SOLUTIONS TO SUPPLEMENTARY EXERCISES

for Chapter 7 (symmetric functions) of

Enumerative Combinatorics, vol. 2

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NOTATION. The symbol ♣ denotes a request to the reader.

1. Let $\nu \vdash k \leq n$. Let λ (respectively, μ) be the partition obtained from ν by adding a part equal to $n-k$ (respectively, $n-k+1$). This bijection $\nu \mapsto (\lambda, \mu)$ shows that $f(n) = p(0) + p(1) + \cdots + p(n)$. See also EC1, second ed., Exercise 1.71.

2. To get a partition of rank r, take an $r \times r$ square S and two partitions λ, μ with largest part at most r. Attach (the diagram of) λ to the right of S and μ' below S. This construction yields

$$F_r(t) = \frac{t^{r^2}}{(1-t)^2(1-t^2)^2\cdots(1-t^r)^2}.$$

See also EC1, second ed., Prop. 1.8.6(b).

3. Note that $p_1 = e_1$.

4. Note that

$$\sum_{k=2}^{n}(-1)^k 2^k e_k e_1^{n-k} - e_1^n = \sum_{k=0}^{n}(-1)^k 2^k e_k e_1^{n-k}.$$

The proof is now immediate from the observation that

$$F_n(x) = \prod_{i=1}^{n}(e_1 - 2x_i).$$

Use the binomial theorem and the formula $e_2 = \frac{1}{2}(p1^2 - p_2)$ (suggested by Darij Grinberg).

6. By Exercise 7.48(f) we have

$$F_{\mathrm{NC}_{n+1}} = \frac{1}{n+1}[t^n]E(t)^{n+1}.$$

But $E(t) = 1/H(-t)$, so

$$F_{\mathrm{NC}_{n+1}} = \frac{1}{n+1}[t^n]H(-t)^{-n-1}.$$

Now

$$\begin{aligned} H(-t)^{-n-1} &= (1 + (-h_1 t + h_2 t^2 - h_3 t^3 + \cdots))^{-n-1} \\ &= \sum_{k \geq 0} \binom{-n-1}{k} (-h_1 t + h_2 t^2 - h_3 t^3 + \cdots)^k. \end{aligned}$$

It is now straightforward to expand each term by the multinomial theorem, etc.

7. *Answer*: $a \geq 2$. Suppose $a \geq 2$. Then $1 + at + t^2 = (1 + \alpha t)(1 + \beta t)$ where $\alpha, \beta \geq 0$. Now expand $F(x)$ as in the solution to Problem 12 to get e-positivity. Conversely, suppose that $a < 2$. Now α and β are *not* nonnegative real numbers. Since $\arg(\alpha^n) = n \arg(\alpha)$, it is easy to see that for some $n > 0$ we have $\Re(\alpha^n) < 0$, where $\Re$ denotes real part. Then the coefficient of $e_{n+1}e_1$ in $F(x)$ is

$$[e_{n+1}e_1]F(x) = \alpha\beta(\alpha^n + \beta^n) = \alpha^n + \beta^n = 2\Re(\alpha^n) < 0.$$

Note the subtlety of this problem: there is no λ for which the coefficient of e_λ in $F(x)$ is negative for *all* $a < 2$. For the generalization to $\prod P(x_i)$ where $P(t)$ is *any* polynomial satisfying $P(0) = 1$, see Exercise 7.91(e). For the even more general (and considerably more difficult) generalization when $P(t)$ is an arbitrary *power series* satisfying $P(0) = 1$, see the references to Edrei and Thoma in the solution to Exercise 7.91(e).

8. It is not difficult to check that the coefficient of $h_{21^{n-2}}$ in e_λ is $\ell(\lambda) - n$. Since this number is negative unless $\lambda = \langle 1^n \rangle$, it follows that we must have $f = \alpha e_1^n = \alpha h_1^n$ for $\alpha \geq 0$.

9. Open.

10. Since $\omega^2 = 1$, the eigenvalues of ω are ± 1. But if $f \neq 0$ and $\omega f = 2f$, then f is an eigenvector for the eigenvalue 2. Hence $f = 0$.

11. (a) We have $\langle e_\lambda, h_\mu \rangle = M_{\lambda\mu}$ and $\langle h_\lambda, h_\mu \rangle = N_{\lambda\mu}$ (as defined in Propositions 7.4.1 and 7.5.1). Clearly from the definitions we have $M_{\lambda\mu} \leq N_{\lambda\mu}$.

    (b) We want to characterize those λ, μ for which every $\mathbb{N}$-matrix with row sum vector λ and column sum vector μ is a $(0,1)$-matrix. For any $\lambda, \mu \vdash n$ it is easy to find an $\mathbb{N}$-matrix $A = (a_{ij})$ with $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$, and $a_{11} = \min\{\lambda_1, \mu_1\}$. Hence a necessary condition is that either $\lambda = \langle 1^n \rangle$ or $\mu = \langle 1^n \rangle$. It is easy to see that this condition is also sufficient.

12. Let $P(x) = (1 + \alpha x)(1 + \beta x) \cdots$. Then

$$\begin{aligned}
\omega \prod_i P(x_i) &= \omega \prod_i (1 + \alpha x_i)(1 + \beta x_i) \cdots \\
&= \omega \left(\sum_{n \geq 0} \alpha^n e_n \right) \left(\sum_{n \geq 0} \beta^n e_n \right) \cdots \\
&= \left(\sum_{n \geq 0} \alpha^n h_n \right) \left(\sum_{n \geq 0} \beta^n h_n \right) \cdots \\
&= \frac{1}{\prod_i (1 - \alpha x_i)(1 - \beta x_i) \cdots} \\
&= \frac{1}{\prod_i P(-x_i)}.
\end{aligned}$$

One could also take logarithms and expand in terms of power sums, etc.

13. Note that $m_{\langle k^j \rangle} = e_j(x_1^k, x_2^k, \ldots)$. Hence by Proposition 7.7.6,

$$m_{\langle k^j \rangle} = \sum_{\lambda \vdash j} \varepsilon_\lambda z_\lambda^{-1} p_{k\lambda},$$

where $k\lambda = (k\lambda_1, k\lambda_2, \ldots)$.

14. (a) By Corollary 7.7.2 the transition matrix $(R_{\lambda\mu})$ between the m_μ's and p_λ's is upper triangular with respect to any linear extension

of dominance order, with diagonal entries $R_{\mu\mu} = d_\mu$. An easy combinatorial argument shows that $R_{\lambda\mu}$ is divisible by d_μ. We can perform integral elementary row operations on the matrix $(R_{\lambda\mu})$, except for multiplying a row by a scalar, without changing the abelian group generated by the rows. Since d_μ divides $R_{\lambda\mu}$ we can obtain the diagonal matrix (d_μ) by such row operations, and the proof follows.

(b) The matrix X_n is the transition matrix between the s_λ's and p_μ's. Since the set $\{s_\lambda\}_{\lambda\vdash n}$ is a basis for $\Lambda^n_{\mathbb{Z}}$ (see the first sentence after the proof of Corollary 7.10.6), the proof follows from (a) and the definition of Smith normal form.

For some further results along these lines, see C. Bessenrodt, J. B. Olsson, and R. P. Stanley, *J. Algebraic Combinatorics* **21** (2005), 163–177; `arXiv:math/040311`.

15. We have

$$\begin{aligned}
\prod_i (1+x_i)^\alpha &= \exp \alpha \sum_i \log(1+x_i) \\
&= \exp \alpha \sum_i \sum_{n\geq 1} (-1)^{n-1} \frac{x_i^n}{n} \\
&= \exp \alpha \sum_{n\geq 1} (-1)^n \frac{p_n}{n} \\
&= \sum_\lambda \epsilon_\lambda \alpha^{\ell(\lambda)} z_\lambda^{-1} p_\lambda,
\end{aligned}$$

by the exponential formula, permutation version (Cor. 5.1.9).

16. Let $C(x,y) = \prod_{i,j} (1-x_i y_j)^{-1}$. By (7.20), we have

$$C(x,y) = \exp \sum_{n\geq 1} \frac{1}{n} p_n(x) p_n(y),$$

from which it is immediate that

$$\frac{\partial}{\partial p_k(x)} C(x,y) = \frac{p_k(y)}{k} C(x,y).$$

17. For fixed $u \in G$, the number of $v \in G$ satisfying $uv = vu$ is (by definition) $\#C(u)$, where $C(u)$ is the centralizer of u. By elementary group theory, $\#C(u) = \#G/\#K_u$, where K_u is the conjugacy class of G containing u. Hence if $\mathcal{C}(G)$ denotes the set of conjugacy classes of G, then

$$\begin{aligned}
\#\{(u,v) \in G \,:\, uv = vu\} &= \sum_{u \in G} \#C(u) \\
&= \sum_{u \in G} \frac{\#G}{\#K_u} \\
&= (\#G) \sum_{K \in \mathcal{C}(G)} \sum_{u \in K} \frac{1}{\#K} \\
&= (\#G)k(G),
\end{aligned}$$

where $k(G)$ denotes the number of conjugacy classes in G. For $G = \mathfrak{S}_n$ the answer becomes $p(n)n!$.

18. Since $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$, the condition that $z_\lambda \not\equiv 0 \pmod{p}$ is equivalent to λ having no parts divisible by p and no parts with multiplicity at least p. Hence

$$\begin{aligned}
F_p(x) &= \prod_{\substack{k \geq 1 \\ p \nmid k}} (1 + x^k + x^{2k} + \cdots + x^{(p-1)k}) \\
&= \prod_{\substack{k \geq 1 \\ p \nmid k}} \frac{1 - x^{pk}}{1 - x^k}.
\end{aligned}$$

Note that [why?] $f_2(n)$ is the number of self-conjugate partitions of n.

19. (a) This is a result of S. Kakeya, *Japanese J. Math.* **4** (1927), 77–85; `www.jstage.jst.go.jp/article/jjm1924/4/0/4_0_77/_pdf`. Kakeya also conjectured the converse. See also item A007323 in the *Encyclopedia of Integer Sequences*, MathOverflow 310210, and R. Dvornicich and U. Zannier, *Advances in Math.* **222** (2009), 1982–2003.

20. (a) Let $H(t) = \sum h_n t^n = \prod (1 - x_i t)^{-1}$. Then

$$\begin{aligned} T &= \frac{H(1) - H(-1)}{H(1) + H(-1)} \\ &= \frac{\exp\left(\sum \frac{p_n}{n}\right) - \exp\left(\sum (-1)^n \frac{p_n}{n}\right)}{\exp\left(\sum \frac{p_n}{n}\right) + \exp\left(\sum (-1)^n \frac{p_n}{n}\right)} \\ &= \frac{\exp\left(\sum_{n\,\mathrm{odd}} \frac{p_n}{n}\right) - \exp\left(-\sum_{n\,\mathrm{odd}} \frac{p_n}{n}\right)}{\exp\left(\sum_{n\,\mathrm{odd}} \frac{p_n}{n}\right) + \exp\left(-\sum_{n\,\mathrm{odd}} \frac{p_n}{n}\right)} \end{aligned}$$

(b) Let

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \sum_{n \geq 0} (-1)^n E_{2n+1} \frac{x^{2n+1}}{(2n+1)!},$$

where E_{2n+1} is an Euler number (the number of alternating permutations of $1, 2, \ldots, 2n+1$, as discussed e.g. in EC1, second ed., §1.6.). Let

$$y = \sum_{n\,\mathrm{odd}} \frac{p_n}{n}.$$

Then by (a),

$$\begin{aligned} T &= \tanh y \\ &= \sum_{n \geq 0} (-1)^n E_{2n+1} \frac{y^{2n+1}}{(2n+1)!}. \end{aligned}$$

From this it follows easily that if $\lambda = \langle 1^{m_1} 3^{m_3} 5^{m_5} \cdots \rangle$ where $\ell(\lambda) = \sum m_i = 2m+1$, then

$$[p_\lambda] T = \frac{(-1)^m E_{2m+1}}{z_\lambda},$$

and otherwise this coefficient is 0. For the algebraic significance of this problem, see Exercise 7.64.

21. See the answer by Darij Grinberg at MathOverflow #403153. Is there a more conceptual proof?

22. Since the coefficient of h_μ for $\mu = (\mu_1, \ldots, \mu_k)$ (so $\ell(\mu) \leq k$) in $(1 + h_1 + h_2 + \cdots)^k$ is just the number of permutations of the entries of

the vector μ, it follows that the desired scalar product is just the total number of $\mathbb{N}$-matrices with k rows and with column sums $\lambda_1, \lambda_2, \ldots$. The number of sequences $(a_1, \ldots, a_k)$ with sum s is $\binom{s+k-1}{s}$. Hence

$$\langle (1+h_1+h_2+\cdots)^k, h_\lambda \rangle = \binom{\lambda_1+k-1}{\lambda_1}\binom{\lambda_2+k-1}{\lambda_2}\cdots.$$

23. (a) Without loss of generality (since F_p is symmetric), consider the coefficient of the monomial $M = x_1^{a_1}\cdots x_k^{a_k}$, where each $a_i > 0$. If a term $T = \left(\sum_{i\in S} x_i\right)^{p-1}$ contains M, then the coefficient of M is the multinomial coefficient $\binom{p-1}{a_1,\ldots,a_k}$. The number of p-subsets S for which the term T contains M is $\binom{2p-1-k}{p-k}$, the number of $(p-k)$-element subsets of $[2p-1]-[k]$. Now

$$\binom{2p-1-k}{p-k} = \frac{(2p-1-k)!}{(p-k)!\,(p-1)!}.$$

Since $1 \le k \le p-1$, the numerator is divisible by p but not the denominator, and the proof follows.

(b) Let $a_1, \ldots, a_{2p-1} \in \mathbb{Z}$. We have by (a) and elementary properties of congruences that

$$\#\{S \subseteq [2p-1] : \sum_{i\in S} i \equiv 0 \,(\mathrm{mod}\, p) \text{ and } \#S = p\}$$

$$\equiv \sum_{\substack{S\subseteq[2p-1]\\ \#S=p}} \left(1 - \left(\sum_{i\in S} a_i\right)^{p-1}\right) \equiv \binom{2p-1}{p} \equiv 1 \,(\mathrm{mod}\, p),$$

and the proof follows. We have obtained the proofs of (a) and (b) from W. D. Gao, *J. Number Theory* **56** (1996), 211–213, and H. Pan, *Amer. Math. Monthly* **113** (2006), 652–654. The original paper of P. Erdős, A. Ginzburg, and A. Ziv appeared in *Bull. Research Council Israel* **10F** (1961), 41–43.

(c) It is not hard to see that if $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ then

$$(m_1!\, m_2! \cdots) m_\lambda \in \mathbb{Z}[p_1, p_2, \ldots].$$

(See Problem 85(a).) Let $\ell(\lambda) = k$ and $\lambda \vdash p-1$. By the solution to (a) the coefficient C of m_λ in F_p is

$$C = \binom{2p-1-k}{p-k} \frac{(p-1)!}{\prod \lambda_i!}.$$

Now

$$\frac{(p-1)!}{\left(\prod \lambda_i!\right)\left(\prod m_i!\right)}$$

is just the number of partitions of $[p-1]$ with block sizes $\lambda_1, \lambda_2, \ldots$ and is hence an integer. Thus

$$\frac{(p-1)!}{\prod \lambda_i!} m_\lambda \in \mathbb{Z}[p_1, p_2, \ldots].$$

Since $\binom{2p-1-k}{p-k} \equiv 0 \pmod{p}$, the proof follows.

24. (a) By linearity it suffices to do this for a *fixed* basis u_λ. The easiest choice is $u_\lambda = p_\lambda$. If $\lambda \vdash n$, then

$$p_\lambda(kx) = k^{\ell(\lambda)} p_\lambda(x).$$

We could also observe that the stated formula is clear when $j, k \in \mathbb{P}$ and use the fact that a polynomial in one variable over a field of characteristic 0 is determined by its values at the positive integers.

(b) Again it suffices to choose $f(x) = p_\lambda(x)$, for which the computation is trivial. Note in fact that $fg(kx) = f(kx)g(kx)$, so the operation $f(x) \mapsto f(kx)$ is in fact an endomorphism (in fact, an automorphism) of $\Lambda_{\mathbb{Q}}$. Hence it actually suffices to take $f = p_n$.

(c) Since both the operations $f(x) \mapsto f(kx)$ and ω are linear, it suffices to take $f(x) = p_\lambda(x)$ (or even $p_n(x)$ as discussed in (b)). Now for $\lambda \vdash n$ we have

$$p_\lambda(-x) = (-1)^{\ell(\lambda)} p_\lambda(x) = (-1)^n \omega p_\lambda(x).$$

Hence $f(-x) = (-1)^n \omega f(x)$ for all $f \in \Lambda_{\mathbb{Q}}^n$.
Note in particular the "reciprocity" $e_n(-x) = (-1)^n h_n(x)$, an extension of $\binom{-N}{n} = (-1)^n \left(\!\binom{N}{n}\!\right)$.

25. Since $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$, it follows from Proposition 7.5.1 that $\langle h_\lambda, h_\mu\rangle = N_{\lambda\mu}$. Hence we want the number of $\mathbb{N}$-matrices A with row and column sum vector $(2, 1^{n-2}, 0, 0, \dots)$. If $A_{11} = 2$, then subtracting 1 from A_{11} yields an $(n-1) \times (n-1)$ permutation matrix, of which there are $(n-1)!$. If $A_{11} = 0$ there are $\binom{n-2}{2}^2 (n-4)!$ matrices. Adding these two numbers gives $(n^2 - n + 2)(n-2)!/4$.

    Another method: use $h_2 = \frac{1}{2}(p_1^2 + p_2)$. Hence

$$\begin{aligned} \langle h_2 h_1^{n-2}, h_2 h_1^{n-2}\rangle &= \frac{1}{4}\langle p_1^n + p_2 p_1^{n-2}, p_1^n + p_2 p_1^{n-2}\rangle \\ &= \frac{1}{4}(z_{\langle 1^n\rangle} + z_{\langle 2,1^{n-2}\rangle}), \end{aligned}$$

    etc.

26. The linear transformation defined by $e_\lambda \mapsto h_\lambda + e_\lambda$ is just $\omega + I$ (where I is the identity transformation), so we want to compute $\mathrm{rank}(\omega + I)$. Now ω is an involution and hence diagonalizable (its minimal polynomial has distinct roots). The eigenvalues of ω are ± 1. Hence $\mathrm{rank}(\omega + I)$ is the number of eigenvalues of ω equal to 1. By the argument near the end of Section 7.7, this is equal to the number $e(n)$ of even conjugacy classes of $\mathfrak{S}_n$. See Exercise 1.22(b) (together with Proposition 1.8.4) for the generating function of $e(n)$.

27. (a) The matrix of φ^{-1} with respect to the basis $\{m_\lambda\}$ is the matrix M of Section 7.4. By a simple result of linear algebra, M and M^{-1} have the same Jordan block sizes. By Theorem 7.4.4 the matrix M is upper triangular with 1's on the main diagonal. Hence the size of the largest Jordan block of φ is the least integer m for which $(M - I)^m = 0$. Again by Theorem 7.4.4 we have $M_{\lambda\mu} = 0$ unless $\mu \leq \lambda'$. By Proposition 7.4.1 and the Gale-Ryser theorem (a basic combinatorial result which everyone should know) the converse is true, i.e., $M_{\lambda\mu} > 0$ if $\mu \leq \lambda'$. Since all entries of M are nonnegative, there is no cancellation in the computation of powers of $M - I$. It follows that m is the size of the longest chain in the poset $\mathrm{Par}(n)$, ordered by dominance. Now use Exercise 7.2(f). (This exercise gives the *length* of the longest chain, so you need to add 1 to get the size.) For some further work on chains in $\mathrm{Par}(n)$, see E. Early, *Discrete Math.* **313** (2013), 2168–2177.

(b) Define $\mu_1, \mu_2, \ldots$ by setting $\mu_1+\mu_2+\cdots+\mu_i$ equal to the number of elements in the largest union of i chains of $\mathrm{Par}(n)$ (with the dominance order). If the nonzero entries of M above the main diagonal were *generic*, then a theorem of Gansner and Saks (independently) shows that the Jordan block sizes of M are $\mu_1, \mu_2, \ldots$. It seems plausible that M is "sufficiently generic" for the Gansner-Saks result to hold for M itself, but we don't know how to show this. (An interesting consequence: $\mathrm{corank}(M-I)$ $(= p(n) - \mathrm{rank}(M-I))$ is the size of the largest antichain in $\mathrm{Par}(n)$. I don't know whether anyone has looked at the problem of determining this size.) Regardless, is there an explicit formula for the numbers μ_1, $\mu_2, \ldots$?

