SUPPLEMENTARY EXERCISES (without solutions)

for Chapter 7 (symmetric functions) of

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1. [2] Find the number $f(n)$ of pairs (λ, μ) such that $\lambda \vdash n$ and μ covers λ in Young's lattice Y. Express your answer in terms of $p(k)$, the number of partitions of k, for certain values of k. Try to give a direct bijection, avoiding generating functions, recurrence relations, induction, etc.

2. [2] Let $p_r(n)$ denote the number of partitions of n of rank r. Find the generating function
$$F_r(t) = \sum_{n\geq 0} p_r(n)t^n.$$

3. [1] Express the symmetric function $p_1 e_\lambda$ in terms of elementary symmetric functions.

4. [2] Let
$$F_n(x) = (-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n) \\ \cdots (x_1 + x_2 + \cdots + x_{n-1} - x_n).$$
Show that
$$F_n(x) = \sum_{k=2}^{n} (-1)^k 2^k e_k e_1^{n-k} - e_1^n,$$
in the ring Λ_n of symmetric functions in n variables.

5. [2+] For what real numbers a is the symmetric formal power series $F(x) = \prod_i (1 + ax_i + x_i^2)$ e-positive, i.e., a nonnegative (infinite) linear combination of the e_λ's?

6. [2+] Find all symmetric functions $f \in \Lambda^n$ that are both e-positive and h-positive.

7. [1+] Find all $f \in \Lambda^n$ for which $\omega f = 2f$.

8. [2] Let $P(x)$ be a polynomial satisfying $P(0) = 1$. Express $\omega \prod_i P(x_i)$ as an infinite product.

9. [2–] Let $j, k \geq 1$. Expand the monomial symmetric function $m_{\langle k^j \rangle}$ as a linear combination of power sums p_λ.

10. (a) [3–] Let $\Lambda_{\mathbb{Z}}^n$ denote the (additive) abelian group with basis $\{m_\lambda\}_{\lambda \vdash n}$. Let $\Pi_{\mathbb{Z}}^n$ denote the subgroup generated by $\{p_\lambda\}_{\lambda \vdash n}$. Thus by the Note after Corollary 7.7.2,

$$[\Lambda_{\mathbb{Z}}^n : \Pi_{\mathbb{Z}}^n] = \prod_{\mu \vdash n} d_\mu,$$

where $d_\mu = \prod_{i \geq 1} m_i(\mu)!$. Show that in fact

$$\Lambda_{\mathbb{Z}}^n / \Pi_{\mathbb{Z}}^n \cong \bigoplus_{\mu \vdash n} \mathbb{Z}/d_\mu \mathbb{Z}.$$

(b) [2] (for readers familiar with Smith normal form) Let X_n denote the character table of $\mathfrak{S}_n$. Deduce from (a) that X_n has the same Smith normal form as the diagonal matrix with diagonal entries d_μ, $\mu \vdash n$.

11. [2–] Let $\alpha \in \mathbb{R}$ (or consider α to be an indeterminate). Expand the product $\prod_i (1 + x_i)^\alpha$ as an (infinite) linear combination of the power sums p_λ.

12. [2] Let $f(x, y) \in \Lambda(x) \otimes \Lambda(y)$, i.e., $f(x, y)$ is symmetric with respect to $x_1, x_2, \ldots$ and separately with respect to $y_1, y_2, \ldots$. Let $\frac{\partial}{\partial p_k(x)} f(x, y)$ denote the partial derivative of $f(x, y)$ with respect to $p_k(x)$ when $f(x, y)$ is written as a polynomial in the $p_i(x)$'s (regard the y_j's as constants). Find a simple formula for

$$\frac{\partial}{\partial p_k(x)} \prod_{i,j} (1 - x_i y_j)^{-1}.$$

13. [2+] Fix $n \geq 1$. Find a simple formula for the number of pairs $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $uv = vu$. Generalize to any finite group G instead of $\mathfrak{S}_n$.

14. [3] Fix $n \geq 1$, and let S be an n-element subset of $\mathbb{P}$. Show that the *field* $\mathbb{Q}(p_1(x_1,\dots,x_n), p_2(x_1,\dots,x_n),\dots)$ of all rational symmetric functions over $\mathbb{Q}$ in the variables $x_1,\dots,x_n$ is generated by $\{p_i(x_1,\dots,x_n) : i \in S\}$ if and only if $\mathbb{P} - S$ is closed under addition.

