_j)$.
- p. 34, Lemma 5.3.9. Delete the first sentence, viz., "Let $w \in \mathcal{A}^*$."
- p. 35, line 10. Change $w \in \mathcal{B}^*$ to $w \in \mathcal{B}^*_{\mathbf{r}}$.
- p. 35, line 8–. Change $\mathbb{A}$ to $\mathcal{A}$.
- p. 36, lines 15–16. Change "beginning with a 1" to "ending with a -1".
- p. 36, line 1–. Insert $+\cdots$ before $=$. (The left-hand side is an infinite sum.)

- p. 51, line 9–. Change $Q_i = \Pi_i^{(2)}$ to "when Q_i is given by Example 5.5.2(d) for $r = 2$".
- p. 59, line 9. Change "Since the rows" to "Since the columns".
- p. 59, line 13. Change "Because the columns" to "Because the rows".
- p. 62. Example 5.6.12, line 5. Change "modulo n" to "modulo 2^n".
- p. 63, line 12. Change "sequence" to "sequences".
- p. 65, line 8. Change "Theorem" to "Lemma".
- p. 97, Exercise 5.51. It is not true that (ii) implies (i), e.g., when $C(x) = c$. One needs to add the hypothesis that $[x]C(x) \neq 0$, so that $(C(x) - c)^{\langle -1 \rangle}$ exists. Substituting $xC(B(x))$ for x in (ii) yields

  $$xC(B(x))/C(A(xC(B(x)))) = x,$$

  so $C(B(x)) = C(A(xC(B(x))))$. Substituting $B(x)^{\langle -1 \rangle}$ for x yields $C(x) = C(A(B(x)^{\langle -1 \rangle}C(x)))$. Subtract c from both sides and apply $(C - c)^{\langle -1 \rangle}$ to get $x = A(B(x)^{\langle -1 \rangle}C(x))$. Applying $A^{\langle -1 \rangle}$ to both sides gives (i). This argument is due to Daniel Giaimo and Amit Khetan and (independently) to Yumi Odama.
- p. 101, line 3. Change $J_0[(2-t)/\sqrt{t-1}]$ to $J_0\left(\sqrt{-t}\,(2-t)/(1-t)\right)$.
- p. 102, Exercise 5.71. It would be better not to specify the degree d of G, since (as stated in the solution) $d = \lambda_1$.
- p. 103, Exercise 5.74(d). Replace the first two sentences with: Show that all vertices have the same outdegree d. (By (c), all vertices then also have indegree d.)
- p. 103, Exercise 5.74(f). For further information, see F. Curtis, J. Drew, C.-K. Li and D. Pragel, *J. Combinatorial Theory (A)* **105** (2004), 35–50, and the references given there.
- p. 137, Exercise 5.45, line 1. Change kxy^k to $(k+1)xy^k$.
- p. 137, Exercise 5.45, line 4. Change this equation to

  $$y = x + 2xy + 3xy^2 + \cdots = \frac{x}{(1-y)^2}.$$

- p. 139, Exercise 5.47(c), line 7. A direct combinatorial proof was given by M. Bousquet-Mélou and G. Schaeffer, *Advances in Applied Math.* **24** (2000), 337–368.

- p. 147, Third Solution. The first two lines should be: Equation (5.530) can be rewritten (after substituting $n+k$ for n)

$$(n+k)[x^n]\frac{1}{k}\left(\frac{F^{\langle -1\rangle}(x)}{x}\right)^k = [x^n]\left(\frac{x}{F(x)}\right)^{n+k}. \qquad (5.140)$$

- p. 162, lines 13– to 12–. Change "Thus any algebraic power series, as defined in Definition 6.1.1" to "Thus any algebraic function, i.e, any solution η to (6.2)".

- p. 175, line 1. Change $\{9,11\}$ to $\{9,14\}$.

- p. 175, line 2. Change x^{11} to x^{14}.

- p. 175, line 4. Change v^{11} to v^{14}.

- p. 175, line 11. Change Theorem 5.4.1 to Theorem 5.4.2.

- p. 175, line 2–. Change $k \in K$ to $k \in \mathbb{Z}$.

- p. 176, line 16. Change intesect to intersect.

- p. 176, line 4–. Change $(n+2)$-gon to $(n+1)$-gon.

- p. 192, line 9–. Change $u(0)=0$ to $v(0)=0$.