28. (a) Since ω is an isometry and an involution, we have

$$\langle \omega f, g\rangle = \langle \omega^2 f, \omega g\rangle = \langle f, \omega g\rangle.$$

Hence ω, and thus also $\omega + aI$, is *self-adjoint*.

(b) It is easy to verify by considering the basis $\{p_\lambda\}$ that the adjoint $p_j^\perp$ to M_j is $j\frac{\partial}{\partial p_j}$.

29. We want to count SSYT's of shape (k,k,k) with $k-1$ 1's, $k-1$ 2's, and one $3, 4, \ldots, k+4$. The 1's must go into the first row. If all the 2's are in the second row, then there are $\binom{k+1}{2}$ choices for the remaining numbers. Suppose there is one 2 in the first row and $k-2$ in the second row. The hook-length formula (Corollary 7.21.6) makes it easy to count the possibilities, but we can also give a naive argument. Given an SSYT of shape (k,k,k) and type $(k-1, k-1, 1^{k+2})$ with all the 2's in the second row (as counted above), interchange the last 2 in the second row with the last element of the first row. This will give an SSYT with one exception, when the last column is $(k+2, k+3, k+4)$, and the correspondence is reversible. Hence

$$\begin{aligned} K_{(k,k,k),(k-1,k-1,1^{k+2})} &= \binom{k+1}{2} + \binom{k+1}{2} - 1 \\ &= k^2 + k - 1. \end{aligned}$$

30. (a) Let T be the 180° rotation of a semistandard tableau of shape $\langle k^n\rangle/\lambda$ and content $\langle (k-1)^n\rangle$. (Note that by Exercise 7.56(a),

for any skew θ and 180° rotation θ^r, and for any partition μ, we have $K_{\theta,\mu} = K_{\theta^r,\mu}$.) Let T' be the tableau of shape λ whose ith column consists of the elements of $[n]$ not in column $k+1-i$ of T, arranged in increasing order. For instance, suppose that $k=4$, $n=6$, $\lambda=(4,2)$, and

$$T = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & & \end{array}.$$

Then

$$T' = \begin{array}{cccc} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{array}.$$

It is easy to check that the map $T \mapsto T'$ is a bijection from SSYT of shape $\langle k^n\rangle/\lambda$ and content $\langle (k-1)^n\rangle$ to SYT of shape λ.

(b) We have

$$\begin{aligned} K_{\langle k^n\rangle,\langle (k-1)^n,1^n\rangle} &= \sum_{\substack{\lambda\subseteq\langle k^n\rangle \\ \lambda\vdash n}} K_{\lambda,\langle 1^n\rangle} K_{\langle k^n\rangle/\lambda,\langle (k-1)^n\rangle} \\ &= \sum_{\substack{\lambda\subseteq\langle k^n\rangle \\ \lambda\vdash n}} \left(f^\lambda\right)^2 \\ &= \sum_{\substack{\lambda\vdash n \\ \lambda_1\leq k}} \left(f^\lambda\right)^2. \end{aligned}$$

By Corollary 7.23.12 (using $f^\lambda = f^{\lambda'}$), the last expression is equal to the number of permutations in $\mathfrak{S}_n$ with no increasing subsequence of length $k+1$.

31. See `math.mit.edu/~rstan/papers/sfcong.pdf`. There are a number of open problems and further directions for research in this area that may be interesting to pursue.

32. *Answer:* $\binom{2}{1}\binom{4}{2}\cdots\binom{2(n-1)}{n-1}$

33. Since $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ we have

$$\begin{aligned}\sum_\lambda g(\lambda) s_\lambda &= \sum_\mu h_\mu \\ &= \prod_{n\geq 1} (1-h_n)^{-1}.\end{aligned}$$

See MathOverflow #18597.

34. (a) Let $E(i)$ denote the expected number of i's among all SSYT of shape λ and largest part at most n, for $1 \leq i \leq n$. Then

$$|\lambda| = d = E(1) + E(2) + \cdots + E(n).$$

Since s_λ is a symmetric function we have $E(1) = E(2) = \cdots = E(n)$. Hence $E(i) = d/n$ for all $1 \leq i \leq n$.

(b) First note that

$$B_{\lambda,n}(k) = \frac{\left.\frac{\partial^k s_\lambda(x_1,\ldots,x_n)}{\partial x_1^k}\right|_{x_1=\cdots=x_n=1}}{k!\, s_\lambda(1^n)}.$$

Thus

$$\sum_{k\geq 0} B_{\lambda,n}(k) t^k = \frac{s_\lambda(t+1, 1^{n-1})}{s_\lambda(1^n)},$$

by Taylor's formula (Exercise 1.167 of EC1, second ed.).

Now in equation (27) put $x_1 = t$ and $x_2 = \cdots = x_n = 0$. Since $s_\mu(t,0,0,\ldots) = 0$ unless $\mu = k$ (where we abbreviate the partition (k) as k), we get

$$\begin{aligned}s_\lambda(t+1, 1^{n-1}) &= \sum_k d_{\lambda k} s_k(t,0,0,\ldots) \\ &= \sum_k d_{\lambda k} t^k.\end{aligned}$$

Hence

$$B_{\lambda,n}(k) = \frac{d_{\lambda k}}{s_\lambda(1^n)}.$$

The solution to Problem 87 shows that

$$d_{\lambda k} = \frac{f^{\lambda/k} \prod_{u \in \lambda/k}(n + c(u))}{(d-k)!}.$$

Then $B_{\lambda,n}(k)$ simplifies straightforwardly to the claimed formula using Corollary 7.21.4.

(c) Using (a) and (b), the expected value of $m_1(T)^2$ is given by

$$2B_{\lambda,n}(2) + B_{\lambda,n}(1) = \frac{2d(d-1)}{n(n+1)} \frac{f^{\lambda/(2)}}{f^\lambda} + \frac{d}{n}.$$

(d) By the combinatorial definition of skew Schur functions (Definition 7.10.1) the right-hand side is equal to $B_{\lambda,n}(k)$, so the proof follows from (b).

35. See Y. Baryshnikov and D. Romik, *Israel J. Math.* **178** (2010), 157–186; `arXiv:0709.0498` (Theorem 1). For a simpler approach to counting $f^{\lambda/\mu}$ for some similar problems including α_n, see R. Stanley, *Electronic J. Combinatorics* **18** (2011-2), P16.

36. Follows from Theorem 10.3 of J. S. Kim, K.-H. Lee, and S.-J. Oh, `arXiv:1703.10321`. (There is a typo in Theorem 10.3: $|S^{(3,3)}_{2m-1}|$ should be $|S^{(3,3)}_{2m+1}|$.) (Private communication from Jang Soo Kim, 30 August 2018.)

37. (a) *Hint.* Apply the exponential specialization.

38. See A. Okounkov and G. Olshanski, *Algebra i Analiz* **9** (1997), 73–146 (Russian); English translation, *St. Petersburg Math. J.* **9** (1998), 239–300; `arXiv:q-alg/9605042` (Thm. 8.1 and Thm. 11.1). A q-analogue of this result was obtained by X. Chen (陈小美) and R. Stanley, *Annals of Combinatorics* **20** (2016), 539–548;

`math.mit.edu/~rstan/papers/q-oo.pdf`.

Problem 102 is a special case.

39. (a) Note that (b) is a restatement of (a) using the code C_λ of Exercise 7.59. We are also using part (b) of Exercise 7.59 (in the reverse form of adding rather than removing a border strip). Hence

it suffices to prove (b). Suppose that there are k integers $j < b$ for which $f(j) = 1$. Then by definition of a and b, the only possible values of j for which $f(j) = 1$ and $f(j - n) = 0$ are the k integers $j < b$ for which $f(j) = 0$, together with the n integers $b+1, b+2, \dots, b+n$. The only ones which fail to have the desired property are those of the form $j + n$ where $j < b$ and $f(j) = 1$. Thus the number of integers j for which $f(j) = 1$ and $f(j-n) = 0$ is $(n + k) - k = n$.

40. Let the successive maximum size border strips be $D_1, \dots, D_k$, so D_k is the "innermost" border strip. Let D_k correspond to the composition α^1, i.e., $D_k = B_{\alpha^1}$ in the notation of §7.23 of EC2. Then D_{k-1} can be decomposed into three (nonempty) border strips in a canonical way, where the middle border strip corresponds to α^1. Let the two other border strips correspond to α^2 and α^3. Continuing this construction to D_{k-2}, we decompose it into three border strips with (an isomorphic copy of) D_{k-1} in the middle, and border strips corresponding to α^4 and α^5 at the ends. The middle D_{k-1} decomposes into border strips corresponding to $\alpha^1, \alpha^2, \alpha^3$. At the end of this process, the skew shape is decomposed into k copies of α^1, $k-1$ copies of α^2 and α^3, $k-2$ copies of α^4 and α^5, etc. The process can be reversed given $\alpha^1, \dots, \alpha^{2k-1}$.

Example. The figure below shows a skew shape of depth three. The innermost border strip has squares labelled 1. The next innermost has squares labelled 1* that duplicate 1, and two further pieces that are border strips, labelled 2 and 3. Finally the outermost border strip (the first one removed) duplicates 1 with 1', 2 with 2', 3 with 3', and has 4 and 5 in addition.

							4
				2	4	4	4
			1	2	2’		
		1	1	1*	2’		
		1	1*	1*	1’		
	3	3	1*	1’	1’		
3	3	3’	3’	1’			
5	3’	3’					
5							

We obtain

$$B_k(x) = \sum_{\alpha^1,\ldots,\alpha^{2k-1}} x^{k|\alpha^1|+(k-1)(|\alpha^2|+|\alpha^3|)+(k-2)(|\alpha^4|+|\alpha^5|)+\cdots+(|\alpha^{2k-2}|+|\alpha^{2k-1}|)},$$

where $\alpha^1,\ldots,\alpha^{2k-1}$ are nonempty compositions. It follows easily that

$$B_k(x) = \frac{x^{k^2}}{(1-2x^k)\prod_{i=1}^{k-1}(1-2x^i)^2}.$$

41. (a) See Figure 7-5 (Section 7.15), where $\nu = \lambda$ and $\rho = \emptyset$.

(b) Using (a) (extended to a product of finitely many Schur functions) and the fact that $h_\nu = s_{\nu_1}s_{\nu_2}\cdots$, we have for suitable ρ,σ that

$$K_{\lambda/\mu,\nu} = \langle s_{\lambda/\mu}, h_\nu\rangle = \langle s_\lambda, s_\mu h_\nu\rangle = \langle s_\lambda, s_{\rho/\sigma}\rangle = c^\rho_{\lambda\sigma}.$$

42. (a) For $\lambda \vdash n$ let φ^λ be an irreducible matrix representation of $\mathfrak{S}_n$ with character χ^λ, such that $\varphi^\lambda(w)$ is a matrix of integers for each $w \in \mathfrak{S}_n$. (It is known that such representations exist. A good reference is B. Sagan, *The Symmetric Group*.) For each $\rho \vdash n$ let $\tilde{c}_\rho$ denote the sum (in the group algebra $\mathbb{Z}[\mathfrak{S}_n]$) of all elements in $\mathfrak{S}_n$ of cycle type ρ. Then $\tilde{c}_\rho$ commutes with each $w \in \mathfrak{S}_n$, so by a basic result in representation theory (Schur's lemma) $\varphi(\tilde{c}_\rho)$ is a scalar multiple of the identity matrix, say

$$\varphi(\tilde{c}_\rho) = \omega^\lambda_\rho I_d, \tag{17}$$

where $\omega_\rho^\lambda \in \mathbb{Z}$ and $d = n!/H_\lambda$, the degree of χ^λ. Taking traces in equation (17) yields

$$\frac{n!}{z_\rho}\chi_\rho^\lambda = \frac{n!}{H_\lambda}\omega_\rho^\lambda.$$

Therefore

$$\omega_\rho^\lambda = \frac{H_\lambda}{z_\rho}\chi_\rho^\lambda$$

is an integer for all λ, ρ. The desired result follows easily.

A p-integral formula for $H_\lambda s_\lambda$ was given by Hanlon, *J. Comb. Theory Series A* **47** (1988), 37–70 (Property 3 on page 63).

(b) Immediate from Problem 38, but this is cheating. For an elementary proof, for $a \in \Lambda^n$ let $a^\perp$ denote the adjoint to multiplication by a, i.e., $\langle a^\perp b, c\rangle = \langle b, ac\rangle$ for all $b, c \in \Lambda$. It's easy to check (by verifying it for $a = p_\lambda$, $b = p_\mu$, and $c = p_\nu$, and using linearity) that if we write a as a polynomial $a(p_1, \ldots, p_n)$ in $p_1, \ldots, p_n$, then

$$a^\perp = a\left(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, 3\frac{\partial}{\partial p_3}, \ldots\right).$$

For a partition ρ, define the differential operator

$$D_\rho = \frac{\partial}{\partial p_{\rho_1}}\frac{\partial}{\partial p_{\rho_2}}\cdots.$$

Then

$$\begin{aligned} s_{\lambda/\mu} &= s_\mu^\perp s_\lambda \\ &= \left(\sum_\rho z_\rho^{-1}\chi^\mu(\rho)1^{m_1(\rho)}2^{m_2(\rho)}\cdots D_\rho\right)\frac{1}{H_\lambda}\sum_\sigma d_{\lambda\sigma}p_\sigma, \end{aligned}$$

where $d_{\lambda\sigma} \in \mathbb{Z}$ by (a). Carrying out the differentiation gives

$$s_{\lambda/\mu} =$$

$$\frac{1}{H_\lambda}\sum_{\sigma,\rho} d_{\lambda\sigma} z_\rho^{-1}\chi^\mu(\rho)1^{m_1(\rho)}2^{m_2(\rho)}\cdots(m_1(\sigma))_{m_1(\rho)}(m_2(\sigma))_{m_2(\rho)}\cdots p_{\sigma-\rho}.$$

Now the falling factorial $(m_i(\sigma))_{m_i(\rho)}$ is divisible by $m_i(\rho)!$. Hence

$$z_\rho^{-1}1^{m_1(\rho)}2^{m_2(\rho)}\cdots(m_1(\sigma))_{m_1(\rho)}(m_2(\sigma))_{m_2(\rho)}\cdots \in \mathbb{Z},$$

so we have

$$s_{\lambda/\mu} = \frac{1}{H_\lambda} \sum g_{\lambda\sigma}\, p_{\sigma-\rho}$$

for some integers $g_{\lambda\sigma}$. Now take the coefficient of $x_1 x_2 \cdots x_{n-k}$ on both sides. The coefficient of $x_1 x_2 \cdots x_{n-k}$ in p_ν is 0 unless $\nu = \langle 1^{n-k} \rangle$, in which case the coefficient is $(n-k)!$. Thus for some integer g we get

$$f^{\lambda/\mu} = \frac{1}{H_\lambda} g \cdot (n-k)!,$$

so

$$g = \frac{H_\lambda\, f^{\lambda/\mu}}{(n-k)!} = \frac{(n)_k\, f^{\lambda/\mu}}{f^\lambda},$$

completing the proof.

Is there a simpler proof?

Greta Panova has pointed out that the proof also follows immediately from Naruse's formula (Problem 43) for the number of SYT of skew shape λ/μ. In fact (using notation from Problem 43),

$$\frac{(n)_j f^{\lambda/\mu}}{f^\lambda} = \prod_{E \in (\lambda/\mu)} \prod_{u \in E} h(u).$$

43. This formula is due to Hiroshi Naruse, *73rd Sém. Lothar. Combin.*, 2014; `www.emis.de/journals/SLC/wpapers/s73vortrag/naruse.pdf`. It has led to a lot of subsequent work. In particular, a q-analogue appears in a paper by A. H. Morales, I. Pak, and G. Panova, *J. Combin. Theory, Ser. A* **154** (2018), 26–49, `arXiv:1512.08348`. Morales *et al.* have written (at least) three sequels to this paper.

44. This result was conjectured by P. McNamara (after the $k = 1$ case was conjectured by S. Assaf and then proved independently by S. Assaf and R. Stanley) and proved combinatorially by S. Assaf (March, 2009). See S. Assaf and P. McNamara (with an appendix by T. Lam), *J. Comb. Theory Ser. A* **118** (2011) 277–290; `arXiv:0908.0345`. For a continuation, see T. Lam, A. Lauve, and F. Sottile, *Int. Math. Res. Not. IMRN* **6** (2011), 1205–1219; `arXiv:0908.3714`. For a skew Murnaghan-Nakayama rule and a q-analogue, see M. Konvalinka, *J. Alg. Combinatorics* **35** (2012), 519–545; `arXiv:1101.5250`. For further work in this area, see V. Tewari and S. van Willigenburg, *Advances in Applied Math.* **100** (2018), 101–121.

45. (a) This result is due to J. N. Bernstein and appears in Macdonald [7.96, Exam. I.5.29.b.5]. See M. Zabrocki, *J. Alg. Comb.* **13** (2000), 83–101, `arXiv:math/9904084`, for a host of related results.

(b) If we apply RSK to w, we get precisely those pairs (P, Q) of SYT with at most $n-k$ columns, and for which the first row of Q is $1, 2, \ldots, n-k$. Hence

$$f_k(n) = \sum_{\lambda \vdash k} f^{(n-k,\lambda)} f^\lambda.$$

By (a),

$$f_k(n) = [x_1 \cdots x_n y_1 \cdots y_k] \sum_{i \geq 0} (-1)^i h_{n-k+i}(x) e_i(x)^\perp \sum_{\lambda \vdash k} s_\lambda(x) s_\lambda(y).$$

By Exercise 7.27(c) and the Cauchy identity we get

$$f_k(n) = [x_1 \cdots x_n y_1 \cdots y_k] \sum_{i \geq 0} (-1)^i h_{k+i}(x) e_i(y) \prod (1 - x_i y_j)^{-1},$$

etc. This result is due to A. Garsia and A. Goupil, *Electronic J. Combinatorics* **16**(2) (2009), R19.

(c) See G. Panova, *Discrete Mathematics and Theoretical Computer Science Proceedings (2010)*, `arXiv:0905.2013`.

46. Let $\ell(\mu) = \ell$. The first column of λ consists of 1 and any $n-k$ of the numbers $2, 3, \ldots, \ell$. Hence

$$K_{(k,1^{n-k}),\mu} = \binom{\ell - 1}{n-k}.$$

47. There are various ways to describe P and Q. One elegant description (though perhaps not the best for proving correctness) is to fill in each tableau in the order $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(2,2)$, $(3,1), \ldots$, always using the least number available that keeps the columns strictly increasing, and ignoring any positions that cannot be filled. For instance, if $m = 4$ and $n = 3$, then the numbers that go into P are 111122223333. First let $P_{11} = 1$, then $P_{12} = 1$, then $P_{2,1} = 2$, etc. Note that no number is available for P_{41}, so we continue with $P_{15} = 2$, etc.

48. (a) Call an SYT T *shiftable* if it becomes an SHSYT by pushing the ith row of T $i-1$ squares to the right, for all $i \geq 1$. The key fact is that for each W-equivalence class X, there is a unique shiftable SYT that is the insertion tableau of some $w \in X$. For further details, see D. Worley, Ph.D. thesis, M.I.T., 1984 (Theorem 6.2.2). Worley's result has never been published.

♣ Has someone else published this result?

(b) One possible example of an "interesting way" is to fix $k \geq 1$ and define two permutations $u, v \in \mathfrak{S}_n$ to be W_k-*equivalent* if they have the same insertion tableau and the same set of their first k elements (when written as words). Or perhaps the first k elements of u (in order) should be the reverse or inverse (after standardizing) of the first k elements of v.