15. (a) [2+] Show that the symmetric power series

$$T = \frac{\sum_{n\geq 0} h_{2n+1}}{\sum_{n\geq 0} h_{2n}}$$

    is a power series in the *odd* power sums p_1, p_3, $p_5, \dots$.

    (b) [3–] Identify the coefficients when T is written as a power series in the power sums.

16. [5–] Define for $n \geq 3$ the symmetric function (in n variables)

$$H_n = \prod (x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n),$$

where the product ranges over all sequences $(\epsilon_2,\dots,\epsilon_n) \in \{-1,1\}^{n-1}$. Show that when $3H_n$ is expanded as a polynomial in the power sums p_j, the coefficients are integers. (This conjecture has been verified for $n \leq 5$ but is probably false.)

17. (a) [2+] Let p be a prime, and define the symmetric function

$$F_p = F_p(x_1,\dots,x_{2p-1}) = \sum_{\substack{S\subseteq[2p-1]\\ \#S=p}} \left(\sum_{i\in S} x_i\right)^{p-1},$$

    where the first sum ranges over all p-element subsets of $1,2,\dots,2p-1$. Show that when F_p is written as a linear combination of monomials, every coefficient is divisible by p.

    (b) [2+] Deduce from (a) the Erdős-Ginzburg-Ziv theorem: given any $(2p-1)$-element subset X of $\mathbb{Z}$, there is a p-element subset Y of X such that $\sum_{i\in Y} i \equiv 0 \pmod{p}$.

    (c) [2+] Show that when F_p is written as a linear combination of power sums p_λ, every coefficient is an integer divisible by p.

18. Given $f \in \Lambda^n_{\mathbb{Q}}$ and $k \in \mathbb{P}$, let $f(kx)$ denote the symmetric function f in k copies of each variable $x_1, x_2, \dots$. Thus for instance $m_1(kx) = km_1(x)$.

(a) [2–] Let $\{u_\lambda : \lambda \vdash n\}$ be a basis for $\Lambda^n_{\mathbb{Q}}$, and let

$$f(kx) = \sum_{\lambda \vdash n} c_\lambda(k) u_\lambda. \tag{1}$$

Show that $c_\lambda(k)$ is a polynomial in k (with rational coefficients). This allows us to use equation (1) to *define* $f(kx)$ for *any* k (in some extension field F of $\mathbb{Q}$, say).

(b) [2–] For any $j \in F$, let $g(x) = f(jx)$. For any $k \in F$ show that $g(kx) = f(jkx)$.

(c) [2] Express $f(-x)$ in terms of $f(x)$ and ω.

19. [2+] Evaluate the scalar product $\langle h_{21^{n-2}}, h_{21^{n-2}} \rangle$.

20. [2+] Fix $n \geq 1$. Find the dimension of the subspace of $\Lambda^n_{\mathbb{Q}}$ spanned by $\{h_\lambda + e_\lambda : \lambda \vdash n\}$.

21. (a) [3–] Define a linear transformation $\varphi : \Lambda^n_{\mathbb{Q}} \to \Lambda^n_{\mathbb{Q}}$ by $\varphi(e_\lambda) = m_{\lambda'}$. Find the size of the largest block in the Jordan canonical form of φ.

    (b) [5–] Find the entire Jordan canonical form of φ.

    (c) [5–] Do the same for such linear transformations as $e_\lambda \mapsto m_\lambda$, $h_\lambda \mapsto m_\lambda$, $p_\lambda \mapsto m_\lambda$, $p_\lambda \mapsto h_\lambda$.

22. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product on $\Lambda_{\mathbb{Q}}$. Two linear transformations $A, B : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ are *adjoint* if $\langle Af, g \rangle = \langle f, Bg \rangle$ for all $f, g \in \Lambda_{\mathbb{Q}}$.

    (a) [2] Find the adjoint to $\omega + aI$, where I denotes the identity transformation and a is a constant.

    (b) [2] Let $\frac{\partial}{\partial p_i} f$ denote the partial derivative of $f \in \Lambda$ with respect to p_i when f is written as a polynomial in $p_1, p_2, \ldots$. Define the linear transformation M_j by $M_j(f) = p_j f$. Express the adjoint of M_j in terms of the operators $\frac{\partial}{\partial p_i}$.