- p. 192, lines 8– to 7–. The example $v = \log(1+x^2)-1$ is confusing since $v(0) \neq 0$. Nevertheless the series $u(v(x)) = \sqrt{\log(1+x^2)}$ is well-defined formally since we can write

$$\sqrt{\log(1+x^2)} = x\sqrt{\frac{\log(1+x^2)}{x^2}}.$$

  It would have been more accurate to define

$$v(x) = \frac{\log(1+x^2)}{x^2} - 1.$$

  The same remarks apply to Exercise 6.59.

- p. 212, line 1. The statement that Catalan number enumeration originated with Segner and Euler in 1760 (or actually 1758/59 in the cited references) is inaccurate. The enumeration of polygon dissections was stated by Euler in a letter to Goldbach in 1751. This letter is printed in P.-H. Fuss, *Correspondance Mathématique et Physique*, Tome. 1, Acad. Sci. St. Petersburg, 1843; reprinted in The Sources of Science, No. 35, Johnson Reprint Corporation, New York and London, 1968, pp. 549–552.

- p. 212, lines 16–17. I have forgotten the source for the statement that Netto was the first to use the term "Catalan number." Can anyone provide a reference?

- p. 219, Exercise 6.16. A combinatorial proof was first given by R. Sulanke, *Electronic J. Combinatorics* **7**(1), R40, 2000. A sharper result was subsequently proved combinatorially by D. Callan, A uniformly distributed parameter on a class of lattice paths, math.CO/0310461.

- p. 221, Exercise 6.19(j).This problem appeared as Problem A5 on the 2003 William Lowell Putnam Mathematical Competition. Many participants found the following bijection with 6.19(i) (Dyck paths from $(0,0)$ to $(2n,0)$): Let D be a Dyck path from $(0,0)$ to $(2n,0)$. If D has no maximal sequence of $(1,-1)$ steps of even length ending on the x-axis, then just prepend the steps $(1,1)$ and $(1,-1)$ to the beginning of D. Otherwise let R be the rightmost maximal sequence of $(1,-1)$ steps of even length ending on the x-axis. Insert an extra $(1,1)$ step at the beginning of D and a $(1,-1)$ step after R. This gives the desired bijection.

- p. 224, item **ii**, line 5. Change $S(w) = w$ to $S(w) = 12\cdots n$.

- p. 228, item **iii**, line 3. To be precise, the displayed sequences should have the intial and final 1's deleted.

- p. 230, Exercise 6.21(b), line 3. Change 5.3.11 to 5.3.12

- p. 230, Exercise 6.23. Brian Rothbach has pointed out that this problem can be given another stipulation: serieshelpmate in 20. By parity considerations (Black cannot lose an odd number of moves with a knight) the solution is the same as before except Black plays Pa7-a6 instead of Pa7-a5. Thus the number of solutions is $C_{10} = 16796$.

- p. 233, Exercise 6.30, line 3. It would be less ambiguous to change "this exercise" to "that exercise".

- p. 235, Exercise 6.34, line 4. At the end of the line it should be mentioned that the polynomial $g(L_n, q)$ of Exercise 3.71(f) is a further q-analogue of C_n. An additional reference for this polynomial is R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Prop. 8.6).

- p. 236, Exercise 6.34(b,c). While (b) is correct as stated (in the paperback edition of 2001), it would be best to change $q^n c_n(q)$ on line 6 to $c_n(q)$ (as it was in the hardcover edition of 1999) and "nonnegative" on line 8 to "nonpositive". In this way part (c) remains valid. If part (b) is kept as it is, then change $c_n(t; q)$ in line 4 of part (c) to $q^n c_n(t; q)$.

- p. 238, Exercise 6.38(d), line 1. Change (n, n) to $(n, 0)$.

- p. 239, Exercise 6.39(h). A period is missing at the end of the sentence.

- p. 241, Exercise 6.41, line 1. Change $S^2(w) = w$ to $S^2(w) = 12 \cdots n$.

- p. 247, Exercise 6.59. See the item above for p. 192, lines 8– to 7–.

- p. 250, Exercise 6.3, line 3. Replace "$r = s + \frac{1}{2}$ for some $s \in \mathbb{Z}$" with "r cannot be a negative integer".

- p. 250, Exercise 6.3, paragraph 3. The earliest proof that $\sum_{n\geq 0} \binom{2n}{n}^t x^n$ isn't algebraic for any $t \in \mathbb{N}$, $t > 1$, appears in the paper P. Flajolet, *Theoretical Computer Science* **49** (1987), 283–309 (page 294). Flajolet shows that if $\sum a_n x^n$ is algebraic and each $a_n \in \mathbb{Q}$, then a_n satisfies an asymptotic formula

$$a_n = \frac{\beta^n n^s}{\Gamma(s+1)} \sum_{i=0}^{m} C_i \omega_i^n + O(\beta^n n^t),$$

where $s \in \mathbb{Q} - \{-1, -2, -3, \ldots\}$, $t < s$, β is a positive algebraic number, and the C_i and ω_i are algebraic with $|\omega_i| = 1$. A simple application of Stirling's formula shows that if $a_n = \binom{2n}{n}^t$, then a_n does not have this asymptotic form when $t \in \mathbb{N}$, $t > 1$.