49. (a) Given $w_1 w_2 \cdots w_n$, insert $w_1, w_2, \ldots, w_n$ successively into an insertion shifted tableau P as follows. Use ordinary row insertion as long as a diagonal element (i.e., the first element of some row) is not bumped. As soon as a diagonal element is bumped, switch to column insertion. Define a recording shifted tableau Q by inserting $1, 2, \ldots, n$ into Q to keep the same shape as P (just as in ordinary RSK). Moreover, if the new position in P was obtained by a column-insertion (i.e., if sometime in the bumping process a diagonal element was bumped) then circle the corresponding element of Q. This sets up a bijection between $\mathfrak{S}_n$ and pairs (P, Q) of shifted SYT of the same shape $\lambda \models n$, and with some subset of the $n - \ell(\lambda)$ nondiagonal elements of Q circled, thereby proving (2). For instance, if $w = 2651743$ then P is built up as follows:

$$\begin{array}{ccccccc} 2 & 26 & 25 & 125 & 1257 & 1247 & 12367 \\ & & \;\;6 & \;\;6 & \;\;6 & \;\;56 & \;\;45 \end{array}.$$

Moreover, Q is given (with an element underlined instead of circled) by

$$\begin{array}{l} 1\,2\,\underline{4}\,5\,\underline{7} \\ \;\;3\,\underline{6} \end{array}.$$

This bijection is due to D. Worley, Ph.D. thesis, M.I.T, 1984 (§6.1) (in the more general context of semistandard shifted tableaux) and B. Sagan, *J. Combinatorial Theory (A)* **45** (1987), 62–103.

(b) Equation (3) is a formal consequence of the following facts, which are not difficult to verify. Let $\mathcal{V}_n$ be the complex vector space with basis $\{\lambda : \lambda \models n\}$. Define linear transformations $U_n : \mathcal{V}_n \to \mathcal{V}_{n+1}$ and $D_n : \mathcal{V}_n \to \mathcal{V}_{n-1}$ by

$$\begin{aligned} U_n(\lambda) &= \sum_\mu \mu + \zeta \sum_\nu \nu \\ D_n(\lambda) &= \sqrt{2} \sum_\sigma \sigma + \bar{\zeta} \sum_\tau \tau, \end{aligned}$$

where (i) $\lambda \subset \mu \models n+1$ and $\ell(\mu) = \ell(\lambda)$, (ii) $\lambda \subset \nu \models n+1$ and $\ell(\nu) = \ell(\lambda)+1$, (iii) $\lambda \supset \sigma \models n-1$ and $\ell(\sigma) = \ell(\lambda)$, and (iv) $\lambda \supset \tau \models n+1$ and $\ell(\tau) = \ell(\lambda) - 1$. Let I_n denote the identity operator on $\mathcal{V}_n$. Then

$$\begin{aligned} D_{n+1}U_n - U_{n-1}D_n &= I_n \\ D_{n+1}\left(\sum_{\lambda \models n+1} \lambda \right) &= U_{n-1}\left(\sum_{\lambda \models n-1} \lambda \right) + \bar{\zeta} \sum_{\lambda \models n} \lambda. \end{aligned}$$

This result is due to Mark Haiman. For further details, see R. Stanley, in *Invariant Theory and Tableaux* (D. Stanton, ed.), The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145–165 (§3).

This problem just scratches the surface of the theory of shifted shapes. Practically anything that can be done for ordinary shapes has a shifted analogue (including the connections with representation theory). See Problem 50 below for further examples.

50. (a) Analogous to the Bender-Knuth proof that s_λ is a symmetric function.

(b) Clearly we want a "shifted analogue" of RSK. See the thesis of Worley or paper of Sagan (Cor. 8.3) cited in #49 above.

(c) Analogous to the proof of Theorem 7.20.1, using the same method of merging two strict partitions with the same number of parts into a single partition. This result is due to O. Foda and M. Wheeler, BKP plane partitions, *J. High Energy Phys.* (2007), 075, and M. Vuletić, Shifted Schur processes and asymptotics of large random strict plane partitions, *Int. Math. Res. Not.* (2007), rnm043. For a generalization, see M. Vuletić, A generalization of MacMahon's formula, *Trans. Amer. Math. Soc.* **361** (2009), 2789–2804.

51. See `mathoverflow.net/questions/193459`.

52. Given a skew SYT of shape $\lambda/2$, add 2 to each entry and fill in the two "missing" squares with 1,2 from left-to-right. This gives a bijection with SYT of shape λ such that 2 appears in the first row. Thus the first sum is equal to the number of $w \in \mathfrak{S}_n$ such that under RSK the insertion tableau has a 2 in the first row. This will be the case if and only if 2 follows 1 in w (i.e., $w^{-1}(2) > w^{-1}(1)$). There are $n!/2$ such permutations, so
$$\sum_{\lambda \vdash n} f^{\lambda/2} f^{\lambda} = n!/2. \tag{18}$$
By similar reasoning, the second sum is given by the number of $w \in \mathfrak{S}_n$ such that $w^{-1}(2) > w^{-1}(1)$ and $w(2) > w(1)$. If neither $w(1)$ nor $w(2)$ equals 1 or 2, then there are $\frac{1}{2}\binom{n-2}{2}(n-2)!$ such permutations. We can never have $w(2) = 1$. If $w(1) = 1$ then there are $(n-1)!$ such permutations. Hence
$$\begin{aligned} \sum_{\lambda \vdash n} \left(f^{\lambda/2}\right)^2 &= \frac{1}{2}\binom{n-2}{2}(n-2)! + (n-1)! \\ &= \frac{1}{4}(n^2 - n + 2)(n-2)!. \end{aligned} \tag{19}$$
Alternative proof of (19). Given a skew SYT of shape $\lambda/2$, add 1 to each entry and fill in the two missing squares with 1's. This gives a bijection with SSYT of shape λ and type $(2, 1^{n-2})$. The RSK algorithm shows that the number of pairs of such SSYT of the same shape is equal to the number of $\mathbb{N}$-matrices with row and column sum vector $(2, 1^{n-2})$. This was enumerated in the solution to Problem 25. (Equation (18) can also be obtained in this way.)

53. (a) See Theorem 4.3 of A. Reifegerste, `arXiv:math/0309266`. The key lemma for the proof is that any Knuth transformation (Appendix 1, Definition A.1.1.3 of EC2) reverses the sign of the recording tableau.

(b) From (a) we get

$$\begin{aligned} 0 &= \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) \\ &= \sum_{\lambda\vdash n} (-1)^{v(\lambda)} \sum_{(P,Q)} \operatorname{sgn}(P)\cdot\operatorname{sgn}(Q) \\ &= \sum_{\lambda\vdash n} (-1)^{v(\lambda)} I_\lambda(-1)^2. \end{aligned}$$

This result was first proved for n even by R. Stanley, *Advances in Applied Math.* **28** (2002), 282–284 (Theorem 3.2(b)); `arXiv:math/0211113`. The result for any n is due independently to Reifegerste (cited above) and (in more general form) by T. Lam, *J. Comb. Theory Ser. A* **107** (2004), 87–115 (Theorem 23); `arXiv:math/0308265`. See also the footnote to Conjecture 3.3 of the paper of Stanley.

54. (a) By Exercise 7.28(a), if $w \in \mathfrak{S}_n$ is an involution and $w \xrightarrow{\text{rsk}} (P, P)$, then the number of fixed points of w is the number of columns of P of odd length. Hence (after transposing P) we see that $\sum_{\lambda\vdash n} o(\lambda) f^\lambda$ is the total number of fixed points of all involutions in $\mathfrak{S}_n$. Since i is a fixed point in t_{n-1} involutions, the total number of fixed points in all involutions in $\mathfrak{S}_n$ is nt_{n-1}.

(b) By Corollary 7.15.9 we get

$$d(\lambda) = \left\langle s_{\lambda/1}, \sum_{\mu\vdash n-1} s_\mu \right\rangle.$$

Hence

$$\begin{aligned}
\sum_{\lambda\vdash n} d(\lambda) f^\lambda &= \left\langle \sum_{\lambda\vdash n} f^\lambda s_\lambda, \sum_{\lambda\vdash n} \left\langle s_{\lambda/1}, \sum_{\mu\vdash n-1} s_\mu \right\rangle s_\lambda \right\rangle \\
&= \left\langle p_1^n, \sum_{\lambda\vdash n} \left\langle s_\lambda,\ p_1 \sum_{\mu\vdash n-1} s_\mu \right\rangle s_\lambda \right\rangle \\
&= \sum_{\lambda} \left\langle s_\lambda, p_1 \sum_{\mu\vdash n-1} s_\mu \right\rangle f^\lambda \quad (\text{since } \langle p_1^n, s_\lambda\rangle = f^\lambda) \\
&= \left\langle p_1^n,\ p_1 \sum_{\mu\vdash n-1} s_\mu \right\rangle.
\end{aligned}$$

Since the operation $\frac{\partial}{\partial p_1}$ is adjoint to multiplication by p_1 (see the solution to Exercise 7.35(a)), we get

$$\begin{aligned}
\sum_{\lambda\vdash n} d(\lambda) f^\lambda &= \left\langle n p_1^{n-1}, \sum_{\mu\vdash n-1} s_\mu \right\rangle \\
&= n \sum_{\mu\vdash n-1} f^\mu \\
&= n t_{n-1}.
\end{aligned}$$

Both results of this exercise, with different proofs, are due to K. Carde, J. Loubert, A. Potechin, and A. Sanborn, 2008 REU report (Corollary 6.3), available at

`www-users.math.umn.edu/~reiner/REU/HanConjReport.pdf`.

Is it just a coincidence that the sums in (a) and (b) have the same value?

Alternative proof (D. Grinberg, 2021). Let $d_n = \sum_{\lambda\vdash n} d(\lambda) f^\lambda$. Now,

$$t_{n+1} = \sum_{\mu\vdash n+1} f^\mu = \sum_{\lambda\vdash n} (d(\lambda)+1) f^\lambda, \tag{20}$$

since each f^μ equals the sum of the f^λ's over all λ that are covered by μ in Young's lattice, and since each partition λ is covered by exactly

$d(\lambda)+1$ partitions. Equation (20) becomes

$$t_{n+1}=\sum_{\lambda\vdash n}(d(\lambda)+1)f^\lambda=\sum_{\lambda\vdash n}d(\lambda)f^\lambda+\sum_{\lambda\vdash n}f^\lambda=d_n+t_n,$$

so that $d_n = t_{n+1}-t_n = nt_{n-1}$ by the standard recurrence for the numbers t_n.

55. This is a minor reformulation of a result of K. Carde, J. Loubert, A. Potechin, and A. Sanborn, `arXiv:0808:0928` (Theorem 1.1, second version), which in turn is a reformulation of a conjecture of G. Han.

56. $1,2,\ldots,k$ will appear in the first row of $\mathrm{ins}(w)$ if and only if $1,2,\ldots,k$ is a subsequence of w (written as a word) [why?]. The probability that $1,2,\ldots,k$ is a subsequence of $w\in\mathfrak{S}_n$ but $1,2,\ldots,k+1$ isn't a subsequence is given by $k/(k+1)!$ if $k<n$, and by $1/n!$ if $k=n$. Hence

$$E_n=\sum_{k=1}^{n-1}k\cdot\frac{k}{(k+1)!}+\frac{n}{n!}.$$

Thus

$$\begin{aligned}\lim_{n\to\infty}E_n &= \sum_{k\geq 1}\frac{k^2}{(k+1)!}\\ &= \sum_{k\geq 1}\frac{k(k+1)-(k+1)+1}{(k+1)!}\\ &= e-(e-1)+(e-2)\\ &= e-1.\end{aligned}$$

Alternative solution (considerably more elegant). The probability that $1,2,\ldots,i$ appear in the first row of $\mathrm{ins}(w)$ is $1/i!$ for $1\leq i\leq n$. Hence

$$E_n=\sum_{i=1}^{n}\frac{1}{i!},$$

etc.

57. See `mathoverflow.net/questions/82353`. Taedong Yun (private communication, 16 April 2012) computed that the total number of permutations $w\in\mathfrak{S}_n$ for which $P(1,3)=n$ is equal to $\binom{2n-2}{n-3}$, from which it

follows that

$$v_{13} = \sum_{n\geq 1} \frac{1}{(n-1)!}\binom{2n-2}{n-3} = 5.090678729\cdots.$$

This gives an alternative formula to the one in the MathOverflow reference above.

(e) The motivation for this conjecture comes from a paper of A. Berele and A. Regev, *Advances in Math.* **64** (1987), 118–175, with further elucidation at `arXiv:1007.3833`. Consider a standard Young tableau P with $P(i,j) = n$. The positions occupied by $1, 2, \ldots, n-1$ are contained in an $(i-1, j-1)$-hook, using the terminology of the papers just cited. Conversely, when n is large compared to i and j, we expect that "most" SYT T of size $n-1$ contained in an $(i-1, j-1)$-hook will contain the squares $(i-1, j)$ and $(i, j-1)$, so we can extend T by adding the square (i,j) and inserting an n. By the result of Berele and Regev, for fixed i, j the number of $w \in \mathfrak{S}_n$ whose shape under RSK is contained in an $(i-1, j-1)$-hook is roughly $(i+j)^{2n}$. (Their formula actually involves $(i+j-2)^{2n}$, but $(i+j)^{2n}$ is a sufficiently accurate estimate for our purposes.) Thus the expected value of $P(i,j)$ is roughly $\sum_n n\frac{(i+j)^{2n}}{n!}$, which is roughly $e^{(i+j)^2}$. Can this argument (or a better similar argument if the present one is flawed) be made precise?

58. See the answer by R. Stanley at MathOverflow,

`mathoverflow.net/questions/331579`.

59. See D. Romik, The number of steps in the Robinson-Schensted algorithm, *Functional Analysis and Its Applications* **39** (2005), 152–155;

`www.stat.berkeley.edu/~romik/paperfiles/schensted.pdf`.

The proof follows fairly straightforwardly from the result of Logan-Shepp and Kerov-Vershik on the "typical" shape of a SYT mentioned in the solution to Exercise 7.109. For further information, see R. Stanley, in *Proc. Internat. Cong. Math. (Madrid, 2006)*, vol. 1, European Mathematical Society, Zurich, 2007, pp. 545–579. For the more subtle problem of the limiting shape of bumping paths, see D. Romik and P. Śniady, *Random Struct. Algorithms* **48** (2016), 171–182; `arXiv:1304.7589`.

60. This result was proved for square shapes (but the proof is exactly the same for rectangles) by D. Romik, *Advances in Applied Math.* **37** (2006), 501–510;

    `www.stat.berkeley.edu/~romik/paperfiles/extremal.pdf`.

    Perhaps the simplest proof is to consider the inverse map $(P, Q) \mapsto w$.

61. Put $x_1 = \cdots = x_i = t$ and $y_1 = \cdots = y_j = 1$ and all other $x_r, y_s = 0$ in the Cauchy identity (Theorem 7.12.1). We get

$$\begin{aligned} f_n(i,j) &= [t^n](1-t)^{-ij} \\ &= \binom{n+ij-1}{ij-1}. \end{aligned}$$

62. By setting $x_i = y_i$ in (7.44) and comparing with (7.20) we get

$$y_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda^2.$$

Write (λ, λ) for the partition $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots)$. By the orthogonality of the power sums (equation (7.34)) we get

$$\begin{aligned} \langle y_n, y_n \rangle &= \sum_{\lambda \vdash n} z_\lambda^{-2} z_{(\lambda,\lambda)} \\ &= \sum_{\lambda = (1^{m_1} 2^{m_2} \cdots) \vdash n} \frac{1^{2m_1} 2^{2m_2} \cdots (2m_1)!\,(2m_2)! \cdots}{1^{2m_1} 2^{2m_2} \cdots (m_1)!^2 (m_2)!^2 \cdots} \\ &= \sum_{\lambda = (1^{m_1} 2^{m_2} \cdots) \vdash n} \binom{2m_i}{m_i}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n \geq 0} \langle y_n, y_n \rangle x^n &= \left(\sum_{m_1 \geq 0} \binom{2m_1}{m_1} t^{m_1} \right) \left(\sum_{m_2 \geq 0} \binom{2m_2}{m_2} t^{2m_2} \right) \cdots \\ &= \frac{1}{\sqrt{1-4x}} \frac{1}{\sqrt{1-4x^2}} \frac{1}{\sqrt{1-4x^3}} \cdots \\ &= P(x,4)^{1/2}. \end{aligned}$$

63. See J. F. Willenbring, *J. Algebra* **242** (2001), 691–708. Willenbring's proof is based on the representation theory of the orthogonal group. The formula given here is simpler than Willenbring's (though of course equal to it). It can be obtained analogously to Problem 62 (though more complicated). To begin, set each $x_i = y_i$ in (7.44) and use (7.20) to get the p-expansion of $\sum s_\mu^2$. For the p-expansion of $\sum s_{2\lambda}$ use Exercise 7.28. The equality of Willenbring's formula with the one of this exercise can be proved by expanding their logarithms.

64. The symmetric function $G \in \hat{\Lambda}_{\mathbb{R}}$ has the desired property if and only if it has the form $G = \alpha F(x_1)F(x_2)\cdots$, where $\alpha \in \mathbb{R}$, $F(x) \in \mathbb{R}[[x]]$ and $F(0) = 1$. Equivalently (as is easy to see), $\log G(x)$ has the form $c_0 + \sum_{k\geq 1} c_k p_k$, where $c_k \in \mathbb{R}$. (See Exercise 7.91 for more on such series.)

    Proof. We may assume [why?] that G has constant term 1. Now assume that $\log G = \sum_{k\geq 1} c_k p_k$. Then, as in Proposition 7.74, we have

$$G = \sum_\lambda z_\lambda^{-1} \left(\prod_i c_{\lambda_i} \right) p_\lambda.$$

Let $f = \sum_\mu a_\mu p_\mu$ and $g = \sum_\nu b_\nu p_\nu$. For any partition ρ let $c_\rho = \prod_i c_{\rho_i}$. Then by the orthogonality of the power sums,

$$\begin{aligned} \langle G, fg \rangle &= \left\langle \sum_\lambda z_\lambda^{-1} c_\lambda p_\lambda, \sum_{\mu,\nu} a_\mu b_\nu p_\mu p_\nu \right\rangle \\ &= \sum_{\mu,\nu} a_\mu b_\nu c_\mu c_\nu \\ &= \left(\sum_\mu a_\mu c_\mu \right) \left(\sum_\mu b_\mu c_\mu \right) \\ &= \langle G, f \rangle \cdot \langle G, g \rangle. \end{aligned}$$

Conversely, let $I = \sum_\lambda z_\lambda^{-1} d_\lambda p_\lambda$, and suppose that $\langle I, fg \rangle = \langle I, f \rangle \cdot \langle I, g \rangle$ for all f, g. In particular,

$$d_{\mu\cup\nu} = \langle I, p_\mu p_\nu \rangle = d_\mu d_\nu,$$

so by iteration $d_\lambda = \prod d_{\lambda_i}$, completing the proof.

65. Since Schur functions $s_\lambda(x_1, \ldots, x_n)$ are the polynomial characters of $\mathrm{GL}(n, \mathbb{C})$ (Appendix 2), it follows by Weyl's "unitary trick" that they are the irreducible characters of the unitary group $U(n)$. Hence

$$\begin{aligned}\langle V_n^{2k}, V_n^{2k}\rangle_n &= \int_{u\in U(n)} V_n^{2k}\overline{V_n^{2k}}\,du \\ &= \int_{u\in U(n)} |V_n|^{4k}\,du.\end{aligned}$$

The Weyl integration formula allows us to express this integral as an integral over a torus. By the residue theorem, it is equal to

$$\frac{1}{n!}\,\mathrm{CT}\prod_{\substack{i,j=1,\ldots,n\\ i\neq j}}(1-x_ix_j^{-1})^{2k+1},$$

where CT denotes constant term. The value of this constant term was conjectured (as a special case of a more general conjecture) by F. J. Dyson to be $(kn)!/(2k+1)!^n$ and given an elegant proof (after an earlier more complicated proof by Gunson and Wilson) by I. J. Good, *J. Math. Phys.* **11** (1970), 1884. See I. G. Macdonald, *SIAM J. Math. Anal.* **13** (1982), 988–1007. For much more on this subject, see the survey by P. J. Forrester and S. Ole Warnaar, *Bull. Amer. Math. Soc.* **45** (2008), 489–534.