23. [2] Let $k \geq 2$. Compute the Kostka number $K_{(k,k,k),(k-1,k-1,1^{k+2})}$.

24. (a) [2+] Let $\lambda \vdash n$ and $\lambda \subseteq \langle k^n \rangle$. Give a bijective proof that

$$K_{\langle k^n \rangle / \lambda, \langle (k-1)^n \rangle} = f^\lambda.$$

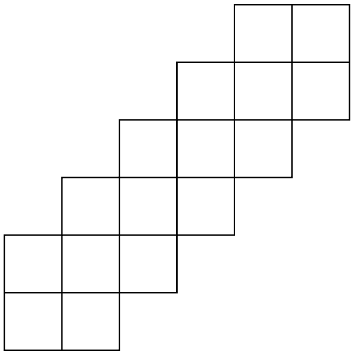
(b) [2] Deduce from (a) that $K_{\langle k^n\rangle,\langle (k-1)^n,1^n\rangle}$ is equal to the number of permutations in $\mathfrak{S}_n$ with no increasing subsequence of length $k+1$.

25. [2–] How many SYT of shape (n^n) have main diagonal $(1,4,9,16,\dots,n^2)$?

26. [3] Let $\delta_m=(m-1,m-2,\dots,1)$. Define skew shapes

$$\begin{aligned}\alpha_n &= (n,n,n-1,n-2,\dots,2)/\delta_{n-1}\\ \beta_n &= (n,n,n,n-1,n-2,\dots,2)/\delta_n\\ \gamma_n &= (n,n,n,n-1,n-2,\dots,1)/\delta_n.\end{aligned}$$

For instance, the diagram below shows α_6.



Show that

$$\begin{aligned}f^{\alpha_n} &= \frac{(3n-2)!E_{2n-1}}{(2n-1)!2^{2n-2}}\\ f^{\beta_n} &= \frac{(3n-1)!E_{2n-1}}{(2n-1)!2^{2n-1}}\\ f^{\gamma_n} &= \frac{(3n)!(2^{2n-1}-1)E_{2n-1}}{(2n-1)!2^{2n-1}(2^{2n}-1)},\end{aligned}$$

where E_{2n-1} denotes an Euler number.

27. [2–] For any partitions λ and μ, express $s_\lambda s_\mu$ as a skew Schur function.

28. [3–] Let λ/μ be a skew shape. Let M_i be the set of all skew shapes obtained from λ/μ by removing a vertical strip of size i from μ (i.e., adding this strip to the inner boundary of λ/μ) and adding a horizontal

strip of size $k-i$ to λ (i.e., adding this strip to the outer boundary of λ/μ). Show that

$$s_k s_{\lambda/\mu} = \sum_{i=0}^{k} (-1)^i \sum_{\rho \in M_i} s_\rho.$$

29. (a) [2+] Let λ be a partition and $m \geq \lambda_1$. Let $\lambda \cup m$ denote the partition obtained by adding a part of length m to λ. Let $e_i^\perp$ denote the linear operator on symmetric functions adjoint to multiplication by e_i. Show that

    $$s_{\lambda \cup m} = \left(\sum_{i \geq 0} (-1)^i h_{m+i} e_i^\perp \right) s_\lambda.$$

    (b) [2+] Let $1 \leq k \leq n/2$. Let $f_k(n)$ be the number of permutations $w = a_1 \cdots a_n \in \mathfrak{S}_n$ such that $a_1 < a_2 < \cdots < a_{n-k}$, and the longest increasing subsequence of w has length exactly $n-k$. Show that

    $$f_k(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n)_{k-i}.$$

    (c) [3–] Is there a "nice" proof of (b) based on the Principle of Inclusion-Exclusion?

30. [2–] Let $1 \leq k \leq n$ and $\lambda = (k, 1^{n-k})$ (called a *hook shape*). For any $\mu \vdash n$ find a simple formula for the Kostka number $K_{\lambda\mu}$.

31. [2+] Let A be the $m \times n$ matrix of all 1's. If $A \overset{\mathrm{rsk}}{\to} (P, Q)$, then describe (with proof) the SSYT's P and Q.