- p. 257, Exercise 6.19(k). Update the reference to *J. Integer Seq.* **4** (2001), Article 01.1.3; available electronically at

  `http://www.research.att.com/~njas/sequences/JIS`.

- p. 258, Exercise 6.19(s), line 1. Change a_i to $a_i - 1$.

- p. 260, Exercise 6.19(ee), lines 10–12. The statement that the first published proof of the enumeration of 321-avoiding permutations is due to D. G. Rogers is inaccurate. Knuth provided such a proof in 1973 in the reference given in the first paragraph of the solution. Moreover, a bijective proof was found by D. Rotem, *Inf. Proc. Letters* **4** (1975), 58–61.

- p. 269, Exercise 4.23. Change the rating to [5].

- p. 278, Exercise 6.53, line 3. Change $Q(x) = x - 2$ to $Q(x) = -x - 2$.

- p. 281, Exercise 6.60. An elegant proof based on Gröbner bases was given by Chris Hillar, *Proc. Amer. Math. Soc.* **132** (2004), 2693–2701.

- p. 293, lines 11-13. Replace ", and such that the ... exist.)" with a period. (The deleted condition automatically holds.)

- p. 298, line 10–. Change "if follows" to "it follows".

- p. 301, line 7. Change 1.1.9(b) to 1.9(b).

- pp. 314–315, proof of Proposition 7.10.4. Change λ to λ/μ throughout proof.

- p. 317, line 12–. Change "clearly impossible" to "clear".

- p. 329, line 15–. Change x's to X's.

- p. 336, line 7 (counting the displayed tableau as a single line). Change 7.8.2(b) to 7.8.2(a).

- p. 346, line 3–. Change "forms a border strip" to "forms a border strip or is empty".

- p. 346, line 1–. Change λ^i/λ^{i+1} to λ^{i+1}/λ^i.

- p. 348, line 9. Change $\chi_\lambda(\mu)$ to $\chi^\lambda(\mu)$.
- p. 352, line 2 of proof of Proposition 7.18.1. Change $\sum_\mu z_\lambda^{-1} f(\lambda) p_\mu$ to $\sum_\lambda z_\lambda^{-1} f(\lambda) p_\lambda$.
- p. 354, line 5. Change "a integral" to "an integral".
- p. 355, line 4. Add a period after "nonnegative".
- p. 359, line 6. Change the subscript α_S to $\mathrm{co}(S)$.
- p. 364, line 1. Change $e(D(T))$ to $e(\mathrm{co}(D(T)))$.
- p. 370, line 3 of second proof. Change 1.22(d) to 1.23(d).
- p. 377, line 7; p. 378, line 8; page 378, line 10–. Change $\pi \in B(r,c,t)$ to $\pi \subseteq B(r,c,t)$.
- p. 379, line 5–. Insert π after the first "partition", and change λ^* to π^*.
- p. 379, line 4–. Change "similary" to "similarly".
- p. 383, line 9. Change "$D(w) = T'$ and $D(w^{-1}) = T$" to "$D(w) = D(T')$ and $D(w^{-1}) = D(T)$".
- p. 395, line 10–. Change "Burnside's theorem" to "Burnside's lemma".
- p. 416, line 7–. Change $u_{i_{t+2}}$ to $u_{j_{t+2}}$.
- p. 418, line 7. Change "subsequences" to "subsequence".
- p. 419, line 16. Change "was" to "is".
- p. 421, line 9–. Insert "a" after "such".
- p. 421, lines 8– to 7–. Change "second statement of Theorem A1.1.4" to "first assertion of Theorem A1.1.6".
- p. 422, line 3. Change A1.1.4 to A1.1.6.
- p. 424, line 11. Delete "by".