66. It is clear from Corollary 7.13.8 that

$$\sum_{\lambda\vdash 2n} s_\lambda = \sum_{\mu\vdash 2n} L_\mu m_\mu,$$

where L_μ is the number of *symmetric* $\mathbb{N}$-matrices $(A_{ij})_{i,j\geq 1}$ with $\mathrm{row}(A) = \mathrm{col}(A) = \mu$. Since $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$, it follows that a_n is the number of $n\times n$ symmetric $\mathbb{N}$-matrices with every row and column sum equal to two. Hence by Example 5.2.7 we get

$$F(t) = \frac{e^{\frac{t^2}{4}+\frac{t}{2(1-t)}}}{\sqrt{1-t}}.$$

67. (a) It follows from Exercise 7.35(a) or from Problem 28(b) that $\frac{\partial}{\partial p_1}s_\lambda = s_{\lambda/1}$. Hence from the case $n=1$ of Theorem 7.15.7 and Corollary 7.15.19, we have

$$T_n = (U+D+DU)^n 1, \tag{21}$$

where $U = p_1$ (i.e., the operator given by multiplication by p_1) and $D = \frac{\partial}{\partial p_1}$. (See Exercise 7.24 for similar reasoning.) From this it is clear that $T_n = g_n(p_1)$, for some polynomial $g_n \in \mathbb{Z}[t]$.
Now let $y = \sum_{n\geq 0} g_n(t)\frac{x^n}{n!}$. Then from (21) we have

$$(t + \frac{\partial}{\partial t} + \frac{\partial}{\partial t}t)y = \frac{\partial y}{\partial x}.$$

a partial differential equation with the initial condition $y(0) = 1$. There are various methods for solving this type of equation. For instance, let $y = e^z$. Then

$$t + z_t + 1 + tz_t = z_x. \tag{22}$$

The general solution to the associated homogeneous equation $u_t + tu_t = u_x$ is $u(x,t) = F(\log(1+t) + x)$ for any (smooth) F. A particular solution is seen by inspection (or by plugging in a linear polynomial with indeterminate coefficients) to be $-t$. Hence the general solution to (22) is given by

$$z(x,t) = -t + F(\log(1+t) + x).$$

The initial condition $z(0,t) = 0$ shows that $F(\log(1+t) + x) = -1 + \exp(\log(1+t) + x)$. Hence

$$y = e^{-1-t+(1+t)e^x}. \tag{23}$$

Thus

$$\begin{aligned}\sum_{n\geq 0} f(n)\frac{x^n}{n!} &= y|_{t=0} \\ &= e^{e^x-1} \\ &= \sum_{n\geq 0} B(n)\frac{x^n}{n!},\end{aligned}$$

where $B(n)$ denotes a *Bell number* (the number of partitions of an n-set; see equation (1.94f) of EC1, second ed.). So we finally get $f(n) = B(n)$. An elegant bijective proof based on growth diagrams was given by M. Korn (unpublished), and was rediscovered (in a more general context) by C. Krattenthaler, *Adv. Appl.*

Math. **37** (2006), 404–431; `arXiv.math/0510676`. For another bijective proof, see Section 4 of William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Du, R. Stanley, and Catherine H. Yan, *Trans. Amer. Math. Soc.* **359** (2007), 1555–1575; `arXiv.math/0501230`.

(b) We have by (23) that

$$\sum_{m,n\geq 0} \langle T_m, T_n\rangle \frac{x^m}{m!}\frac{y^n}{n!} = \left\langle e^{-1-p_1+(1+p_1)e^x}, e^{-1-p_1+(1+p_1)e^y}\right\rangle$$

$$\begin{aligned}
&= \exp(e^x+e^y-2)\ \langle \exp(p_1(e^x-1), \exp(p_1(e^y-1)\rangle \\
&= \exp(e^x+e^y-2)\ \left\langle \sum_{m\geq 0} p_1^m \frac{(e^x-1)^m}{m!}, \sum_{n\geq 0} p_1^n \frac{(e^y-1)^n}{n!}\right\rangle \\
&= \exp(e^x+e^y-2)\ \sum_{n\geq 0}(e^x-1)^n(e^y-1)^n \frac{1}{n!} \\
&= \exp\left(e^x+e^y-2+(e^x-1)(e^y-1)\right) \\
&= \exp(e^{x+y}-1) \\
&= \sum_{k\geq 0} B(k)\frac{(x+y)^k}{k!} \\
&= \sum_{m,n\geq 0} B(m+n)\frac{x^m}{m!}\frac{y^n}{n!},
\end{aligned}$$

whence $\langle T_m, T_n\rangle = B(m+n)$.

The following much slicker argument was given by T. Lam. The result of part (a) may be restated as

$$\langle T_n, 1\rangle = \langle (U+D+DU)^n 1, 1\rangle = B(n).$$

But U and D are adjoint with respect to $\langle , \rangle$, so $U+D+DU$ is self-adjoint. Hence

$$\begin{aligned}
\langle T_m, T_n\rangle &= \langle (U+D+DU)^m 1, (U+D+DU)^n 1\rangle \\
&= \langle (U+D+DU)^{m+n} 1, 1\rangle \\
&= B(m+n).
\end{aligned}$$

Solutions to (a) and (b) can also be given using induction, working directly with the operators U and D and avoiding generating functions.

This problem is not as contrived as it may first appear. There is a semisimple algebra $\mathcal{P}_n$, called the *partition algebra*, whose irreducible representations σ^λ are indexed by (certain) partitions λ, such that $\dim \sigma^\lambda = \langle T_n, s_\lambda \rangle$. See P. Martin, *J. Knot Theory Ramifications* **3** (1994), 51–82; P. Martin, *J. Algebra* **183** (1996), 319–358; P. Martin and G. Rollet, *Compositio Math.* **112** (1998), 237–254; P. Martin and D. Woodcock, *J. Algebra* **203** (1998), 91–124; C. Xi, *Compositio Math.* **119** (1999), 99–109; W. Doran and D. Wales, *J. Algebra* **231** (2000), 265–330; P. Martin, *J. Phys. A* **33** (2000), 3669–3695; T. Halverson, *J. Algebra* **238** (2001), 502–533.

68. (a) It is easy to see that

$$\left(g(p) + h(p)\frac{\partial}{\partial p} \right) F(x,p) = \frac{\partial}{\partial x} F(x,p),$$

with the initial condition $F(0,p) = 1$. It is then routine to check that the stated formula for $F(x,p)$ does satisfy this differential equation and initial condition.

(b) We have $L(t) = L^{\langle -1 \rangle}(t) = t$ and $M(t) = \frac{1}{2}t^2$. Hence by (a),

$$\begin{aligned} F(x,p) &= \exp\left[-\frac{1}{2}p^2 + \frac{1}{2}(x+p)^2 \right] \\ &= \exp\left(px + \frac{1}{2}x^2 \right). \end{aligned}$$

(c) We have [why?]

$$\sum_{n \geq 0} \sum_{\lambda \in \mathrm{Par}} g_\lambda(n) s_\lambda \frac{x^n}{n!} = F(x,p),$$

where $F(x,p)$ corresponds to $g(t) = 1+t$ and $h(t) = 1+t$. Hence $L(t) = \log(1+t)$, $L^{\langle -1 \rangle}(t) = e^t - 1$, and $M(t) = t$, so

$$\begin{aligned} L^{\langle -1 \rangle}(x + L(p)) &= (1+t)e^x - 1 \\ F(x,p) &= \exp\left[-p + e^{x + \log(1+t)} - 1 \right] \\ &= \exp(-1 - p + (1+p)e^x). \end{aligned}$$

(d) We have $g(t) = h(t) = 1/(1-t)$, $L(t) = t - \frac{1}{2}t^2$, $L^{\langle -1\rangle}(t) = 1-\sqrt{1-2t}$, and $M(t) = t$. We get

$$\begin{aligned}\sum_{n\geq 0} g_{\emptyset}(n)s_\lambda \frac{x^n}{n!} &= F(x,p)\\ &= \exp\left(-p+1-\sqrt{1-2\left(x+p-\frac{1}{2}p^2\right)}\right)\\ &= \exp\left(1-p-\sqrt{(1-p)^2-2x}\right).\end{aligned}$$

(e) We have $g(t) = 1/(1-t)$, $h(t) = 1$, $L(t) = L^{\langle -1\rangle}(t) = t$, and $M(t) = -\log(1-t)$. Hence

$$\begin{aligned}F(x,p) &= \exp\left(\log(1-p) - \log(1-x-p)\right)\\ &= \frac{1}{1-\frac{x}{1-p}}\\ &= \sum_{n\geq 0} \frac{x^n}{(1-p)^n},\end{aligned}$$

etc. Is there a nice bijective proof?

69. One first shows that $\mathrm{ins}(w)$ can be obtained by row inserting a_{n+1}, then *column* inserting a_n, then row inserting a_{n+2}, then column inserting a_{n-1}, etc., ending with a row insertion of a_{2n} followed by a column insertion of a_1. It can be shown that for each $1 \leq i \leq n$ the shape increases by the addition of one domino after inserting a_{n+i} and a_{n+1-i}, and the proof follows. Alternatively, by Theorem A.1.2.10 we have $P = \mathrm{evac}(P)$. Now consider the growth diagram for computing $\mathrm{evac}(P)$ (Figure A1-13), or see Section 3 of R. Stanley, Promotion and evacuation, *Electronic J. Comb.* **15(2)** (2008–2009). For a good overview of this topic, see M. Shimozono and D. E. White, *Electronic J. Combinatorics* **8**(1) (2001), R21.

70. If $d \nmid n$ and $\lambda \vdash n$, then some λ_j is not divisible by d. Hence

$$p_{\lambda_j}(1,\zeta,\ldots,\zeta^{d-1}) = \frac{1-\zeta^{d\lambda_j}}{1-\zeta^{\lambda_j}} = 0.$$

Thus $p_\lambda(1, \zeta, \ldots, \zeta^{d-1}) = 0$, and the proof follows (since the p_λ's for $\lambda \vdash n$ are a basis for $\Lambda^n_{\mathbb{Q}}$).

For stronger results about $s_\lambda(1, \zeta, \ldots, \zeta^{d-1})$, note that

$$\sum_\lambda s_\lambda(1, \zeta, \ldots, \zeta^{d-1}) s_\lambda(x) = \frac{1}{\prod_k (1 - x_k^d)} = \sum_{n \geq 0} h_n(x_1^d, x_2^d, \ldots)$$

and see Exercise 7.61.

71. See I. P. Goulden, *Canad. J. Math.* **42** (1990), 763–775 (Theorem 2.3(a)).

72. We have

$$\sum_{u \in \lambda/\mu} h_{\lambda/\mu}(u) = \sum_{u \in \lambda/\mu} \sum_{v \in H(u)} 1. \tag{24}$$

Given $u, v \in \lambda/\mu$, it is easy to see that

$$v \in H_{\lambda/\mu}(u) \Longleftrightarrow u \in H_{(\lambda/\mu)^r}(v).$$

Hence reversing the order of summation in (24) gives $\sum_{u \in (\lambda/\mu)^r} h_{(\lambda/\mu)^r}(u)$, as desired.

73. (a) This identity was conjectured by T. Amdeberhan on MathOverflow 312771 on October 13, 2018, with solutions by G. Zaimi and S. Hopkins.

74. (a) Let

$$f(n) = \sum_{\lambda \vdash n} \eta_k(\lambda).$$

It is an easy consequence of Exercise 7.59(a,b) that the number of ways to add a border strip of size k to λ is k more than the number of border strips (or hooks) of λ of size k. (In fact, this is a direct consequence of Exercise 7.59(e) and the observation that in Y^k, an element λ is covered by k more elements than it covers. See Exercise 3.51(c) of EC1, second ed.) It follows that $f(n+k) = kp(n) + f(n)$, where $p(n)$ denotes the number of partitions of n. Let $F(x) = \sum_{n \geq 0} f(n)x^n$ and $P(x) = \sum_{n \geq 0} p(n)x^n$. We get

$x^{-k}F(x) = kP(x) + F(x)$, whence $F(x) = kx^kP(x)/(1-x^k)$. It is easy to see that

$$\sum_{n\geq 0}\left(\sum_{\lambda\vdash n} m_k(\lambda)\right)x^k = \frac{x^kP(x)}{1-x^k},$$

and the proof follows. This result is due to R. Bacher and L. Manivel, *Sém. Lotharingien de Combinatoire* **47** (2002), B47d; `arXiv.math/0108199`.

(b) This is a result of C. Bessenrodt and G. Han, *Discrete Math.* **309** (2009), 6070–6073. The proof follows easily from the following stronger result, which is proved by a combinatorial argument. Given $u = (i,j) \in \lambda$, let $a(u) = \lambda_i - i$ (the *arm length* of u) and $l(u) = \lambda'_j - j$ (the *leg length* of u), so $h(u) = a(u) + l(u) + 1$. Let $f_n(a,l,r)$ denote the number of ordered pairs (λ, u) such that $\lambda \vdash n$, $u \in \lambda$, $a(u) = a$, $l(u) = l$, $r(u) = r$. Then

$$\sum_{n\geq 0} f_n(a,l,r)q^n = \frac{1}{\prod_{i\geq a+1}(1-q^i)}\binom{\boldsymbol{l+a}}{\boldsymbol{a}}\binom{\boldsymbol{r+a}}{\boldsymbol{a}}q^{(m+1)(l+1)+1}.$$

(c) Reformulation of (a).

75. See Lemma 3.1 of K. Liu, C. Yan, and J. Zhou, *Sci. China Ser. A* **45** (2002), 420–431. A combinatorial proof was later given by G. Warrington, *J. Combinatorial Theory Ser. A* **116** (2009), 379–403.

76. Easy.

77. By Theorem 7.15.1 we have

$$\begin{aligned} s_\lambda(x_1^2,\dots,x_n^2) &= \frac{a_{2(\lambda+\delta)}}{a_{2\delta}} \\ &= \frac{a_{(2\lambda+\delta)+\delta}}{a_\delta}\cdot\frac{a_\delta}{a_{2\delta}} \\ &= \frac{s_{2\lambda+\delta}(x_1,\dots,x_n)}{s_\delta(x_1,\dots,x_n)}, \end{aligned}$$

whence

$$s_\delta(x_1,\dots,x_n)s_\lambda(x_1^2,\dots,x_n^2) = s_{2\lambda+\delta}(x_1,\dots,x_n).$$

For some related results, see Exercise 7.30.

78. By the solution to Exercise 7.69(a) we have

$$\left(\sum s_\lambda\right)^t = \exp\left(t\left(\sum_{\substack{n\geq 1\\ n\ \text{odd}}} \frac{1}{n}p_n + \sum_{\substack{n\geq 1\\ n\ \text{even}}} \frac{1}{n}p_{n/2}^2\right)\right).$$

We can now apply the exponential formula as in the solution to Exercise 7.69(a).

79. Let $N = 2n$ and $\lambda = \delta_N = (N, N-1, \dots, 1)$. Thus in Exercise 7.40, $r = n$ and $B(i,j) = B_{(2^{N+1-i-j},1)}$, the "zigzag" border strip corresponding to the composition $(2^{N+1-i-j}, 1)$. The number $f^{B(i,j)}$ of SYT of (skew) shape $B(i,j)$ is $E_{|B(i,j)|} = E_{2N+3-2i-2j}$ (see Exercise 7.64(a)). Take the formula of Exercise 7.40, reflect the matrix through the antidiagonal (which doesn't effect the determinant) and apply ex (the exponential specialization) to obtain

$$A_n = \frac{f^{\delta_N}}{|\delta_N|!}.$$

Thus by the hook-length formula (Corollary 7.21.6) we get

$$A_n = \frac{1}{\prod_{u\in\delta_N} h(u)} = \frac{1}{1^N\, 3^{N-1}\, 5^{N-2} \cdots (2N-1)^1}.$$

Exactly analogous reasoning for for $N = 2n-1$ yields that

$$B_n = \frac{1}{1^N\, 3^{N-1}\, 5^{N-2} \cdots (2N-1)^1}.$$

80. Let $\tau = \tau_n$ be the "zigzag shape" of Exercise 7.64. By Corollary 7.23.8 we have $f(n) = \langle \tau, \tau\rangle$. Let $\chi = \mathrm{ch}(\tau)$. Since ch is an isometry (Prop. 7.18.1) we have $f(n) = \langle \chi, \chi\rangle$. By Exercise 7.64 we get for n odd that

$$\begin{aligned} \langle \chi, \chi\rangle &= \sum_{\mu\vdash n} z_\mu^{-1}\chi(\mu)^2 \\ &= \sum_{\mu} z_\mu^{-1} E_{2r}^2, \end{aligned}$$

where μ ranges over all partitions of n with $2r+1$ odd parts and no even parts. The proof now follows from a routine application of Corollary 5.1.8. The proof for even n is similar, using a simple modification of Corollary 5.1.8.

This result is due to R. Stanley, *J. Combinatorial Theory Ser. A* **114** (2007), 436–460 (Theorem 3.1); `arXiv:math/0603520`.

81. See R. Stanley, *J. Combinatorial Theory (A)* **114** (2007), 436–460; `arXiv.math/0603520` (Corollary 5.7).

82. This was a conjecture of R. Stanley, presented on June 25, 2003, at the 15th FPSAC meeting in Vadstena, Sweden, and available at

    `math.mit.edu/~rstan/transparencies/fpsacproblem.pdf`.

    It was proved by M. Ishikawa, *Ramanujan J.* **16** (2008), 211–234; `arXiv.math/0408204`.

83. (a) By (7.66) we have

$$s_\lambda(x,y) = \sum_{\mu\subseteq\lambda} s_\mu(x)s_{\lambda/\mu}(y).$$

Now apply ω_y and use Theorem 7.15.6.

(b) It is clear from the definition (4) that

$$p_n(x/y)_{x_1=t,\, y_1=-t} = p_n(x/y)_{x_1=y_1=0}.$$

Since ω_y and the substitutions $x_1=t, y_1=-t$ and $x_1=y_1=0$ are homomorphisms (into suitable rings), it follows that $g(x,y)$ satisfies (6) whenever $g\in \mathrm{im}(\omega_y)$.

Conversely, suppose that $g(x,y)\in\Lambda(x)\otimes\Lambda(y)$ and that $g(x,y)$ satisfies (6). Thus by hypothesis

$$g((t,x),(-t,y)) = g(x,y), \tag{25}$$

where $(t,x)=(t,x_1,x_2,\dots)$ and similarly for (t,y). Define

$$\begin{aligned} u_i &= \frac{1}{2}\left(p_i(x)+(-1)^i p_i(y)\right)\\ v_i &= \frac{1}{2}\left(p_i(x)+(-1)^{i-1} p_i(y)\right). \end{aligned}$$

It is clear that $g(x,y)$ can be written uniquely as a polynomial $P(u_1, u_2, \dots; v_1, v_2, \dots)$ in the u_i's and v_i's, and we want to show that no u_i appears. Now

$$g((t,x),(-t,y)) = P(u_1 + t, u_2 + t^2, \dots; v_1, v_2, \dots). \tag{26}$$

By considering the coefficient of t in (26) and using (25), we see that u_1 does not appear in P. Now by considering the coefficient of t^2 is follows that u_2 does not appear in P, etc. This proof is due to John Stembridge (unpublished).

(c) See J. R. Stembridge, *J. Algebra* **95** (1985), 439–444 (Theorem 1).

(d) This is immediate from

$$p_{2i}(x/x) = p_{2i}(x) + (-1)^{2i-1} p_{2i}(x) = 0.$$

(e) From (a) it is easy to see that

$$[x^\alpha y^\beta] s_\lambda(x/y) = \langle h_\alpha e_\beta, s_\lambda \rangle.$$

This corresponds to building up the shape λ by first adding horizontal strips of sizes $\alpha_1, \alpha_2, \dots$ and then vertical strips of sizes $\beta_1, \beta_2, \dots$. But multiplication of symmetric functions is *commutative*, so we could first add a horizontal strip H_1 of size α_1, then a vertical strip V_1 of size β_1, then a horizontal strip H_2 of size α_2, then a vertical strip V_2 of size β_2, etc. To get $s_\lambda(x/x)$, fill in each square of $H_i \cup V_i$ with i, getting an array T, and sum all x^T's. Now the array T is precisely a supertableau. How many times does a particular supertableau T get counted? For each component C of $H_i \cup V_i$, there are exactly two ways to decompose C into a horizontal strip H followed by a vertical strip V. (The last square in the top row can be part of H or V, but there are no other choices.) Hence the number of times T gets counted is $2^{c(T)}$, completing the proof.