32. (a) [3] Let λ be a partition with distinct parts. A *shifted standard tableau* (SHSYT) of shape λ is defined just like an ordinary standard Young tableau of shape λ, except that each row is indented one space to the right from the row above. An example of an SHSYT of shape $(5,4,2)$ is given by

    $$\begin{array}{|c|c|c|c|c|c}\hline 1 & 2 & 3 & 5 & 9 & \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & 8 & 11 & \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & 10 & \multicolumn{1}{|c}{} & \\ \cline{3-4} \end{array}.$$

Call two permutations $u, v \in \mathfrak{S}_n$ *W-equivalent* if they belong to the same equivalence class of the transitive closure of the following relation: either (i) they have the same insertion tableau under the RSK-algorithm or (ii) $u(1) = v(2)$ and $u(2) = v(1)$. For instance, the W-equivalence classes for $n = 3$ are $\{123, 213, 231, 321\}$ and $\{312, 132\}$. Show that the number of W-equivalence classes in $\mathfrak{S}_n$ is equal to the number of SHSYT of size n.

(b) [5–] Can this be generalized in an interesting way?

33. Let λ be a partition of n with distinct parts, denoted $\lambda \models n$. Let g^λ denote the number of shifted SYT of shape λ, as defined in Problem 32.

(a) [3–] Prove by a suitable modification of RSK that

$$\sum_{\lambda \models n} 2^{n-\ell(\lambda)} \left(g^\lambda\right)^2 = n!. \tag{2}$$

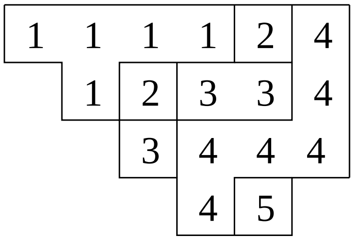
(b) [3] The "shifted analogue" of Corollary 7.13.9 is the following curious result. Let $\zeta = (1+i)/\sqrt{2} = e^{2\pi i/8}$. Let

$$u(n) = \sum_{\lambda \models n} \zeta^{\ell(\lambda)} 2^{(n-\ell(\lambda))/2} g^\lambda.$$

Show that

$$\sum_{n \geq 0} u(n) \frac{t^n}{n!} = e^{\zeta t + \frac{1}{2} t^2}. \tag{3}$$

34. (a) [2+] Let λ be a partition with distinct parts. A *strict shifted SSYT* (or S4YT) of shape λ is a way of filling the squares of the shifted diagram of λ with positive integers such that each row and column is weakly increasing, and there are no 2×2 squares of equal entries. (This last condition is equivalent to every diagonal from the upper left to lower right is strictly increasing.) A *component* of an S4YT is a maximal connected set of equal entries. For instance, the diagram



is an S4YT with seven components, where we have outlined each component. Given an S4YT T, let $k(T)$ denote its number of components. Define

$$Q_\lambda(x) = \sum_T 2^{k(T)} x^T,$$

summed over all S4YT of shape λ. Show that $Q_\lambda(x)$ is a symmetric function. For instance

$$Q_{31}(x) = 4m_{31} + 8m_{22} + 16m_{211} + 32m_{1111}.$$

(b) [3] Show that

$$\sum_\lambda 2^{-\ell(\lambda)} Q_\lambda(x) Q_\lambda(y) = \prod_{i,j} \frac{1+x_iy_j}{1-x_iy_j},$$

where the sum is over all partitions of all $n \geq 0$.

(c) [3–] A *diagonal-strict plane partition* (DSPP) is a plane partition such that there are no 2×2 squares of equal *positive* entries. If π is an DSPP, then define $k(\pi)$ as before (ignoring 0 entries). Let $|\pi|$ denote the sum of the parts of π. Use (b) to show that

$$\begin{aligned}\sum_\pi 2^{k(\pi)} q^{|\pi|} &= \prod_{j\geq 1} \left(\frac{1+q^j}{1-q^j}\right)^j \\ &= 1 + 2q + 6q^2 + 16q^3 + 38q^4 + 88q^5 + 196q^6 + \cdots,\end{aligned}$$

where π ranges over all DSPP.