- p. 426, line "tableaux in (A1.137)" to "tableau defined by (A1.137)".
- p. 439, line 7. Delete comma after 156.
- p. 442, Theorem A2.4, line 9. Change "Hence" to "Moreover,".
- p. 443, line 11. Change

$$\mathrm{char}\,\varphi = (x_1 \cdots x_n)^{-1} = (x_1 \cdots x_n)^{-1} s_{\emptyset}$$

to

$$\mathrm{char}\,\varphi = x_1^{-1} + \cdots + x_n^{-1} = (x_1 \cdots x_n)^{-1} s_{1^{n-1}}$$

- p. 444, line 12. Delete "char".
- p. 444, line 11–. Change "given by (A2.156)" to "generated (as a $\mathbb{C}$-algebra) by (A2.156)".
- p. 447, line 3–. Change $s_1(x_1^{\lambda_i})$ to $s_1(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots)$.
- p. 451, Exercise 7.13(a). This exercise is stated incorrectly. For instance, $K_{777,6654} = 1$, contrary to the statement of the exercise. One way to state the correct result is as follows. Let the parts of λ' be given by as follows. Let the parts of λ' be given by

$$\lambda'_1 = \cdots = \lambda'_{n_1} > \lambda'_{n_1+1} = \cdots = \lambda'_{n_2} > \lambda'_{n_2+1} = \cdots$$
$$> \lambda'_{n_{k-1}+1} = \cdots = \lambda'_{n_k} > 0.$$

Define $\lambda^{(j)} = (\lambda'_{n_{j-1}+1}, \ldots, \lambda'_{n_j})$ (with $n_0 = 0$), so $\lambda^{(j)}$ is a partition of rectangular shape. Let μ be a partition with $|\mu| = |\lambda|$, and let

$$\mu^{(j)} = (\mu_{n_{j-1}+1}, \ldots, \mu_{n_j}).$$

Then $K_{\lambda\mu} = 1$ if and only if $\lambda \geq \mu$ (dominance order) and

(i) $|\lambda^{(j)}| = |\mu^{(j)}|$ and $\lambda^{(j)} \geq \mu^{(j)}$ for all j.

(ii) For all $1 \leq j \leq k$ either $0 \leq \mu'_{n_{j-1}+1} - \lambda'_{n_{j-1}+1} \leq 1$ or $0 \leq \lambda'_{n_j} - \mu'_{n_j} \leq 1$.

- p. 452, line 6. Change "k times" to "n times".

- pp. 452–453, Exercise 7.16(b,e). The formulas for $y_i(n)$ and $u_i(n)$ have been extended to $i \le 6$ by F. Gascon, *Fonctions de Bessel et combinatoire*, Publ. LACIM **28**, Univ. du Québec à Montréal, 2002 (page 75). In particular,

$$y_6(2n) = 6(2n)! \sum_{k=0}^{n} \frac{(10n-13k+8)C_{k+1}}{(n-k+2)!\,(n-k)!\,(k+4)!\,k!},$$

where C_{k+1} denotes a Catalan number.

- p. 459, Exercise 7.30(c), line 4. Change $d-1$ to d.

- p. 460, Exercise 7.37. For further information on expanding a_δ^2 in terms of Schur functions, see

`http://www.phys.uni.torun.pl/~bgw/vanex.html`.

- p. 461, Exercise 7.42, line 2. Change $s_{\tilde{\lambda}}(y)$ to $s_{\widetilde{\lambda'}}(y)$.

- p. 466, line 3–. Change $(\lambda_i-1)!(\lambda'_i-1)!$ to $(\lambda_i-i)!(\lambda'_i-i)!$.

- p. 467, Exercise 7.59. In order for the bijection $\lambda \mapsto (\lambda^0, \lambda^1, \dots, \lambda^{p-1})$ given in the solution to part (e) (page 517) to be correct, it is necessary to define a specific indexing of the terms of C_λ. Namely, index a term a by c_i if $i = i_1 - i_0$, where i_1 is the number of 1's weakly to the left of a, and i_0 is the number of 0's strictly to the right of a (so if $a = 1$, then this contributes to i_1). The sequence becomes $\cdots c_{-2}c_{-1}c_0c_1c_2\cdots$ as before, so it suffices to define the indexing by letting the first 1 be c_{i_0-1}, where i_0 is the number of 0's following this 1.

  Example. If $\lambda = (4,3,3,3,1)$, then $C_\lambda = \cdots 0010110001011\cdots$. The first 1 in this sequence is $c_{1-4} = c_{-3}$. On the other hand, if $\lambda = (3,3,3,2,2,1)$, then $C_\lambda = \cdots 0010100100011\cdots$. Now the first 1 is $c_{1-6} = c_{-5}$.

- p. 468, Exercise 7.59(e), line 3. Change Y^k to Y^p.

- p. 469, Exercise 7.61, line 2. Change "0 or 1" to "0 or ± 1".

- p. 474, Exercise 7.70, line 3. Under the second summation sign insert a space between "in" and $\mathfrak{S}_n$.