(f) First note that $s_{(n^m)}(x_1, \dots, x_m / y_1, \dots, y_n)$ is a homogeneous polynomial of degree mn. If $\mu \subseteq (n^m)$, then either $\ell(\mu) = m$ or $(n^m)/\mu$ contains a row of length n. In the former case $s_\mu(x_2, x_3, \dots, x_m) = 0$ and in the latter $s_{(n^m)'/\mu'}(y_2, y_3, \dots, y_n) = 0$. Thus by (5) we have $s_{(n^m)}(0, x_2, \dots, x_m / 0, y_2, \dots, y_n) = 0$, so by (6) we have

$$s_{(n^m)}(x_1, x_2, \dots, x_m / y_1, y_2, \dots, y_n)\big|_{x_1 = -y_1} = 0.$$

Hence $s_{(n^m)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n)$ is divisible by x_1+y_1. By symmetry, $s_{(n^m)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n)$ is divisible by $x_i + y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This accounts for mn factors, so $s_{(n^m)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n) = c\prod(x_i + y_j)$ for some constant c. It is easy to see that $c = 1$, e.g., by considering the coefficient of $x_1^n x_2^n \cdots x_m^n$, completing the proof. This result is due to D. E. Littlewood and can be found in [7.88, XVIII on page 115].

Bijective proof. Immediate from Exercise 7.42 (with the typo $s_{\tilde{\lambda}}(y)$ replaced with $s_{\tilde{\lambda}'}(y)$).

(g) This result is due to A. Berele and A. Regev, *Advances in Math.* **64** (1987), 118–175. A bijective proof was given by J. B. Remmel, *Linear and Multilinear Algebra* **28** (1990), 119–154. For another proof, see [7.96, Exam. I.3.23(4)].

84. See P. Clifford and R. Stanley, *Electronic J. Combinatorics* **11**(1) (2004), R67; `arXiv.math/0311382`.

85. (a) Let $\lambda, \mu \vdash n$. A monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is obtained from p_μ by picking out a term from each factor p_{μ_i} of p_μ. If say $\lambda_{t+1} = \cdots = \lambda_{t+r}$, then we can permute the choices giving rise to these exponents in $r!$ ways. It follows that p_μ is a $\mathbb{Z}$-linear combination of the $\tilde{m}_\lambda$'s.

A noncomputational way to prove the converse is the following. Recall that Λ^n denotes the abelian group (under addition) of $\mathbb{Z}$-linear combinations of the m_λ's, where $\lambda \vdash n$. Let Υ^n denote the $\mathbb{Z}$-linear combinations of the $\tilde{m}_\lambda$'s, where $\lambda \vdash n$. Let Π^n denote the $\mathbb{Z}$-linear combinations of the p_λ's, where $\lambda \vdash n$. In the previous paragraph we have shown that $\Pi^n \subseteq \Upsilon^n$. Clearly by the definition of $\tilde{m}_\lambda$, the index of Υ^n in Λ^n is $\prod_{\lambda \vdash n} \prod_i r_i(\lambda)!$. By the Note following Corollary 7.7.2, this number is also the index of Π^n in Λ^n, so $\Upsilon^n = \Pi^n$.

(b) One could conjecture, for instance, that $\gamma(s_\lambda)$ is s-positive; and if $\lambda_1 \leq 2$, then $\gamma(s_\lambda)$ is e-positive.

86. (a) We have $\langle s_k^\perp p_\lambda, p_\mu \rangle = \langle p_\lambda, s_k p_\mu \rangle$. The proof now follows from $s_k = \sum_{\mu \vdash k} z_\mu^{-1} p_\mu$ and the orthogonality relation $\langle p_\lambda, p_\nu \rangle = \delta_{\lambda\nu} z_\lambda$.

(b) See `mathoverflow.net/questions/376494` (answer by R. Stanley).

87. *Sketch of proof.* Write ϑ_n for ϑ specialized to $t = n$, where $n \in \mathbb{P}$. Note that

$$\vartheta_n(p_k)(x_1, \ldots, x_n) = p_k(x_1 + 1, \ldots, x_n + 1).$$

Hence

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_n) = s_\lambda(x_1 + 1, \ldots, x_n + 1).$$

Now

$$\begin{aligned} s_\lambda(x_1 + 1, \ldots, x_n + 1) &= \frac{a_{\lambda+\delta}(x_1 + 1, \ldots, x_n + 1)}{a_\delta(x_1 + 1, \ldots, x_n + 1)} \\ &= \frac{a_{\lambda+\delta}(x_1 + 1, \ldots, x_n + 1)}{a_\delta(x_1, \ldots, x_n)}. \end{aligned}$$

By expanding the entries of $a_{\lambda+\delta}(x_1 + 1, \ldots, x_n + 1)$ and using the multilinearity of the determinant we get (see I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Example I.3.10 on page 47)

$$s_\lambda(x_1 + 1, \ldots, x_n + 1) = \sum_{\mu \subseteq \lambda} d_{\lambda\mu} s_\mu, \tag{27}$$

where

$$d_{\lambda\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1 \leq i,j \leq n}.$$

We can factor out factorials from the numerators of the row entries and denominators of the column entries of the above determinant. These factorials altogether yield $\prod_{u \in \lambda/\mu}(n + c(u))$. What remains is exactly the determinant for $f^{\lambda/\mu}/|\lambda/\mu|!$ given by Corollary 7.16.3, and we obtain

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \left(\prod_{u \in \lambda/\mu} (n + c(u))\right) s_\mu(x_1, \ldots, x_n).$$

We can set $x_{i+1} = \cdots = x_n = 0$ to obtain the above equation in i variables. In other words,

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_i) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \left(\prod_{u \in \lambda/\mu} (n + c(u))\right) s_\mu(x_1, \ldots, x_i)$$

for all $n \geq i$. Both sides are polynomials in n, so they are equal as polynomials, and we can replace n with the indeterminate t. Now let $i \to \infty$.

Is there a more conceptual proof that doesn't involve the evaluation of a determinant?

Equation (27) (without the evaluation of the determinant $d_{\lambda\mu}$) goes back to A. Lascoux, *C. R. Acad. Sci. Paris Sér. A-B* **286** (1978), 385–387. See also I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed. (Example I.3.10 on pages 47–48). The evaluation of $d_{\lambda\mu}$ is due to R. Stanley. For further aspects of this result, see S. C. Billey, B. Rhoades, and V. Tewari, *Int. Math. Research Not. IMRN* (2019); `arXiv:1920.11165` (Section 4).

88. See Lemma 3.2 of R. Stanley, *Ramanujan J.* **23** (2010), 91–105; `arXiv:0807.0383`.

89. Part (a) follows from Section 4.2 of

`www.mat.univie.ac.at/~slc/wpapers/FPSAC2018/`
`50-Thiel-Williams.html`

by M. Thiel and N. Williams. The key to doing the entire exercise is to first show that for any symmetric function f, we have

$$[z^r]f(z+1,\overbrace{1,\cdots,1}^{n-1},0,0,\ldots) = \frac{1}{n\cdot r!}\psi_r f,$$

where $[z^r]g$ denotes the coefficient of z^r in g (when expanded as a power series in z). This exercise is based on unpublished work of X. Li, L. Mu, and R. Stanley.

90. This is the special case of Exercise 7.47(j) where P is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free. It was shown by M. Guay-Paquet, `arXiv:1306.2400`, that this special case actually implies the full conjecture.

For the case $\mathcal{I} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$ (i.e., no two consecutive elements of $i_1 i_2 \cdots i_n$ are equal), see Exercise 7.47(k). Ben Joseph claimed to have a proof based on the involution principle for the case $\mathcal{I} = \{\{1,2,3\},\{2,3,4\},\ldots,\{n-2,n-1,n\}\}$, but this was never written up. A different proof is due to S. Dahlberg, *Sém. Lotharingien de Combinatoire* **82B** (2019), Article #59, 12 pp.

David Gebhard and Bruce Sagan, *J. Algebraic Comb.* **13** (2001), 227–255, have generalized Exercise 7.47(k) to the case $\mathcal{I} = \{\{1,2,\ldots,k_1\},\{k_1,k_1+1,\ldots,k_2\},\{k_2,k_2+1,\ldots,k_3\},\ldots,\{k_r,k_r+1,\ldots,n\}\}$. For further work on chromatic symmetric functions and their quasisymmetric generalization, see the web page

`www.symmetricfunctions.com/chromaticQuasisymmetric.htm`

of Per Alexandersson.

91. (a) We have

$$\begin{aligned} a(m,n) &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\mu f^\nu f^\lambda c^\lambda_{\mu\nu} \\ &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\mu f^\nu f^\lambda \langle s_\mu s_\nu, s_\lambda\rangle \\ &= \left\langle \left(\sum_{\mu\vdash m} f^\mu s_\mu\right)\left(\sum_{\nu\vdash n-m} f^\nu s_\nu\right), \sum_{\lambda\vdash n} f^\lambda s_\lambda \right\rangle \\ &= \langle p_1^m p_1^{n-m}, p_1^n\rangle \\ &= n!. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} b(m,n) &= \left\langle p_1^n, \sum_{\lambda\vdash n} s_\lambda \right\rangle \\ &= t(n), \end{aligned}$$

since e.g. if $f\in\Lambda^n$ then $\langle p_1^n, f\rangle = [x_1x_2\cdots x_n]f$, while $\sum_{\lambda\vdash n} f^\lambda = t(n)$ by Corollary 7.13.9. In the same way we obtain

$$c(m,n) = \left\langle \left(\sum_{\mu\vdash m} s_\mu\right) p_1^{n-m}, p_1^n \right\rangle.$$

An elegant way to proceed is to use the fact (Problem 28(b)) that multiplication by p_1 is adjoint to $\frac{\partial}{\partial p_1}$. Hence

$$\begin{aligned}
c(m,n) &= \left\langle \sum_{\mu\vdash m} s_\mu, \frac{\partial^{n-m}}{\partial^{n-m}p_1} p_1^n \right\rangle \\
&= \left\langle \sum_{\mu\vdash m} s_\mu, (n)_{n-m} p_1^m \right\rangle \\
&= (n)_{n-m} t(m).
\end{aligned}$$

Finally,

$$\begin{aligned}
d(m,n) &= \left\langle \left(\sum_{\mu\vdash m} s_\mu\right)\left(\sum_{\nu\vdash n-m} s_\nu\right), p_1^n \right\rangle \\
&= [x_1x_2\cdots x_n]\left(\sum_{\mu\vdash m} s_\mu\right)\left(\sum_{\nu\vdash n-m} s_\nu\right) \\
&= \binom{n}{m} t(m)t(n-m).
\end{aligned}$$

(b) See C. Ikenmeyer, *J. Alg. Combinatorics* **44** (2016), 1–29 (Theorem 1.1); `arXiv 1209.1521`.

(c) We have (pointed out by B. Tenner)

$$\begin{aligned}
e(m,n) &= \left\langle \left(\sum_{\mu\vdash m} s_\mu\right)\left(\sum_{\nu\vdash n-m} f^\nu s_\nu\right), \sum_{\lambda\vdash n} s_\lambda \right\rangle \\
&= \left\langle \left(\sum_{\mu\vdash m} s_\mu\right) p_1^{n-m}, \sum_{\lambda\vdash n} s_\lambda \right\rangle \\
&= \sum_{\mu\vdash m}\sum_{\lambda\vdash n} \langle p_1^{n-m} s_\mu, s_\lambda \rangle \\
&= \sum_{\mu\vdash m}\sum_{\lambda\vdash n} \langle p_1^{n-m}, s_{\lambda/\mu} \rangle \\
&= \sum_{\mu\vdash m}\sum_{\lambda\vdash n} f^{\lambda/\mu}.
\end{aligned}$$

Now use Exercise 7.27(a).

92. (a) Let

$$Y_n = \left(\sum_{\mu \vdash n} s_\mu(x) s_\mu(y) \right)^2 .$$

It is easy to see that the coefficient of q^n in the left-hand side of (7) is just $\langle Y_n, Y_n \rangle$ (scalar product in the ring $\Lambda(x) \otimes \Lambda(y)$). Expanding Y_n in terms of power sums yields

$$\langle Y_n, Y_n \rangle = \sum_{|\mu|+|\nu|=n} z_\mu^{-1} z_\nu^{-1} z_{\mu\cup\nu},$$

from which it is easy to complete the proof. The original proof of Jeb Willenbring (private communication of 5 June 2003) is based on representation theory. See P. E. Harris and J. F. Willenbring, in *Symmetry: representation theory and its applications*, Birkhäuser, New York, 2014, pp. 305–326.

(e) Follows easily from Exercise 7.71(c) and letting $n \to \infty$ in part (d) of the present problem.

93. Conjectured by Stijn Lievens and Neli Stiolova, and proved by Ron King in July, 2007, in a manuscript entitled "Notes on certain sums of Schur functions."

♣ Can this manuscript be accessed online? Is there another reference?

94. This conjecture is due to S. Sundaram. It has been checked for even $n \leq 20$.

95. See T. Lam, A. Postnikov, and P. Pylyavskyy, *Amer. J. Math.* **129** (2007), 1611–1622; `arXiv:math/0502446`.

96. This result is the famous "saturation conjecture" for Littlewood-Richardson coefficients. The first proof was by given by A. Knutson and T. Tao, *J. Amer. Math. Soc.* **12** (1999), 1055–1090; `math.RT/9807160`. An exposition of this proof was given by A. S. Buch, *Enseign. Math.* **46** (2000), 43–60; `arXiv.math/9810180`. Later proofs were given by H. Derksen and J. Weyman, *J. Amer. Math. Soc.* **13** (2000), 467–479, and P. Belkale, *J. Alg. Geom.* **15** (2006), 133–173; `math.AG/0208107`. For some generalizations of the saturation conjecture, see A. N. Kirillov, *Publ. Res. Inst. Math. Sci.* **40** (2004), 1147–1239; `arXiv.math/0404353`.

97. This result is closely related the saturation conjecture (Problem 96 above) and to the problem of characterizing the possible eigenvalues of hermitian matrices $A, B, A+B$. For a nice survey of this area see W. Fulton, *Bull. Amer. Math. Soc.* **37** (2000), 209–249; `math.AG/9908012`.

98. (a) This result follows from the theory of Hall polynomials, not discussed in EC2. See (4.3) on page 188 of Macdonald, *Symmetric Functions and Hall Polynomials*, second ed. To complete the proof, one needs to show that $g^\lambda_{\mu\nu}(p) \neq 0$ (when p is prime). This fact follows from a result of F. M. Maley, *J. Algebra* **184** (1996), 363–371, that $g^\lambda_{\mu\nu}(t)$ has nonnegative coefficients when expanded in powers of $t-1$.

    (b) This result (if true) would give an algebraic strenghtening of the Saturation Conjecture (Exercise 96 above). For a somewhat more general context, see `mathoverflow.net/questions/212368`.

99. (a) One method is to note that by the Murnaghan-Nakayama rule we have $\chi^\lambda((n)) \neq 0$ (if and) only if λ is a hook. But the only hooks λ with a border strip of size $n-1$ (otherwise $\chi^\lambda(n-1,1) = 0$) are (n) and (1^n). Since $\chi^n(\mu) = \varepsilon_\mu \chi^{1^n}(\mu) = 1 \neq 0$ for all $\mu \vdash n$, it follows that $\lambda = (n)$ or $\lambda = (1^n)$.

100. Let μ_1 be the size of the largest border strip B_1 of λ. Thus $\mu_1 = h(1,1) = \lambda_1 + \lambda'_1 - 1$, the hook length of square (1,1). Let μ_2 be the size of the largest border strip B_2 after we remove B_1 from λ. Thus $\mu_1 = h(2,2) = \lambda_2 + \lambda'_2 - 3$. Continue in this way to obtain $\mu = (\mu_1, \mu_2, \dots)$. The number of parts of μ is rank(λ). Since each B_i is unique, there is only one border strip tableau of type μ. Thus by the Murnaghan-Nakayama rule $\chi^\lambda(\mu) = \pm 1$.

101. Given $w \in \mathfrak{S}_n$, let

$$\mathrm{sq}(w) = \#\{u \in \mathfrak{S}_n \,:\, u^2 = w\},$$

the number of square roots of w. Let $\varphi : \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ be the linear transformation (not a ring homomorphism) defined by $\varphi(s_\lambda) = 1$. Thus $\varphi^{\otimes k}$ (the kth tensor power of φ) is a linear transformation from $\Lambda_{\mathbb{Q}}(x^{(1)}) \otimes \cdots \otimes \Lambda_{\mathbb{Q}}(x^{(k)})$ to $\mathbb{Q}$ defined by

$$\varphi(s_{\lambda^1}(x^{(1)}) \cdots s_{\lambda^k}(x^{(k)})) = 1.$$

By the solution to Exercise 7.69(b) we have $\varphi(p_\lambda) = \mathrm{sq}(w)$, where $\rho(w) = \lambda$. Now apply $\varphi^{\otimes k}$ to both sides of (7.186).

102. Equivalent to equation (4.7) of J. B. Lewis, V. Reiner, and D. Stanton, *J. Alg. Combinatorics* **40**(3) (2014), 663–691; `arXiv:1308.1468`, where the result is attributed to Kerov and to Garsia-Haiman. Several persons subsequently found elegant combinatorial proofs, the first being X. Chen (陈小美).

103. See R. Stanley, *Ramanujan J.* **23** (2010), 91–105, `arXiv:0807.0383`; G. Olshanski, *Electronic J. Math.* **17** (2010), R43, `arXiv:0905.1304`; and G. Panova, *Ramanujan J.* **27** (2012), 349–356, `arXiv:0811.3463`.

104. Let λ be a p-core with $\lambda_1 = k$. Remove from λ all rows such that the first square $(i, 1)$ of the row satisfies $h(i, 1) \equiv h(1, 1) \pmod{p}$, where $h(i, j)$ is the hook length of the square (i, j). One can check that this sets up a bijection with $(p-1)$-cores μ with $\mu_1 \leq k$. Thus if $f_p(k)$ denotes the number of p-cores with largest part k, then

$$f_p(k) = f_{p-1}(1) + f_{p-1}(2) + \cdots + f_{p-1}(k),$$

from which the proof follows easily. This result was explained by M. Vazirani in a conversation at MSRI on March 24, 2008.

105. Conjectured by D. Armstrong, C. R. H. Hanusa, and B. C. Jones, *Europ. J. Combinatorics* **41** (2014), 205–220; `arXiv:1308:0572`. For proofs see W. Y. C. Chen, H. H. Y. Huang, and L. X. W. Wang, *Proc. Amer. Math. Soc.* **144** (2016), 1391–1399, `arXiv:1405.2175`, and P. D. Johnson, *Electron. J. Combin.* **25** (2019), Paper 3.47, `arXiv:1502.07834`.

106. (a) We have

$$\langle \chi^{\lambda^1} \cdots \chi^{\lambda^k}, \chi^{(n)} \rangle = \langle \chi^{\lambda^1} \cdots \chi^{\lambda^{k-1}}, \chi^{\lambda^k} \rangle.$$

Thus [why?]

$$\begin{aligned} u_k(n) &= \sum_{\lambda^1,\dots,\lambda^{k-1}} \langle \chi^{\lambda^1}\cdots\chi^{\lambda^{k-1}}, \chi^{\lambda^1}\cdots\chi^{\lambda^{k-1}}\rangle \\ &= \sum_{\lambda^1,\dots,\lambda^{k-1}} \left\langle \left(\chi^{\lambda^1}\right)^2\cdots\left(\chi^{\lambda^{k-1}}\right)^2, \chi^{(n)}\right\rangle \\ &= \left\langle \left(\sum_{\mu\vdash n}(\chi^\mu)^2\right)^{k-1}, \chi^{(n)}\right\rangle. \end{aligned}$$

By Exercise 7.82(a),

$$\mathrm{ch}\sum_{\mu\vdash n}(\chi^\mu)^2 = \sum_{\mu\vdash n} p_\mu.$$

Hence [why?]

$$u_k(n) = \left\langle \left(\sum_{\mu\vdash n} p_\mu\right)^{*(k-1)}, h_n\right\rangle,$$

where $*(k-1)$ denotes the $(k-1)$th power with respect to $*$. Now since

$$p_\lambda * p_\mu = z_\mu p_\mu \delta_{\lambda\mu},$$

and $h_n = \sum_{\mu\vdash n} z_\mu^{-1} p_\mu$, finally we get [why?]

$$u_k(n) = \sum_{\mu\vdash n}(z_\mu)^{k-1}.$$

Thanks to Sam Hopkins for asking about the value of $u_2(n)$. Some references are J. B. Geloun and S. Ramgoolan, *Ann. Inst. Henri Poincaré D* **1** (2014), 77–138, `arXiv:1307.6490` (equation (18)); OEIS A110143, MO 41337, and MO 162428, where MO = MathOverflow.