35. [2+] Evaluate the sums

$$\sum_{\lambda\vdash n} f^{\lambda/2} f^\lambda \quad \text{and} \quad \sum_{\lambda\vdash n} \left(f^{\lambda/2}\right)^2.$$

36. Let $o(\lambda)$ denote the number of odd parts of the partition λ, and $d(\lambda)$ the number of distinct parts. Let t_{n-1} denote the number of involutions in $\mathfrak{S}_{n-1}$.

   (a) [2] Use the RSK algorithm to show that

$$\sum_{\lambda \vdash n} o(\lambda) f^\lambda = n t_{n-1}.$$

   (b) [2+] Show that

$$\sum_{\lambda \vdash n} d(\lambda) f^\lambda = n t_{n-1}.$$

37. [3] With $o(\lambda)$ as in Problem 36(a), show that

$$\sum_{\lambda \vdash n} f^\lambda \left(\frac{1+q}{1-q}\right)^{o(\lambda)} = \sum_{\lambda \vdash n} f^\lambda \prod_{u \in \lambda} \frac{1+q^{h(u)}}{1-q^{h(u)}},$$

where $h(u)$ denotes the hook length of u. Equivalently,

$$\exp\left(\frac{1+q}{1-q}t + \frac{1}{2}t^2\right) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\lambda \vdash n} f^\lambda \prod_{u \in \lambda} \frac{1+q^{h(u)}}{1-q^{h(u)}}.$$

38. [2+] Given an SYT T, let $\sigma(T)$ be the largest integer k such that $1, 2, \ldots, k$ appear in the first row of T. Let E_n denote the expected value of $\sigma(\mathrm{ins}(w))$, where w is a random (uniform) permutation in $\mathfrak{S}_n$ and $\mathrm{ins}(w)$ denotes the insertion tableau of w under the RSK algorithm. Thus

$$E_n = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \sigma(\mathrm{ins}(w)).$$

Find $\lim_{n\to\infty} E_n$.

39. [3] Show that as $n \to \infty$, for almost all (i.e., a $(1-o(1))$-fraction) permutations $w \in \mathfrak{S}_n$ the number of bumping operations performed in applying RSK to w is

$$(1+o(1))\frac{128}{27\pi^2} n^{3/2}.$$

Moreover, the number of comparison operations performed is

$$(1+o(1))\frac{64}{27\pi^2} n^{3/2} \log_2 n.$$

40. (a) [2+] Let $n = pq$, $w \in \mathfrak{S}_n$ and $w \overset{\text{rsk}}{\to} (P, Q)$. Suppose that the shape of P and Q is a $p \times q$ rectangle. Show that when the RSK algorithm is applied to w, every bumping path is vertical (never moves strictly to the left).

(b) [2] Let $P = (a_{ij})$ and $Q = (b_{ij})$ in (a). Deduce from (a) that $w(b_{ij}) = a_{p+1-i,j}$.

41. [2] Let $i, j, n \geq 1$. Evaluate the sum

$$f_n(i, j) = \sum_{\lambda \vdash n} s_\lambda(1^i) s_\lambda(1^j).$$

42. [3–] Let $y_n = \sum_{\lambda \vdash n} s_\lambda^2$. Find the generating function

$$F(q) = \sum_{n \geq 0} \langle y_n, y_n \rangle q^n.$$

Express your answer in terms of the generating function $P(x, t) = \prod_{i \geq 1} (1 - tx^i)^{-1}$.

43. [3] Let

$$f(n) = \left\langle \sum_{\mu \vdash n} s_\mu^2, \sum_{\lambda \vdash n} s_{2\lambda} \right\rangle,$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. Thus

$$(f(0), f(1), \dots, f(10)) = (1, 1, 3, 5, 12, 20, 44, 76, 157, 281, 559).$$

Show that

$$\sum_{n \geq 0} f(n) q^n = \prod_{i \geq 1} \frac{1}{\sqrt{1 - 2q^i}} \cdot \prod_{j \geq 1} \frac{1}{(1 - q^{2j})^{2^{j-2}}}.$$

44. [3+] Let $V_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Show that for $k \geq 0$,

$$\langle V_n^{2k}, V_n^{2k} \rangle_n = \frac{((2k+1)n)!}{(2k+1)!^n \, n!}.$$

The notation $\langle \, , \rangle_n$ indicates that the scalar product is taken in the ring Λ_n, i.e., the Schur functions $s_\lambda(x_1, \dots, x_n)$ with $\ell(\lambda) \leq n$ form an orthonormal basis.