- p. 485, line 3–. The asymptotic formula for $a(n)$ should be multiplied by a factor of $1/\sqrt{3\pi}$. The factor $1/\sqrt{\pi}$ was included by Wright and omitted here by mistake. The additional factor $1/\sqrt{3}$ was omitted by Wright, though his proof makes it clear that it should appear. See L. Mutafchiev and E. Kamenov, `math.CO/0601253`.

- p. 491, Exercise 7.9, line 1. Insert ε_λ before $a_{\lambda\mu}e_\lambda$.

- p. 493, Exercise 7.13(a). For another proof, see A. N. Kirillov, *Europ. J. Combinatorics* **21** (2000), 1047–1055, arXiv:hep-th/9304099 (Prop. 2.2).

- p. 494, line 4. Change 169–172 to 175–177.

- p. 494, Figure 7-20. Change the labels R_1h6, R_1h5, and R_2h6 to R_1a6, R_1a5, and R_2a5, respectively.

- p. 496, equation (7.199). Change $(m_i(\lambda)!)^{-1}$ to $[\prod_i (m_i(\lambda)!)^{-1}]$.

- p. 498, Exercise 7.22(h), line 7. Update the Fomin and Greene reference to *Discrete Math.* **193** (1998), 179–200.

- p. 502, Exercise 7.27, first displayed equation. Change $(n)_m$ to $(n)_{n-m}$.

- p. 505, Exercise 7.32(a). Stembridge's more general result appears in "Computational aspects of root systems, Coxeter groups, and Weyl characters," in *Interactions of Combinatorics and Representation Theory*, MSJ Memoirs **11**, Math. Soc. Japan, Tokyo, 2001, pp. 1–38 (Theorem 7.4).

- p. 514, Exercise 7.47(m), lines 1–3. Update the reference to R. Stanley, *Discrete Math.* **193** (1998), 267–286.

- p. 514, Exercise 7.48(b), lines 2–4. Update the reference to R. Simion and R. Stanley, *Discrete Math.* **204** (1999), 369–396.

- p. 514, line 1–, and p. 515, line 1. "*Ibid.*" refers to the reference in the item above, not to the previous reference in the book.

- p. 515, line 3. "the reference" refers to the reference two items above, viz., R. Stanley, *Electron. J. Combinatorics* **3**, R6 (1996), 22 pp.

- p. 515, Exercise 7.49. Update this reference to C. Lenart, *J. Algebraic Combin.* **11** (2000), 69–78.
- p. 516, line 8. Change $(\lambda_i - 1)!(\lambda'_i - 1)!$ to $(\lambda_i - i)!(\lambda'_i - i)!$.
- p. 517, Exercise 7.59(e), line 3. Change Y^k to Y^p.
- p. 517, Exercise 7.59(e), line 9. Change $Y_\emptyset$ to $Y_{p,\emptyset}$, and change Y^k to Y^p.
- p. 518, Exercise 7.59(h), line 1. Change $Y_\emptyset$ to $Y_{p,\emptyset}$, and change Y^k to Y^p.
- p. 518, Exercise 7.59(h), line 2. Change Y^k to Y^p (three times).
- p. 518, Exercise 7.59(h), line 3. Change Y^k to Y^p.
- p. 520, line 3–. Change $\sum_{n\geq 0} h_{2n+1}t^{2n+1}$ to $\sum_{n\geq 0}(-1)^n h_{2n+1}t^{2n+1}$
- p. 535, lines 7–10. Replace the sentence "No proof ... are known." with "A bijective proof of the unimodality of $s_\lambda(1, q, \ldots, q^n)$ was given by A. N. Kirillov, *C. R. Acad. Sci. Paris*, Sér. I **315** (1992), 497–501."
- p. 537, Exercise 7.78(f), line 6. Change $s_\mu(x)$ to $s_\mu(y)$ and $s_\nu(x)$ to $s_\nu(z)$.
- p. 576, line 7. Change work to word.
- p. 584, fifth item. Update the reference to K. S. Kedlaya, *Proc. Amer. Math. Soc.* **129** (2001), 3461–3470.
- p. 584, eleventh item. Update this reference to J. H. Przytycki and A. S. Sikora, *J. Combinatorial Theory(A)* **92** (2000), 68–76, math.CO/9811086.
- p. 584, first item. Add the additional reference P. J. Larcombe and P. D. C. Wilson, *Congr. Numerantium* **149** (2001), 97–108.
- p. 584, third item. Update the reference to M. Haiman, *J. Amer. Math. Soc.* **14** (2001), 941–1006; `math.AG/0010246`.
- p. 585, ninth item. Update the reference to P. Hersh, *J. Combinatorial Theory (A)* **97** (2002), 1–26.