(b) We now have [why?]

$$\begin{aligned} v_k(n) &= \sum_{\lambda^1,\dots,\lambda^k} \left\langle \chi^{\lambda^1}\cdots\chi^{\lambda^k}, \chi^{(n)}\right\rangle \\ &= \left\langle \left(\sum_{\lambda\vdash n} s_\lambda\right)^{*k}, h_n\right\rangle. \end{aligned}$$

Now by Exercise 7.69(a) we have

$$\begin{aligned}\sum_{\lambda\vdash n} s_\lambda &= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} p_{\rho(w^2)}\\ &= \sum_{\mu\vdash n} z_\mu^{-1}\mathrm{sq}(w_\mu)p_\mu,\end{aligned}$$

where w_μ is a permutation in $\mathfrak{S}_n$ of cycle type μ. Hence [why?]

$$\begin{aligned}v_k(n) &= \sum_{\mu} z_\mu^{-1}\mathrm{sq}(w_\mu)^k\\ &= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{sq}(w)^k.\end{aligned}$$

Note the special case $k=2$:

$$\frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{sq}(w)^2 = p(n).$$

(For a different generalization, see Problem 101.)

Note. For any finite group G of order d with k conjugacy classes we have (Problem 17)

$$\#\{(u,v)\in G\times G\,:\,uv=vu\} = kd.$$

Hence

$$\#\{(u,v)\in\mathfrak{S}_n\times\mathfrak{S}_n\,:\,uv=vu\} = \#\{(u,v)\in\mathfrak{S}_n\times\mathfrak{S}_n\,:\,u^2=v^2\}.$$

(In fact, this is true for any finite group G all of whose complex representations are equivalent to real representations.) Is there a combinatorial proof?

(c) Let $k=3$ in equation (10). The number of terms in the two sums is around $e^{c\sqrt{n}}$. The largest term on the right-hand side is $n!$, coming from $\mu=\langle 1^n\rangle$. Hence the largest term on the left-hand side is $n!$, up to a factor of the order $e^{c\sqrt{n}}$. From this observation and Stirling's formula for $n!$ the proof follows easily. For further information, including the partitions λ, μ, ν achieving the maximum, see I. Pak, G. Panova, and D. Yeliussizov, *J. Combinatorial Theory Ser. A* **165** (2019), 44–77; `arXiv:1804.04693`.

107. (a) See R. Stanley, *Advances in Applied Math.* **30** (2003), 283–294; `arXiv.math/0106115` (Theorem 2.2). The proof consists simply of applying the exponential specialization ex (EC2, pp. 304–306) to the identity of Exercise 7.27(e). Since this identity has a bijective proof *via* a skew version of RSK, its exponential specialization is therefore just a special case of this bijective proof. A somewhat different bijective proof was given by Aaron D. Jaggard, *Electronic J. Combinatorics* **12** (2005), R14 (Theorem 3.2); `arXiv.math/0107130`.

(b) See R. Stanley, *ibid.* (Theorem 2.1).

108. (b) See D. White, *Advances in Math.* **50**, 160–186, though there may be earlier references.

(c) Easy consequence of (b), the isomorphism φ, and the Murnaghan-Nakayama rule.

109. (a) See R. Stanley, *Sém. Lotharingien de Combinatoire* **50** (2003), B50d; `arXiv.math/0109093` (Theorem 1).

(b) See V. Féray, *Ann. Combinatorics* **13** (2010), 453–461; `arXiv.math/0612090`. There is also an exposition in P.-L. Méliot, *Representation Theory of Symmetric Groups*, Chapman and Hall/CRC, 2017.

110. (a) We may assume $q \in \mathbb{P}$. In Exercise 7.70, let $k = 3$ and $x^{(3)} = 1^q$. Take the scalar product of both sides with $p_n\big(x^{(1)}\big)\, p_n\big(x^{(2)}\big)$. The right-hand side becomes after some simplification

$$\mathrm{RHS} = n \sum_{\rho(w)=(n)} q^{\kappa(w(1,2,\ldots,n))}.$$

The left-hand side becomes (using Corollary 7.21.4)

$$\mathrm{LHS} = \sum_{\lambda \vdash n} \chi^\lambda((n))^2 \prod_{u \in \lambda} (q + c(u)).$$

Now e.g. by the Murnaghan-Nakayama rule we have

$$\chi^\lambda((n)) = \begin{cases} (-1)^{n-i}, & \text{if } \lambda = (i, 1^{n-i}) \\ 0, & \text{otherwise.} \end{cases}$$

From this we easily get

$$\begin{aligned} P_n(q) &= \frac{1}{n}\sum_{i=1}^{n}(q+i-1)_n \\ &= \frac{1}{n(n+1)}\left((q+n)_{n+1}-(q)_{n+1}\right). \end{aligned}$$

(b) If $P_n(z)=0$ then

$$|(z+1)(z+2)\cdots(z+n)|^2 = |(z-1)(z-2)\cdots(z-n)|^2. \quad (28)$$

Let $z=a+bi$ where $a,b\in\mathbb{R}$ and $a>0$. Then for $j>0$ we have

$$|(z+j)|^2-|(z-j)|^2 = 4aj>0.$$

Hence the left-hand side of (28) is greater than the right. The reverse inequality holds if $a<0$. Hence if (28) holds then $a=0$, and the proof follows.

(c) See R. X. F. Chen and C. M. Reidys, *SIAM J. Discrete Math.* **30** (2016), 1660–1684, `arXiv:1502.07674`, and the references therein.

(d) (sketch) As in (a) we obtain

$$P_\lambda(q) = \sum_{i=1}^{n}\chi^{\langle n-i,1^i\rangle}(\mu)(-1)^{i-1}(q+n-i)_n.$$

Now if we apply the specialization (homomorphism) $\psi(p_n)=1-(-t)^n$, then

$$\psi(s_\lambda) = \begin{cases} t^k(1+t), & \lambda=\langle n-k,1^k\rangle,\ 0\le k\le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence from $p_\mu=\sum_\lambda \chi_\lambda(\mu)s_\lambda$ we get

$$\frac{\prod(1-(-t)^{\mu_i})}{1+t} = \sum_{k=0}^{n-1}\chi^{\langle n-k,1^k\rangle}(\mu)t^k,$$

so

$$P_\lambda(q) = \left.\frac{\prod(1-(-t)^{\mu_i})}{1+t}\right|_{t^k\to(-1)^k(q+n-k-1)_n}.$$

It is now not hard to complete the proof using Lemma 9.3 of A. Postnikov and R. Stanley, *J. Combinatorial Theory (A)* **91** (2000), 544–597.

For this entire exercise, see R. Stanley, *Europ. J. Combinatorics* **32** (2011), 937–943; `arXiv:0901.2008`.

111. (a) Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. Define

$$m \circ \alpha = (\alpha_1, [\alpha_2, \ldots, \alpha_{k-1}, \alpha_k + \alpha_1]^{m-1}, \alpha_2, \ldots, \alpha_k),$$

where $[\gamma]^m$ denotes the concatenation of γ with itself m times. For instance, $3 \circ (2,1,3,2) = (2,1,3,4,1,3,4,1,3,2)$. Define

$$(\beta_1, \ldots, \beta_h) \circ \alpha = (\beta_1 \circ \alpha, \ldots, \beta_h \circ \alpha).$$

For instance, $(1,3,2) \circ (2,1) = (2,1,2,3,3,1,2,3,1)$. Given a composition α, there is unique way to write it as $\beta^1 \circ \beta^2 \circ \cdots \circ \beta^j$, where each β^i is irreducible with respect to $\circ$ (i.e., $\beta \neq \gamma \circ \delta$, where γ and δ are compositions of integers greater than one). If $\beta = (\beta_1, \ldots, \beta_h)$, let $\overline{\beta} = (\beta_h, \ldots, \beta_1)$. L. Billera and S. van Willigenburg, *Advances in Math.* **204** (2006), 204–240, `arXiv.math/0405434`, have shown that the compositions equivalent to α are exactly those of the form $(\beta^1)' \circ (\beta^2)' \circ \cdots \circ (\beta^j)'$, where each $(\beta^i)'$ is β^i or $\overline{\beta^i}$. For instance, since $12132 = 12 \circ 12$, the compositions equivalent to 12132 are 12132, $12 \circ 21 = 21231$, $21 \circ 12 = 13212$, and $21 \circ 21 = 23121$.

(b) See Theorem 3.4 of M. Rubey, *Sém. Lotharingien de Combinatoire* **64** (2011), B64c; `arXiv:1008.2501`.

112. See R. Stanley, *J. Combinatorial Theory (A)* **100** (2002), 349–375; `arXiv.math/0109092` (Corollary 5.3). A direct combinatorial argument was given by Thomas Lam.

113. See R. Stanley, *J. Combinatorial Theory (A)* **100** (2002), 349–375; `arXiv.math/0109092` (Proposition 2.2 and Theorem 3.2). For the last item, see W. Y. C. Chen and A. L. B. Yang, *Trans. Amer. Math. Soc.* **360** (2008), 3121–3131; `arXiv.math/0509181`.

114. See N. Eriksen and A. Hultman, Estimating the expected reversal distance after a fixed number of reversals, *Advances in Applied Math.* **32** (2004), 439–453 (Theorem 3).

115. (a,b) See B. Joseph, Ph.D. thesis, M.I.T, 2001 (Chapter 3), available at `dspace.mit.edu/handle/1721.1/8225`. This proof was never published.

(c) Immediate from (a) and the definition of the scalar product $\langle f_n, \mathrm{sgn}\rangle$. This result is attributed to Paul D. Hanna, 9 March 2013, in OEIS A033917.

116. (a) We have

$$\sum_{\lambda\vdash n} s_\lambda(x)s_\lambda(y) = \sum_{\lambda\vdash n} z_\lambda^{-1} p_\lambda(x)p_\lambda(y). \tag{29}$$

Set $y_i = q^{i-1}$ and multiply by $(1-q)(1-q^2)\cdots(1-q^n)$. By Proposition 7.19.11 we obtain

$$\sum_{\lambda\vdash n}\left(\sum_{\mathrm{sh}(T)=\lambda} q^{\mathrm{maj}(T)}\right) s_\lambda = \sum_{\lambda\vdash n} z_\lambda^{-1}\frac{(1-q)\cdots(1-q^n)}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_\ell})}p_\lambda,$$

where $\ell = \ell(\lambda)$. Now set $q=-1$. The left-hand side becomes R_n. The term indexed by λ on the right-hand side vanishes unless $\lambda = (2^m)$ or $\lambda = (2^m, 1)$. Moreover, $z_{(2^m)} = z_{(2^m,1)} = 2^m m!$, and

$$\lim_{q\to -1}\frac{(1-q)(1-q^2)\cdots(1-q^{2m})}{(1-q^2)^m}$$
$$= \lim_{q\to -1}\frac{(1-q)(1-q^2)\cdots(1-q^{2m+1})}{(1-q)(1-q^2)^m} = 2^m m!.$$

Hence

$$R_n = \begin{cases} p_2^m, & n=2m \\ \\ p_1p_2^m, & n=2m+1. \end{cases}$$

(b) For definiteness let $n=2m$; the case $n=2m+1$ is essentially the same. By (a) we need to characterize all $\lambda\vdash n$ for which $\langle p_2^m, s_\lambda\rangle \neq 0$. By Corollary 7.17.4, if $\langle p_2^m, s_\lambda\rangle \neq 0$ then there exists a border strip tableau of shape λ and type (2^m). Moreover, a simple argument shows that all border strip tableaux of shape λ and type (2^m) have the same parity of horizontal (and hence of vertical) dominos. Thus there is no cancellation of signs in the computation of $\chi^\lambda(2^m) = \langle p_2^m, s_\lambda\rangle$, so $\langle p_2^m, s_\lambda\rangle \neq 0$ if and only if there exists a border strip tableau of shape λ and type (2^m). (This fact also follows from Problem 108.) Such a border strip tableau defines a covering of the shape of λ with m (disjoint) dominos. Moreover, it's easy to see that the dominos of any covering of λ with m dominos can be ordered so that they define a border strip tableau, and the proof follows. (A crucial point in extending the argument to odd n is that a border strip tableau of type $(2, 2, \ldots, 2, 1)$ always has the same square $(1, 1)$ as the border strip of size 1. This fact breaks down for skew shapes.)

For a generalization to posets, not involving symmetric functions, see Theorem 5.1 of R. Stanley, *Advances in Applied Math.* **34** (2005), 880–902; `arXiv:math/0211113`.

(c) For the first statement, see Prop. 5.3 of the previous reference, which in fact is valid for more general labelled posets than Schur labelled skew shapes.

For the second statement, let $\lambda/\mu = 43/2$. Then we can place $\lfloor 5/2 \rfloor = 2$ disjoint dominos on the diagram of $43/2$, but $E(43/2) = 5$ and $O(43/2) = 4$.

(d) Similar to (a)–(c); details omitted.

117. (a) For the case of rectangular shapes, see D. E. White, *J. Combinatorial Theory (A)* **95** (2001), 1–38.

(b) This was a conjecture of R. Stanley. For proofs see J. Sjöstrand, *J. Combinatorial Theory, Ser. A* **111** (2005), 190–203, `arXiv.math/0309231`, and T. Lam, *J. Combinatorial Theory Ser. A* **107** (2004), 87–115, `arXiv.math/0308265`.

118. (a) For any homogeneous symmetric function Y of degree n, $\langle Y, p_1^n \rangle$ is equal to $[x_1 x_2 \cdots x_n]Y$ (the coefficient of $x_1 x_2 \cdots x_n$ in Y). For any n-vertex graph G, $[x_1 x_2 \cdots x_n]X_G$ is equal to the number of proper colorings of G using the colors $1, 2, \ldots, n$ once each. Hence $\langle X_G, p_1^n \rangle = n!$, so the proof follows since F_n is a sum of C_n X_G's.

(b) For any n-vertex graph G with $X_G = \sum_{\lambda \vdash n} d_\lambda e_\lambda$, Exercise 7.47(g) asserts that $\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} d_\lambda$ is the number of acyclic orientations of G with k sinks. Hence $c_{\langle 1^n \rangle}$ is the total number of acyclic orientations of n-vertex unit interval graphs (always up to isomorphism) with

n sinks. The only such graph is the one with no edges, and it has one acyclic orientation. Hence $c_{\langle 1^n \rangle} = 1$.

(c) Since $\langle 2, 1^{n-2} \rangle$ is the only partition of n with $n-1$ parts, we want the total number of acyclic orientations with $n-1$ sinks of n-vertex unit interval graphs. Such a graph can have exactly one edge (with two acyclic orientations having $n-1$ sinks), or have two incident edges (with one acyclic orientation having $n-1$ sinks). There are $n-1$ n-vertex unit interval graphs with exactly one edge, and $n-2$ with two incident edges. Hence

$$c_{\langle 2,1^{n-2} \rangle} = 2(n-1) + (n-2) = 3n-4.$$

(d) Let the unit interval graph G have vertices $1, 2, \ldots, n$ in the order of the unit intervals that define it. (The unit intervals are ordered by the order of their left endpoints.) Let vertex i be adjacent to d_i vertices $j < i$ (so $d_1 = 0$). It is easy to see that

$$\chi_G(q) = \prod_{i=1}^{n} (q - d_i).$$

It is well-known that $(-1)^n \chi_G(-1) = \mathrm{ao}(G)$, the number of acyclic orientations of G. Hence $\mathrm{ao}(G) = \prod (d_i + 1)$. It is also easy to see that these sequences $(d_1, \ldots, d_n) \in \mathbb{P}^n$ are characterized by $d_1 = 1$ and $d_{i+1} \leq 1 + d_i$. Let $\mathcal{D}_n$ denote the set of all such sequences. (We have $\#\mathcal{D}_n = C_n$ by Exercise 80 in R. Stanley, *Catalan Numbers*.) We need to show that

$$\prod_{(d_1, \ldots, d_n) \in \mathcal{D}_n} d_i = (2n-1)!!.$$

Let $e_i = 2i - d_i$. It is not hard to see that the number of complete matchings M on $[2n]$ such that $e_1, \ldots, e_n$ are the smaller vertices of the edges of M is equal to $d_1 \cdots d_n$. Moreover, all possible such "smaller vertex" sets $\{e_1, \ldots, e_n\}$ are obtained in this way, and the proof follows.

(e) Let G be any n-vertex graph and v any vertex of G. Greene and Zaslavsky (1983) showed that the number of acyclic orientations of G having v as the only sink is equal to $[q]\chi_G(q)$. Using the notation of (d) above, it follows that

$$c_{(n)} = n \sum_{(a_1,\dots,a_n)\in\mathcal{D}_n} (d_2-1)(d_3-1)\cdots(d_n-1).$$

The set of sequences $(d_2-1,\dots,d_n-1)$ with no term equal to 0 is just the set $\mathcal{D}_{n-1}$, so the proof follows from (d).

(g) *Answer.* Write a matching M on $[2n]$ in the canonical form $((a_1,b_1),\dots,(a_n,b_n))$, where $a_1 < a_2 < \cdots < a_n$ and $a_i < b_i$. Let w_M be the standardization of the sequence $b_1,\dots,b_n$. Then

$$\omega F_n = \sum_M p_{\rho(w_M)},$$

summed over all matchings on $[2n]$.

Example. When $n=2$, the matchings are $(12,34)$, $(13,24)$, and $(14,23)$. The sequences b_1,b_2 are $24,34,43$, with standardizations $12,12,21$. Hence

$$\omega F_2 = 2p_{\rho(12)} + p_{\rho(21)} = 2p_1^2 + p_2.$$

119. Follows from the solution to Exercise 7.47(f), or equivalently, from Theorem 3.1 and equation (8) of R. Stanley, *Advances in Math.* **111** (1995), 166–194. See also T. Y. Chow, `arXiv:math/9712229`. (The difficulty rating of this exercise assumes no knowledge of Exercise 7.47(f).)

120. (a) Let $\mathcal{M} = x_1^{a_1}\cdots x_k^{a_k}$ be a monomial of degree n, with each $a_i > 0$. Suppose that $\mathcal{M}$ appears in $L_{\mathrm{Xdes}(w)}$. Write w as a juxtaposition (not the product of permutations in $\mathfrak{S}_n$) $v_1v_2\cdots v_k$, where v_i has length a_i. The partial permutations v_i therefore have no X-descents. For $\pi\in\mathfrak{S}_k$ let $\pi(\mathcal{M}) = x_1^{a_{\pi(1)}}\cdots x_k^{a_{\pi(k)}}$. Then $\pi(\mathcal{M})$ appears in the permutation $v_{\pi(1)}\cdots v_{\pi(k)}$. It follows easily that U_X is a symmetric function.

Moreover, if $r_j(\mathcal{M})$ of the b_i's are equal to j, then the v_i's of length j can be permuted in $r_j(\mathcal{M})!$ ways. Hence the coefficient of $\mathcal{M}$ in U_x is divisible by $r_1(\mathcal{M})!\cdots r_k(\mathcal{M})!$. In other words, U_X is a $\mathbb{Z}$-linear combination of the augemented monomial symmetric functions $\tilde{m}_\lambda$ of Problem 84(c). Now use Problem 85 to deduce that U_X is p-integral.

(b) *Answer:* $U_{\overline{X}} = \omega U_X$.

(c) Conjectured by R. Stanley and proved by I. Gessel (private communication dated 5 November 2021).

Here is a sketch of Gessel's proof. For any finite multiset M of positive integers define

$$U_X^{(M)} = \sum_w L_{\mathrm{XDes}(w)},$$

where the sum is over all permutations w of the multiset M. Write just U_X when $M = [n]$.

Define the generating function

$$G(t) = \sum_{m\geq 0} t^m \sum_{i_1,\ldots,i_m} y_{i_1} \cdots y_{i_m},$$

where the inner sum is over all sequences $i_1 i_2 \cdots i_m$ with no X-descents. One first shows (proof omitted) that if $M = \{1^{j_1}, 2^{j_2}, \ldots\}$ then $U_X^{(M)}$ is the coefficient of $y_1^{j_1} y_2^{j_2} \cdots$ in the infinite product

$$P := \prod_{k\geq 1} G(x_k).$$

For simplicity we will ignore powers of y_i greater than 1, i.e., we are working in the ring $\mathbb{Q}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]/(y_1^2, y_2^2, \ldots)$. Let

$$g(t) = \log G(t) := \sum_{n\geq 1} g_m t^m.$$

Then

$$\log P = \sum_{k\geq 1} g(x_k) = \sum_{k,m\geq 1} g_m x_k^m = \sum_{m\geq 0} g_m p_m,$$

so

$$P = \exp\left(\sum_{m\geq 0} g_m p_m\right). \tag{30}$$

We claim that $g(t)$ counts words (without repetitions) with no X-descents that start with their smallest entry. By the exponential formula, this is equivalent to the existence of a bijection from sets of words (with disjoint distinct entries) that start with their smallest entry to words (with no repetitions) with no X-descents: given a set of such words, we concatenate them in increasing order of their first element. The inverse is that we cut each word before each left-to-right maximum. We only need to check that concatenating words doesn't introduce any new X-descents. This follows from the property that if $(i, j) \in X$ then $i > j$. Thus when we exponentiate in equation (30) we recover all words without X-descents, but now weighted as in the problem.