45. [2+] Let

$$a_n = \left\langle h_2^n, \sum_{\lambda \vdash 2n} s_\lambda \right\rangle .$$

Find the generating function $F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$. (A results in Chapter 5 may prove useful.)

46. (a) [3–] Find the number $f(n)$ of ways to move from the empty partition $\emptyset$ to $\emptyset$ in n steps, where each step consists of either (i) adding a box, (ii) removing a box, or (iii) adding and then removing a box, always keeping the diagram of a partition (even in the middle of a step of type (iii)). For instance, $f(3) = 5$, corresponding to the five sequences

$$\begin{array}{cccc} \emptyset & (1,\emptyset) & (1,\emptyset) & (1,\emptyset) \\ \emptyset & (1,\emptyset) & 1 & \emptyset \\ \emptyset & 1 & (2,1) & \emptyset \\ \emptyset & 1 & (11,1) & \emptyset \\ \emptyset & 1 & \emptyset & (1,\emptyset) \end{array} .$$

Express your answer as a familiar combinatorial number and not, for instance, as a sum.

(b) [3–] Given a partition λ, let $f_\lambda(n)$ be the same as in (a), except we move from $\emptyset$ to λ in n steps. Define

$$T_n = \sum_\lambda f_\lambda(n) s_\lambda .$$

For instance,

$$T_3 = 5 + 10 s_1 + 6 s_2 + 6 s_{11} + s_3 + 2 s_{21} + s_{111} .$$

Find $\langle T_m, T_n \rangle$. As in (a), express your answer as a familiar combinatorial number.

47. (a) [2+] Let $h(t) \in \mathbb{C}[[t]]$ with $h(0) \neq 0$, and let $g(t) \in \mathbb{C}[[t]]$. Write $p = p_1 = \sum x_i$. Let Ω be the operator on $\Lambda_{\mathbb{C}}$ defined by

$$\Omega = g(p) + h(p) \frac{\partial}{\partial p} .$$

Define

$$F(x,p)=\sum_{n\geq 0}\Omega^n(1)\frac{x^n}{n!}.$$

Show that

$$F(x,p)=\exp\left[-M(p)+M(L^{\langle -1\rangle}(x+L(p)))\right],$$

where

$$\begin{aligned} L(t) &= \int_0^t \frac{ds}{h(s)} \\ M(t) &= \int_0^t \frac{g(s)ds}{h(s)}, \end{aligned}$$

and where $L^{\langle -1\rangle}$ denotes the compositional inverse of L.

(b) [1+] Let $g(t)=t$ and $h(t)=1$, so $\langle\Omega^n(1),s_\lambda\rangle$ is the number of oscillating tableaux of shape λ and length n, as defined in Exercise 7.24(d). Show that

$$F(x,p)=\exp\left(px+\frac{1}{2}x^2\right).$$

(c) [2] Let $f_\lambda(n)$ be the number of ways to move from the empty partition $\emptyset$ to λ in n steps, where the steps are as in Problem 46. Use (a) to show that

$$\sum_{n\geq 0}\sum_{\lambda\in\mathrm{Par}} f_\lambda(n)s_\lambda\frac{t^n}{n!}=\exp\left(-1-p+(1+p)e^x\right).$$

(d) [2–] Let $g_\lambda(n)$ be the number of ways to move from $\emptyset$ to λ in n steps, where each step consists of adding one square at a time any number i of times (including $i=0$) to the current shape and then either stopping or deleting one square (always maintaining the shape of a partition). Show that

$$\sum_{n\geq 0}\sum_{\lambda\in\mathrm{Par}} g_\lambda(n)s_\lambda\frac{x^n}{n!}=\exp\left(1-p-\sqrt{(1-p)^2-2x}\right).$$

In particular,

$$\sum_{n\geq 0} g_\emptyset(n)\frac{x^n}{n!}=\exp\left(x+\sum_{k\geq 2}(2k-3)!!\frac{x^k}{k!}\right).$$

(e) [2–] Let $j_\lambda(n)$ be the number of ways to move from $\emptyset$ to $\lambda \vdash k$ in n steps, where each step consists of adding one square at a time any number i of times (including $i = 0$) to the current shape or else deleting one square (always maintaining the shape of a partition). Show that

$$j_\lambda(n) = n!\left(\!\!\binom{n}{k}\!\!\right) f^\lambda,$$

where as usual f^λ denotes the number of SYT of shape λ.