(d) No. Gessel noted that if $X = \{(1,3),(2,1),(3,1),(3,2)\}$, then $U_X = p_1^3 - p_2 p_1 + p_3$.

(e) Equivalent to Problem 119 above.

(f) Expand the right-hand side of equation (11) in terms of the fundamental quasisymmetric functions L_S. We need to show that the coefficient of L_S is equal to the number of permutations $w \in \mathfrak{S}_n$ with $\mathrm{XDes}(w) = S$. Now the L-expansion of $s_{i,1^{n-i}}$ is given by

$$s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} L_S.$$

Thus we need to show that the number of $w \in \mathfrak{S}_n$ with $\mathrm{XDes}(w) = S \in \binom{[n-1]}{n-i}$ is equal to f_i. Hence given $S \in \binom{[n-1]}{n-i}$ we want to define a bijection

$$\{w \in \mathfrak{S}_n \,:\, \mathrm{XDes}(w) = S\} \to \{u \in \mathfrak{S}_i \,:\, \mathrm{XDes}(u) = \emptyset\}.$$

Given w, factor it (as a word) into maximal factors of the form $k, k-1, \ldots, k-i$ to obtain u. For instance, if $w = 3247651$ (so $S = \{1,4,5\}$) then we factor w as $32 \cdot 4 \cdot 765 \cdot 1$. Now replace the factor with the least elements by 1, the factor with the next least elements by 2, etc. For our example we replace 1 by 1, 32 by 2, 4 by 3, and 765 by 4 to get $u = 2341$. It is easy to check that this procedure gives the desired bijection.

(g) Similar to (f).

(h) Similar to (f) and (g).

121. (a) Immediate from the definition of U_X and Problem 120(b).

(b) It suffices to verify it for $f = p_i$, which is routine.

(c) Follows from (a), (b), and the p-integrality of U_X (Problem 120(a)). This result appears as Problem 7.7 in I. Tomescu, *Problems in Combinatorics and Graph Theory*, John Wiley & Sons, Chichester, 1985. The proof given here is due to D. Grinberg, private communication, 5 April 2022.

122. (a) Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ and $\mathrm{XDes}(w) = S = \{i_1, \ldots, i_k\}$. Let $w^r = a_n a_{n-1} \cdots a_1$, $S' = \{n - i_1, \ldots, n - i_k\}$, and $\overline{S} = [n-1] - S$. Since X is a tournament, $\overline{\mathrm{X}}\mathrm{Des}(w^r) = S'$ (where the meaning of $\overline{\mathrm{X}}\mathrm{Des}$ should be clear). Suppose that $f = \sum_{S \subseteq [n-1]} c_S F_S \in \Lambda^n$. It is easily seen that $\omega f = \sum_{S \subseteq [n-1]} c_S F_{\overline{S}}$. On the other hand, by Proposition 7.19.2, $c_S = c_{S'}$. From these observations the result follows.

(b) This remarkable formula was proved by Darij Grinberg (private communication, 19 April 2022).

(c) Note that $\mathrm{ham}(\overline{X})$ is the sum of the coefficients in the power sum expansion of U_X. Moreover, for any X the coefficient of p_1^n is 1. If (b) holds then all other coefficients are even, so the sum of the coefficients is odd, and the proof follows.

The result that every tournament has an odd number of Hamiltonian cycles is due to L. Rédei, *Acta Litteraria Szeged* **7** (1934), 39–43.

123. (b) This result is equivalent to a result of I. Schur, *Math. Z.* **1** (1918), 184–207, as may be seen by expanding each $p_{\rho(w)}$ in the definition of $\mathrm{sfdet}(A)$ in terms of Schur functions and taking the coefficient of s_λ.

(d) Similarly, this result is equivalent to a result of J. Stembridge, *Bull. London Math. Soc.* **23** (1991), 422–428. For a generalization of (b) and (c), see B. Kostant, *J. Amer. Math. Soc.* **8** (1995), 181–186.

(e) Similarly, this assertion is equivalent to Stembridge, *Canad. J. Math.* **44** (1992), 1079–1099 (Conjecture 2.1).

(f) M. Guay-Paquet, `arXiv:1306.2400`, has reduced Exercise 7.47(j) to the case where P is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free, i.e., a *semiorder* (solution to Exercise 6.19(ddd)). It follows from the sentence preceding Conjecture 5.1 of R. Stanley, *Advances in Math.* **111** (1995), 166–194, that Exercise 7.47(j) is equivalent to the h-positivity of sfdet(A), where if G has n vertices then A is the $n \times n$ $(0,1)$-matrix with the 0's occupying a certain rotated Young diagram justified into the lower left-hand corner and lying below the main diagonal. Such a matrix is totally nonnegative, so Exercise 7.47(j) is a special case of (d). For another combinatorial application of the total nonnegativity of A, see the final paragraph of R. Stanley, *J. Combinatorial Theory (A)* **74** (1996), 169–172.

124. See T. Lam and P. Pylyavskyy, *International Math. Research Notices* **2007** (2007), rnm125, `arXiv:0705.2189` (§9). Item (c) first appeared as Proposition 14 of M. Shimozono and M. Zabrocki, Stable Grothendieck symmetric functions and Ω-calculus, unpublished manuscript dated February 4, 2003. The result also appears as Corollary 15 of A. Amanov and D. Yeliussizov, `arXiv:2003.03907v1`. Closely related results appear in D. Yeliussizov, *J. Algebraic Combinatorics* **45** (2017), 295–344 (§10); `arXiv:1601.01581`.

125. The concept of set-valued tableau is due to A. Buch, *Acta Math.* **189** (2002), 37–78, `arXiv:math/0004137`, who showed (a). For (b,c) see Lam and Pylyavskyy, *ibid.* For (d), see Proposition 6 of Shimozono and Zabrocki, *ibid.* A further paper of interest is J. Bandlow and J. Morse, *Electronic J. Combinatorics* **19** (2012), P39; `arXiv:1106.1594`. Somewhat different (yet analogous) formulas appear in T. Hudson, T. Ikeda, T. Matsumura, and H. Naruse, *Advances in Math.* **320** (2017), 115–156, `arXiv:1504.02828`; T. Matsumura, *Proc. Japan Acad. Ser. A Math. Sci.* **93** (2017), 82–85, `arXiv:1611.06483`; and J. S. Kim, `arXiv:2003.00540`.

126. (a) Equation (13) is due to I. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Theorem 2.1).

  (b) Immediate from (a), the definition of L_λ, and the fact that $L_4 = L_{2,1,1}$.

  (c) The values of $f(n)$ for $1 \le n \le 15$ are 1, 2, 3, 4, 6, 10, 13, 19, 26, 38, 52, 70, 91, 123, 161. See OEIS A121152. For $n = 4$ there is one linear dependence, namely, $L_4 = L_{211}$. For $n = 5$

the unique linear dependence is $L_{41} = L_{32} + L_{2111}$. For $n = 6$ it's $L_6 + L_{33} = L_{321} + L_{3111}$. For $n = 7$, there are two (independent) relations: $L_{43} = L_{3211}$ (a consequence of multiplying the relation $L_4 = L_{211}$ by L_3) and $L_{61} + L_{331} = 2L_{43} + L_{322} + L_{31111}$.

127. (a) *First proof.* Sum equation (13) on all $\lambda \vdash n$. The right-hand side becomes

$$\sum_{w \in \mathfrak{S}_n} F_{\mathrm{co}(w)} = p_1^n.$$

Second proof. We can give a purely computational proof, avoiding the Gessel-Reutenauer result, as follows. By equation (7.191) and the definition $L_{\langle k^r \rangle} = h_r[L_k]$ of Exercise 7.89(f), for fixed k we have

$$\sum_{r \geq 0} L_{\langle k^r \rangle} = \exp \sum_{n \geq 1} \frac{1}{nk} \sum_{d|k} \mu(d) p_{nd}^{k/d}. \qquad (31)$$

By the multiplicative property $L_\lambda = L_{\langle 1^{m_1} \rangle} L_{\langle 2^{m_2} \rangle} \cdots$ of Exercise 7.89(f), we have

$$\sum_{\lambda \in \mathrm{Par}} L_\lambda = \prod_{k \geq 1} \left(\sum_{r \geq 0} L_{\langle k^r \rangle} \right).$$

Hence by equation (31) we get

$$\sum_{\lambda \in \mathrm{Par}} L_\lambda = \exp \sum_{n \geq 1} \sum_{k \geq 1} \frac{1}{kn} \sum_{d|k} \mu(d) p_{nd}^{k/d}.$$

Set $N = dn$ and $j = k/d$. We get

$$\sum_{\lambda \in \mathrm{Par}} L_\lambda = \exp \sum_{N \geq 1} \sum_{j \geq 1} \frac{1}{jN} p_N^j \sum_{d|N} \mu(d).$$

Since $\sum_{d|N} \mu(d) = \delta_{1N}$ (Kronecker delta), we finally obtain

$$\begin{aligned} \sum_{\lambda \in \mathrm{Par}} L_\lambda &= \exp \sum_{j \geq 1} \frac{1}{j} p_1^j \\ &= \exp \log(1 - p_1)^{-1} \\ &= \frac{1}{1 - p_1} \\ &= \sum_{n \geq 0} p_1^n, \end{aligned}$$

and the proof follows.

(b) Similar to the second proof of (a).

(c) Also similar to the second proof of (a). Let

$$Y=\prod_{k\geq 1}\left(\sum_{r\geq 0}(-1)^{(k-1)r}L_{\langle k^r\rangle}\right).$$

Since $\varepsilon_\lambda=1$ if and only if λ has an even number of even parts, we have (using (a))

$$\sum_{\substack{\lambda\in\mathrm{Par}\\ \varepsilon_\lambda=1}}L_\lambda=\frac{1}{2}\left(\frac{1}{1=p_1}+Y\right)$$

Now

$$Y=\exp\sum_{n\geq 1}\sum_{k\geq 1}\frac{(-1)^{(k-1)n}}{kn}\sum_{d|k}\mu(d)p_{nd}^{k/d}.$$

Making the same substitutions $N=dn$ and $j=d/k$ gives

$$Y=\exp\sum_{N\geq 1}\sum_{j\geq 1}\frac{1}{jN}p_N^j\sum_{d|N}(-1)^{N(jd-1)/d}\mu(d).$$

Note that

$$\sum_{d|N}(-1)^{N/d}\mu(d)=\begin{cases}-1, & n=1\\ 2, & n=2\\ 0, & n\geq 3.\end{cases}$$

It follows that

$$\begin{aligned}Y &= \exp\sum_{j\geq 1}\left(\frac{(-1)^{j-1}}{j}p_1^j+\frac{1}{j}p_2^j\right)\\ &= \exp\left(\log(1+p_1)-\log(1-p_2)\right)\\ &= \frac{1+p_1}{1-p_2},\end{aligned}$$

and the proof follows.

128. (a) Follows readily from Problem 126 above and the formula for $L_\alpha(1^m)$ preceding Proposition 7.19.12.

(b) Note that $f_j(n) = L_j(1^n)$. For any $f, g \in \Lambda$ we have

$$f[g](1^n) = f(1^n)|_{n\to g(1^n)}\,, \tag{32}$$

i.e., take the composition of the polynomial $f(1^n)$ with the polynomial $g(1^n)$. Equation (32) is proved by verifying it for $f = p_\lambda$ and using the linearity of $f[g]$ with respect to f. Moreover, $h_k(1^n) = \left(\!\binom{n}{k}\!\right)$ (Section 7.5). The proof now follows from the definition of L_λ in Exercise 7.89.

(c) Follows readily from Problem 127(c) and the formula for $L_\alpha(1^m)$ near the end of Section 7.19. This result is due to J. Fulman, G. B. Kim, S. Lee, and T. K. Petersen, *Electronic J. Comb.* **28** (2021), P3.37 (Theorem 1.1). This paper contains numerous other results related to the joint distribution of descents and signs of permutations in the symmetric group and hyperoctahedral group.

129. (a) By Problems 126 and 127(c) we have

$$\begin{aligned}\gamma_n(S) &= \left\langle \sum_{\substack{\lambda\vdash n\\ \varepsilon_\lambda=1}} L_\lambda, s_{B_S}\right\rangle.\\ &= \left\langle \frac{1}{2}\left(p_1^n + p_2^{n/2}\right), s_{B_S}\right\rangle.\end{aligned}$$

We have that $\langle p_1^n, s_{B_S}\rangle = f^{B_S} = \beta_n(S)$. In order to compute $\left\langle p_2^{n/2}, s_{B_S}\right\rangle$, we use the Murnaghan-Nakayama rule (version given by Corollary 7.17.5 and equation (7.75)). Let D be the unique domino tiling of B_S. Clearly border-strip tableaux of shape B_S and type $\langle 2^{n/2}\rangle$ correspond to standard Young tableaux of shape $B_{S/2}$, and all have weight $(-1)^{v(B_S)}$. The proof follows.

Note the curious consequence: if n is even then we can never have $\gamma_n(S) = \frac{1}{2}\beta_n(S)$. Of course this is nontrivial only when $\beta_n(S)$ is even.

(b) Now we must compute $\left\langle p_1 p_2^{(n-1)/2}, s_{B_S}\right\rangle$ via the Murnaghan-Nakayama rule. First remove a border strip of size one (i.e., a *corner square*, or a square with no square below it or to the right) from B_S; then cover the remaining squares with dominos. When we remove a corner square u the diagram B_S breaks up into two border strips B_T and B_U (one of which is possibly empty, or when $n = 1$ both are empty). Only the case when B_T (and hence also B_U) have an even number of squares will contribute to the answer, in which case we call u an *even corner square*. If B_S has m squares, then the contribution will be

$$g(u) := (-1)^{v(B_T)+v(B_U)} \binom{(n-1)/2}{m/2} \beta_{m/2}(S_T)\beta_{(n-m-1)/2}(S_U).$$

Thus

$$\gamma_n(S) = \frac{1}{2}\left(\beta_n(S) + \sum_u g(u)\right),$$

where u ranges over all even corner squares of B_S.

(c) Let n be even. This is the special case $S = \{1, 3, 5, \ldots, n-1\}$ of part (a). All $n/2$ dominos are vertical, so $v(B_S) = (-1)^{n/2}$. Moreover, $B_{S/2}$ is just a single row of size $n/2$, and the proof follows.

Let n be odd, so $S = \{1, 3, 5, \ldots, n-2\}$. For $n > 1$ there is no corner square u that splits B_S into two pieces of even size, so $E^*(n) = \frac{1}{2}E_n$.

(d) Let n be even. Thus $S = \{2, 4, 6, \ldots, n-2\}$ and $v(B_S) = 0$. Moreover $S/2$ is a single column of size $n/2$, and the proof follows.

Let n be odd. Then $S = \{2, 4, 6, \ldots, n-1\}$. There are $(n+1)/2$ even corner squares u. Label them $0, 1, \ldots, (n-1)/2 = m$ from bottom to top. When we remove corner square i, we get the contribution $(-1)^i\binom{m}{i}$. Since $\sum_{i=0}^{m}(-1)^i\binom{m}{i} = 0$ for $m > 1$, the proof follows.

NOTE. The value of E_n^* and E_n' when n is odd is also an immediate consequence of Theorem 1.1 (due to F. Ruskey) or Corollary 2.2 of R. Stanley, *Advances in Applied Math.* **34** (2005), 880–902; `arXiv:math/0211113`. It is also easy to see that the formula for either of E_n^* and E_n' implies the other, since a reverse alternating permutation in $\mathfrak{S}_n$ is obtained from an alternating permutation in $\mathfrak{S}_n$ by multiplying on the right by $w_0 = n, n-1, \ldots, 1$. (Consider whether w_0 is even or odd.)

(e) The condition on S is equivalent to $B_{S/2} = \emptyset$. Moreover, $v(B_S) = \#S$. The proof follows easily.

130. Let $\mathfrak{o}$ be an orbit of the action of $\mathfrak{S}_k$ on $\mathcal{C}_k$. Let $G_\mathfrak{o}(p_1, p_2, \dots)$ be the Frobenius characteristic symmetric function of the action of $\mathfrak{S}_k$ on $\mathfrak{o}$. Thus $\mathfrak{o}$ corresponds to an unlabelled structure σ. It follows from Theorem A2.8 that the action of $\mathfrak{S}_{kn}$ on n disjoint copies of σ is given by $h_n[G_\mathfrak{o}]$. Now it follows directly from the definition of plethysm that if $f(x_1, x_2, \dots) \in \hat{\Lambda}$, then

$$p_n[f] = f[p_n] = f(x_1^n, x_2^n, \dots).$$

Hence by equation (34),

$$\begin{aligned} F_\mathfrak{o} &:= \sum_{n\geq 0} h_n[G_\mathfrak{o}]t^{kn} \\ &= \exp \sum_{n\geq 1} \frac{1}{n} p_n[G_\mathfrak{o}]t^{kn} \\ &= \exp \sum_{n\geq 1} \frac{1}{n} G_\mathfrak{o}(p_n, p_{2n}, p_{3n}, \dots)t^{kn}. \end{aligned}$$

It follows from Proposition 7.18.2 that

$$F_n = \prod_{m\geq 1} \prod_{\mathfrak{o}} F_\mathfrak{o},$$

where $\mathfrak{o}$ ranges over all orbits of the action of $\mathfrak{S}_m$ on $\mathcal{C}_m$, and the proof follows.

See equation (20)c on page 46 of F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-Like Structures.*

131. (a) Suppose that $f(n) = \beta(S,T)$. By Corollary 7.23.8 we have $\beta(S,T) = \langle s_{B_S}, s_{B_T} \rangle$. Now

$$\langle s_{B_S}, s_{B_S} \rangle - 2\langle s_{B_S}, s_{B_T} \rangle + \langle s_{B_T}, s_{B_T} \rangle = \langle s_{B_S} - s_{B_T}, s_{B_S} - s_{B_T} \rangle \geq 0.$$

Hence $\frac{1}{2}(\beta(S,S) + \beta(T,T)) \geq \beta(S,T)$. Therefore either $\beta(S,S) \geq \beta(S,T)$ or $\beta(T,T) \geq \beta(S,T)$, and the proof follows.

(b) This problem was raised by Ira Gessel (private communication, 2007).

132. (a) This result can be deduced from #126, using the fact that a multiset permutation (of a totally ordered set) and its standardization have the same descent set, together with [why?]

$$h_\alpha = \sum_{S_\beta \subseteq S_\alpha} s_{S_\beta}.$$

(b) See Theorem 2.3 of R. Stanley, *Electronic J. Combinatorics* **4** (1997), R20.

(c) The set of all parking functions is a union of sets $\mathfrak{S}_M$ of all permutations of a multiset M. The proof follows from (a) and the definition of PF_n.

(d) We have $L_n = L_{(n)} = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d}$ (equation (7.191)) and

$$\mathrm{PF}_n = \sum_{\lambda \vdash n}(n+1)^{\ell(\lambda)-1}z_\lambda^{-1}p_\lambda$$

(by applying ω to the first formula of Exercise 7.48(f)). The proof follows from the orthogonality relation $\langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda\mu}$ (Propostition 7.9.3).

133. (a,b,d) See Section 5 (the case $r = 1$) of R. Stanley and Y. Wang (王颖慧), *J. Combinatorial Theory, Ser. A* **159** (2018), 54–78.

(e) By Problem 132(b) above and Exercise 7.48(f), we have that the two series

$$\begin{aligned} F(t) &= t + \mathrm{PF}_1 t^2 + \mathrm{PF}_2 t^3 + \cdots, \text{ and} \\ G(t) &= t - e_1 t^2 + e_2 t^3 - \cdots \end{aligned}$$

are compositional inverses. Equating coefficients of t^{n+1} on both sides of the identity $t = F(G(t))$ yields an expansion of PF_n as a polynomial in the symmetric functions $(-1)^n e_n$. Doing the same for $t = G(F(t))$ yields an expansion of $(-1)^n e_n$ as a polynomial in the symmetric functions PF_n with exactly the same coefficients. A simple homogeneity argument shows the same is true using e_n instead of $(-1)^n e_n$. Since both e_λ and PF_λ are multiplicative bases, it follows that $R_n = R_n^{-1}$.