48. [3] Let $w = a_1a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$. Suppose that $a_i + a_{2n+1-i} = 2n + 1$ for all $1 \leq i \leq n$. Show that the shape of the insertion tableau $\mathrm{ins}(w)$ can be covered with n dominos.

49. [2–] Let $d, n \geq 1$ and $\zeta = e^{2\pi i/d}$, a primitive dth root of unity. Let $f \in \Lambda^n$. Show that $f(1, \zeta, \ldots, \zeta^{d-1}) = 0$ unless $d|n$.

50. [3–] Let $(n-3)/2 \leq m \leq n-1$. Show that

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} f^\lambda = t(n) - \sum_{\substack{i,j,l \geq 0 \\ 2i+j+2l = n-m-1}} \frac{(-1)^i (n)_{i+j}}{i!\, j!} t(j),$$

where $t(j)$ denotes the number of involutions in $\mathfrak{S}_j$.

51. [2–] Let u be a square of the skew shape λ/μ. We can define the *hook* $H(u) = H_{\lambda/\mu}(u)$ just as for ordinary shapes, viz., the set of squares directly to the right of u and directly below u, counting u itself once. Similarly we can define the *hook length* $h(u) = h_{\lambda/\mu}(u) := \#H(u)$. Let $(\lambda/\mu)^r$ denote λ/μ rotated 180°, as in Exercise 7.56. Show that

$$\sum_{u \in \lambda/\mu} h_{\lambda/\mu}(u) = \sum_{u \in (\lambda/\mu)^r} h_{(\lambda/\mu)^r}(u).$$

52. (a) [3–] Let $\eta_k(\lambda)$ be the number of hooks of length k of the partition λ. Show that

$$\sum_{\lambda \vdash n} \eta_k(\lambda) = k \sum_{\lambda \vdash n} m_k(\lambda).$$

As usual, $m_k(\lambda)$ denotes the number of parts of λ equal to k. Note that Problem 1 is equivalent to the case $k = 1$. Is there a simple bijective proof similar to the solution to Problem 1?

(b) [3–] Part (a) can be rephrased as follows. For $u = (i, j) \in \lambda$, let $r(u) = \lambda_i$, the length of the row in which u appears. Then the statistics $h(u)$ and $r(u)$ have the same distribution over all squares of all $\lambda \vdash n$, i.e.,

$$\sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{h(u)} = \sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{r(u)}.$$

Show in fact that $h(u)$ and $r(u)$ have a symmetric joint distribution, i.e., if

$$F(x, y) = \sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{h(u)} y^{r(u)},$$

then $F(x, y) = F(y, x)$.

53. [3+] Given a partition λ and $u \in \lambda$, let $a(u)$ and $\ell(u)$ denote the arm and leg lengths of u as in Exercise 7.26. Define

$$\gamma(\lambda) = \#\{u \in \lambda \,:\, a(u) - l(u) = 0 \text{ or } 1\}.$$

Show that

$$\sum_{\lambda \vdash n} q^{\gamma(\lambda)} = \sum_{\lambda \vdash n} q^{\ell(\lambda)},$$

where $\ell(\lambda)$ denotes the length (number of parts) of λ.

54. [2] Let $\delta = (n-1, n-2, \ldots, 0)$ as usual, and let $\lambda \in \mathrm{Par}$ with $n \geq \ell(\lambda)$. Find the Schur function expansion of the product

$$s_\delta(x_1, \ldots, x_n) s_\lambda(x_1^2, \ldots, x_n^2).$$

55. [2] Let E_k denote an Euler number (the number of alternating permutations of $1, 2, \ldots, k$). Evaluate the determinants

$$A_n = \left| \frac{E_{2i+2j-1}}{(2i+2j-1)!} \right|_{i,j=1}^{n}$$

and

$$B_n = \left| \frac{E_{2i+2j-3}}{(2i+2j-3)!} \right|_{i,j=1}^{n}.$$

HINT. Use Exercise 7.40.