(f) Immediate from $R_n^2 = I$.

(h) Follows from applying ω to Problem 6 and using Problem 132(b) and part (f) of the present problem.

(i) It is easy to see that the e-expansion of PF_n has the form $\mathrm{PF}_n = C_n e_1^n + \cdots$. The proof follows from the definition $\mathrm{PF}_\lambda = \mathrm{PF}_{\lambda_1}\mathrm{PF}_{\lambda_2}\cdots$.

(j) Generalizing (i) above, it is not hard to show that

$$\mathrm{PF}_n = C_n e_1^n - \frac{n-1}{2} C_n e_2 e_1^{n-2} + \cdots .$$

Thus if $\lambda \vdash n$, then the coefficient of $e_2 e_1^{n-2}$ in PF_λ is

$$-\frac{1}{2}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - 1)\cdot \prod C_{\lambda_i} = -\frac{1}{2}(n-\ell(\lambda)\prod C_{\lambda_i}.$$

134. See F. Chyzak, M. Mishna, and B. Salvy, *J. Combinatorial Theory, Ser. A* **112** (2005), 1–43, `arXiv.math/0310132` (Proposition 9.1).

135. *Answer:* the column vector $[\chi^\lambda(\nu)]_{\lambda\vdash n}$. Once the answer is guessed, the verification is straightforward.

136. (a) There are many approaches. We give an elementary argument. The character χ of the defining representation is given by $\chi(w) = \#\mathrm{Fix}(w)$, the number of fixed points of w. Thus

$$s_n + s_{n-1,1} = \sum_{\lambda\vdash n} z_\lambda^{-1} m_1(\lambda) p_\lambda,$$

where $m_1(\lambda)$ is the number of parts of λ equal to one. Hence by definition of internal product,

$$(s_n + s_{n-1,1})^{*k} = \sum_{\lambda\vdash n} z_\lambda^{-1} m_1(\lambda)^k p_\lambda.$$

Now $s_n = \sum_{\lambda\vdash n} z_\lambda^{-1} p_\lambda$, so

$$\sum_{\lambda\vdash n} z_\lambda^{-1} m_1(\lambda)^k p_\lambda = \left(p_1 \frac{\partial}{\partial p_1}\right)^k s_n.$$

It is formal consequence of the commutation relation

$$p_1\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_1}p_1 = 1$$

(where the 1 on the right-hand side denotes the identity operator) that

$$\left(p_1 \frac{\partial}{\partial p_1}\right)^k = \sum_{i=1}^{k} S(k,i) p_1^i \frac{\partial^i}{\partial p_1^i}. \tag{33}$$

(See Exercise 3.209.) Since $\frac{\partial^i}{\partial p_1^i} s_n = s_{n-i}$, the proof follows.

A rather similar argument, but formulated in terms of representation theory, is given by Dan Petersen at MathOverflow 284054. The earliest related reference of which we are aware is A. Goupil and C. Chauve, *Sémin. Lothar. Comb.* **54** (2005), B54j; `arXiv:math/0503307`.

♣ Who first looked at this problem? What are some other references?

(b) The value of the character of this $\mathfrak{S}_n$-action on a permutation of cycle type λ is $\binom{m_1}{2} + m_2$. Hence

$$(s_n + s_{n-1,1} + s_{n-2,2})^{*k} = \left(\frac{1}{2} p_1^2 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial}{\partial p_2}\right)^k s_n.$$

Thus in analogy with equation (33) we need to find the coefficients c_{ij} so that

$$\left(\frac{1}{2} p_1^2 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial}{\partial p_2}\right)^k = \sum_{i,j} c_{ij} p_1^i p_2^j \frac{\partial^i}{\partial p_1^i} \frac{\partial^j}{\partial p_2^j}.$$

We can then use $\frac{\partial^i s_n}{\partial p_1^i} = s_{n-i}$ (as before) and $\frac{\partial^j s_n}{\partial p_2^j} = 2^{-j} s_{n-2j}$.

137. Open.

138. This result is equivalent to a result of E. Egge, N. Loehr, and G. Warrington, *European J. Combin.* **31** (2010), 2014–2027. An elucidation was given by A. M. Garsia and J. B. Remmel, `arXiv:1802.09686`, with a simplification by I. M. Gessel, *Electronic J. Combin.* **26** (2019), P4.50, `arXiv:1803.09271`.

139. Immediate from Theorem 7.19.7 and Corollary 7.23.6 (setting $f = s_\lambda$).

140. (a) Let $\alpha = \mathrm{co}(S)$ and $\overline{B}_\alpha$ the rotation of B_α by 180° as in Exercise 7.56. Hence by this exercise we have $s_{B_\alpha} = s_{\overline{B}_\alpha}$. Given an

SSYT of shape $\overline{B}_\alpha$ such as

$$\begin{array}{rrrr} & & 2\,2\,3 \\ & 3\,3 & \\ & 4 & \\ 2\,6 & & \end{array}$$

for $n = 8$ and $S = \{2, 3, 5\}$, simply read the entries from left-to-right and bottom-to-top. For example above we get the sequence 26433223. This sets up a bijection between sequences $u = u_1 \cdots u_n$ with descent set S and terms $x_{u_1} \cdots x_{u_n}$ of s_{B_α}.

(b) Straightforward consequence of EC1, second ed., Exercise 4.40.

141. (a) Let $\lambda \vdash n$. The largest $\alpha \in \mathrm{Comp}(n)$ in lexicographic order such that $[L_\alpha]s_\lambda \neq 0$ is given by $\alpha = \lambda$. It follows that [why?] $f = \sum_{\mu \leq_L \lambda} a_\mu s_\mu$, where $a_\mu \in \mathbb{Z}$ and $\leq_L$ denotes lex order. Similarly the smallest $\beta \in \mathrm{Comp}(n)$ in lex order such that $[L_\beta]s_\lambda \neq 0$ is defined by $[n-1] - S_\beta = S_{\lambda'}$. Thus $f = \sum_{\nu \geq_L \lambda} b_\mu s_\mu$ where $b_\mu \in \mathbb{Z}$. Hence $f_\lambda = c s_\lambda$ for some $c \in \mathbb{Q}$. Since the s_λ's form an integral basis, we must have $c = 0$ or 1. This result and proof are due to L. Billera.

(b) The smallest example is

$$s_{31} + s_{211} - s_{22} = L_{31} + L_{13} + L_{211} + L_{112}.$$

It is an open (and probably very difficult) problem to find all L-positive symmetric functions u such that if $u = f + g$, where $f, g \in \Lambda$ and f, g are L-positive, then $f = 0$ or $g = 0$.

142. Since $h_n = \frac{1}{n!}\sum_{w \in \mathfrak{S}_n} p_{\rho(w)}$ and $e_n = \frac{1}{n!}\sum_{w \in \mathfrak{S}_n} \varepsilon_w p_{\rho(w)}$, we get

$$h_n + e_n = \frac{2}{n!}\sum_{w \in \mathfrak{A}_n} p_{\rho(w)}.$$

Hence $Z_{\mathfrak{A}_n} = h_n + e_n = s_n + s_{1^n}$.

143. We have $c_\mu = \langle \mathrm{ch}(\chi), h_\mu \rangle$. Now h_μ is the Frobenius characteristic of the induction 1_μ of the trivial representation $1_{\mathfrak{S}_\mu}$ from $\mathfrak{S}_\mu$ to $\mathfrak{S}_n$. Hence $\langle \mathrm{ch}(\chi), h_\mu \rangle = \langle \chi, 1_\mu \rangle$. By Frobenius reciprocity this is $\langle \chi|_\mu, 1_{\mathfrak{S}_\mu} \rangle$, as desired. This result for permutation representations is due to V. Dotsenko, `arXiv:0802.1340`.

144. (a) This follows from the fact that for $\lambda \vdash n$, $\mathrm{ex}(s_\lambda)|_{t=1} = f^\lambda/n! > 0$, while

$$\mathrm{ex}\,(p_\lambda)|_{t=1} = \begin{cases} 1, & \lambda = (1^n) \\ 0, & \text{otherwise.} \end{cases}$$

Here ex denotes the exponential specialization (EC2, pp. 304–306).

145. (a) Clearly $\mathfrak{S}_n$ acts transitively on the set $\mathfrak{S}_n/G$ of left cosets of G. (In fact, for any finite group K, the transitive permutation representations of K correspond to the actions of K on left cosets of subgroups G.) On the other hand, by Burnside's lemma (Lemma 7.24.5) and the fact that $Z_G = \mathrm{ch}\,1_G^{\mathfrak{S}_n}$ (equation (7.119)), we have that the number of orbits is $\langle Z_G, s_n\rangle$. Thus $\langle Z_G, s_n\rangle = 1$. If $Z_G = \sum a_\lambda h_\lambda$, then $a_\lambda \in \mathbb{Z}$ since $\mathrm{ch}\,\chi^\lambda = s_\lambda$ and both the s_λ's and h_λ's are an integral basis. Moreover, if $\lambda \vdash n$ then $\langle h_\lambda, s_n\rangle = K_{n,\lambda} = 1$. Hence if $Z_G = \sum_\lambda a_\lambda h_\lambda$ and Z_G is h-positive, then

$$1 = \langle Z_G, s_n\rangle = \sum_\lambda a_\lambda,$$

where each $a_\lambda \in \mathbb{N}$. It follows that $Z_G = h_\lambda$ for some $\lambda \vdash n$.

(b) Assume that $Z_G = h_\lambda = Z_{\mathfrak{S}_\lambda}$, and let $\ell = \ell(\lambda)$. The number of orbits of G itself acting on $1, 2, \ldots, n$ is by Burnside's lemma the average number of fixed points of elements of G and hence is determined by Z_G. Since the action on left cosets of $\mathfrak{S}_\lambda$ has ℓ orbits, the same is true for G. Now $\mathfrak{S}_\lambda$ contains an element of cycle type λ, so G also contains such an element w. Since w has exactly ℓ cycles and G has ℓ orbits, it follows that $G \subseteq H$, where H is a subgroup of $\mathfrak{S}_n$ conjugate to $\mathfrak{S}_\lambda$. But $[p_1^n]Z_G = 1/\#G$, so $\#G = \#H$. Thus $G = H$, as desired.

146. The identity is equivalent to

$$1 + \left(\sum_{n\geq 1} \widetilde{Z}_{\mathfrak{I}_n} x^n\right)\left(\sum_{n\geq 0} n! h_n x^n\right) = \sum_{n\geq 0} n! h_n x^n.$$

Equating coefficients of x^n gives

$$\sum_{k=1}^{n} \tilde{Z}_{\mathfrak{I}_k}(n-k)!\, h_{n-k} = n!\, h_n.$$

For every permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$, there is a unique $1 \le k \le n$ for which $u := a_1 \cdots a_k \in \mathfrak{I}_k$. For fixed k, the cycle indicator of all such u is $\tilde{Z}_{\mathfrak{I}_k}$. For any such u, the cycle indicator of the remaining terms $a_{k+1} \cdots a_n$ is $(n-k)!\, h_{n-k}$ by Exercise 7.111(a). Hence for fixed k, the cycle indicator of all such w is $\tilde{Z}_{\mathfrak{I}_k}(n-k)!\, h_{n-k}$, and the proof follows by summing on k. This proof is completely analogous to the sketched proof in the solution to Exercise 1.128(a) of EC1, second ed.

147. (a,b) This is due to Brendan Pawlowski. See MathOverflow #254782.

(c) Take $n = 4$ and the three partitions to be 12-3-4, 13-2-4, and 14-2-3. Take the three characters to be trivial. We get

$$f = p_1^4 + 3p_1^2 p_2 + 3p_1 p_3 + p_4 = 8s_4 + 5s_{31} - s_{22} + s_{211}.$$

148. S. C. Billey, B. Rhoades, and V. Tewari, Boolean product polynomials, Schur positivity, and Chern plethysm, *Int. Math. Res. Not. IMRN* (2019); `arXiv:1902.11165`.

149. (a) Suppose that the cycle c of $w_1 \cdots w_k$ containing 1 intersects A_j in i_j elements, so the length of this cycle is $i_1 + \cdots + i_k - k + 1$. The cycle c will have the form

$$c = (1, u_1, u_2, \ldots, u_k),$$

where u_j is a sequence of $i_j - 1$ distinct elements of $A_j - \{1\}$. Hence there are $\prod_{j=1}^k (a_j - 1)_{i_j - 1}$ choices for c. The elements of A_j not in c can be any permutation of size $a_j - i_j$. The generating function by cycle type of these permutations is

$$\sum_{w \in \mathfrak{S}_{a_j - i_j}} p_{\rho(w)} = (a_j - i_j)! h_{a_j - i_j}.$$

Thus the terms of the symmetric function (15) for fixed $a_1, \ldots, a_k, i_1, \ldots, i_k$ yield

$$\left(\prod_{j=1}^k (a_j - 1)_{i_j - 1} (a_j - i_j)! \right) \cdot p_{i_1 + \cdots + i_k - k + 1} x_1^{i_1} \cdots x_k^{i_k} h_{a_1 - i_1}(x_1) \cdots h_{a_k - i_k}(x_k).$$

Note that $(a_j - 1)_{i_j - 1}(a_j - i_j)! = (a_j - 1)!$. Divide by $\prod (a_j - 1)!$ and sum on $i_1, \ldots, i_k \ge 1$ and $a_1, \ldots, a_k \ge 0$ to complete the proof.

(b) This result is due to Miriam Farber in 2015. The s-expansion uses the Schur function expansion of $h_i h_j$ and the Murnaghan-Nakayama rule. The h-expansion uses the s-expansion and the Jacobi-Trudi identity. Are there more conceptual proofs?

(c) $G_{2,2,2} = 8s_4 + 5s_{31} - s_{22} + s_{211}$

(d) This conjecture is due to M. Farber.

(e) Farber has a few more sporadic results and conjectures.

150. *Answer.* Suppose that $G \subseteq \mathfrak{S}_S$. Suppose our set of colors is $X \cup \overline{X}$, where $X = \{c_1, c_2, \dots\}$ and $\overline{X} = \{\overline{c}_1, \overline{c}_2, \dots\}$. Let $f : S \to X \cup \overline{X}$ be a coloring of S. Define

$$H_f = \{w \in G \,:\, w \cdot f = f\},$$

the subgroup of G fixing the coloring f, and let $\overline{H}_f$ be the restriction of H_f to $f^{-1}(\overline{X})$. Call f a *G-super coloring* if $\overline{H}_f$ does not contain an odd permutation. For instance, let $S = \{1, 2, 3, 4\}$ and let G be generated by the 4-cycle $(1, 2, 3, 4)$. Write $c_1 = a$ and let f be the coloring $a\overline{a}a\overline{a}$ of 1234. Then $H_f = \{(1)(2)(3)(4), (1, 3)(2, 4)\}$ and $\overline{H}_f = \{(2)(4), (2, 4)\} \not\subseteq \mathfrak{A}_2$ (although $H_f \subseteq \mathfrak{A}_4$). Hence f is not a G-super coloring.

Theorem. *The coefficient of $x^\alpha y^\beta$ in $Z_G(x/y)$ is equal to the number of G-orbits of G-super colorings f such that $\alpha_i = \#f^{-1}(c_i)$ and $\beta_i = \#f^{-1}(\overline{c}_i)$ for all i.*

151. (a) See R. Stanley, *Advances in Applied Math.* **34** (2005), 880–902; `arXiv:math/0211113` (Prop. 3.1).

(b) See R. Stanley, Problem 10969, *Amer. Math. Monthly* **109** (2002), 760 (published solution by O. P. Lossers, **111** (2004), 536–539). For a generalization, see C. E. Boulet, *Ramanujan J.* **12** (2006), 315–320, `arXiv.math/0308012`, repeated in EC1, second ed., Exercise 1.83.

152. Let $F(t) = \operatorname{ex} f$ and $G(t) = \operatorname{ex} g$. We claim that $\operatorname{ex} f[g] = F(G(t))$. Since the maps ex and $f \mapsto f[g]$ are homomorphisms, it suffices to take $f = p_n$, in which case the computation is straightforward.

153. There are many approaches. One way is to note that $h_2 = \frac{1}{2}(p_1^2 + p_2)$. Hence

$$h_2[h_n] = \frac{1}{2}(h_n^2 + h_n(x_1^2, x_2^2, \dots)).$$

Now

$$\begin{aligned}\sum_{n\geq 0} h_n(x_1^2, x_2^2, \dots) &= \frac{1}{\prod(1-x_i^2)} \\ &= \frac{1}{\prod(1-x_i)(1+x_i)} \\ &= \left(\sum_{j\geq 0} h_j\right)\left(\sum_{k\geq 0}(-1)^k h_k\right),\end{aligned}$$

whence

$$h_2[h_n] = \frac{1}{2}\left(h_n^2 + \sum_{k=0}^{2n}(-1)^k h_k h_{2n-k}\right).$$

Expand $h_k h_{2n-k}$ into Schur functions by Theorem 7.17.5 (Pieri's rule) and collect terms to get

$$h_2[h_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} s_{(2n-2k,2k)}.$$

This result has been extended e.g. to $h_3[h_n]$ (see for instance S. P. O. Plunkett, *Canad. J. Math.* **24** (1972), 541–552), but becomes increasingly unmanageable for $h_4[h_n], h_5[h_n]$, etc.

154. Since $e_1^2 = \sum_{i,j} x_i x_j$ and

$$\sum_{n\geq 0} e_n = \prod(1+x_i),$$

we have

$$\sum_{n\geq 0} e_n[e_1^2] = \prod_{i,j}(1+x_i x_j).$$

Setting $y_i = x_i$ in the dual Cauchy identity gives

$$\prod_{i,j}(1+x_i x_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(x).$$

Taking the degree $2n$ part gives

$$e_n[e_1^2] = \sum_{\lambda \vdash n} s_\lambda(x) s_{\lambda'}(x).$$

155. Follows from the fact that the operation $g \mapsto g[f]$ is a (continuous) ring homomorphism and the formula

$$\sum_{n \geq 0} h_n t^n = \exp \sum_{n \geq 1} \frac{p_n}{n}. \tag{34}$$

If f and g are sums of monomials, the proof also follows from the fact that when M is a multiset of monomials and $r = \sum_{u \in M} u$, then

$$\sum_{n \geq 0} h_n[r] = \frac{1}{\prod_{u \in M}(1-u)}.$$

See Lemma 2.3 of I. Gessel and Y. Zhuang, *Adv. Math.* **375** (2020), 107370; `arXiv:2001.00654`.

Note that the exponential specialization of equation (16) is equivalent to $e^{a+b} = e^a e^b$.

156. (a) *Answer.* Let $F(x) = \sum a_n x^n$ and let $F(x)^{\langle -1 \rangle} = \sum b_n x^n$ be its compositional inverse. Then the plethystic inverse of f is $\sum b_n p_1^n$ (easy to prove e.g. from the fact that $p_1^n[p_1^m] = p_1^{mn}$).

(b) *Answer.* Define $\delta : \mathbb{P} \to \mathbb{Z}$ by $\delta(n) = \delta_{1n}$ (the Kronecker delta). Note that δ is the identity for $*$, i.e., $F * \delta = \delta * F = F$ for all F. Since $a_1 \neq 0$, the function a_n possesses a unique *Dirichlet inverse* b_n, i.e., $a * b = \delta$. (For instance, if $a_n = 1$ for all n then $b_n = \mu(n)$, the number-theoretic Möbius function.) Then the plethystic inverse of f is $\sum_{n \geq 1} b_n p_n$.

157. See S. Sundaram, The plethystic inverse of the odd Lie representations Lie_{2n+1}, `arXiv:2003.10700`.

158. Let E_{ij} be the matrix in $\mathrm{Mat}(n, \mathbb{C})$ with a 1 in the (i,j)-position and 0's elsewhere, as in Appendix A2, following Theorem A2.4. If $A = \mathrm{diag}(\theta_1, \ldots, \theta_n)$ then $AE_{ij} = \theta_i E_{ij}$. Hence the eigenvalues of A acting on $\mathrm{Mat}(n, \mathbb{C})$ are just $\theta_1, \ldots, \theta_n$, n times each, so the character of the action is just $n(x_1 + \cdots + x_n) = n s_1(x_1, \ldots, x_n)$.