56. [2+] Let $f(n)$ be the number of permutations $w \in \mathfrak{S}_n$ such that both w and w^{-1} are alternating. Let

$$\begin{aligned} L(x) &= \frac{1}{2}\log\frac{1+x}{1-x} \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots . \end{aligned}$$

Use Corollary 7.23.8 and Exercise 7.64 to show that

$$\begin{aligned} \sum_{k\geq 0} f(2k+1)x^{2k+1} &= \sum_{k\geq 0} E_{2k+1}^2 \frac{L(x)^{2k+1}}{(2k+1)!} \\ \sum_{k\geq 0} f(2k)x^{2k} &= \frac{1}{\sqrt{1-x^2}} \sum_{k\geq 0} E_{2k}^2 \frac{L(x)^{2k}}{(2k)!}, \end{aligned}$$

where E_n denotes an Euler number.

57. [3–] Let $a(n)$ denote the number of alternating involutions in $\mathfrak{S}_n$, i.e., the number of involutions in $\mathfrak{S}_n$ that are alternating permutations in the sense of the last two paragraphs of Section 3.16. Let E_m denote an Euler number. Use Problem 89 below and Exercise 7.64 to show that

$$\sum_{k\geq 0} a(2k+1)x^{2k+1} = \sum_{i,j\geq 0} \frac{E_{2i+2j+1}}{(2i+1)!\,j!\,4^j} \left(\tan^{-1} x\right)^{2i+1} \left(\log\frac{1+x^2}{1-x^2}\right)^j$$

$$\sum_{k\geq 0} a(2k)x^{2k} = \frac{1}{\sqrt[4]{1-x^4}} \sum_{i,j\geq 0} \frac{E_{2i+2j}}{(2i)!\,j!\,4^j} \left(\tan^{-1} x\right)^{2i} \left(\log\frac{1+x^2}{1-x^2}\right)^j .$$

58. [3+] Let $\lambda \vdash n$, and let a, b, c, d be (commuting) indeterminates. Define

$$w(\lambda) = a^{\sum\lceil \lambda_{2i-1}/2\rceil} b^{\sum\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum\lceil \lambda_{2i}/2\rceil} d^{\sum\lfloor \lambda_{2i}/2\rfloor}.$$

For instance, if $\lambda = (5,4,4,3,2)$ then $w(\lambda)$ is the product of the entries below in the diagram of λ:

$$\begin{array}{l} a\,b\,a\,b\,a \\ c\,d\,c\,d \\ a\,b\,a\,b \quad . \\ c\,d\,c \\ a\,b \end{array}$$

Let $y = \sum_\lambda w(\lambda)s_\lambda$, where λ ranges over all partitions. Show that

$$\log(y) - \sum_{n\geq 1} \frac{1}{2n} a^n(b^n - c^n)p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Note that if we set $a = qt$, $b = q^{-1}t$, $c = qt^{-1}$, $d = q^{-1}t^{-1}$ and then set $q = t = 0$, then y becomes $\sum_\lambda s_\lambda$, where λ ranges over all partitions such that each λ_i and λ'_i is even.

59. Let $\omega_y : \Lambda(x, y) \to \Lambda(x) \otimes \Lambda(y)$ be the algebra homomorphism defined by

$$\omega_y p_n(x, y) = p_n(x) + (-1)^{n-1} p_n(y). \qquad (4)$$

Equivalently, ω_y is the automorphism ω acting on the y-variables only. Write $\omega_y f(x, y) = f(x/y)$. In particular, $s_\lambda(x/y)$ is called a *super Schur function*. Let

$$\Sigma = \mathrm{im}(\omega_y) = \{f(x/y) \,:\, f \in \Lambda\},$$

a subalgebra of $\Lambda(x) \otimes \Lambda(y)$.

(a) [2–] Show that

$$s_\lambda(x/y) = \sum_{\mu\subseteq\lambda} s_\mu(x) s_{\lambda'/\mu'}(y). \qquad (5)$$

(b) [3–] Let $g(x, y) \in \Lambda(x) \otimes \Lambda(y)$, and let t be an indeterminate. Show that $g \in \Sigma$ if and only if

$$g(x, y)|_{x_1=t,\, y_1=-t} = g(x, y)|_{x_1=y_1=0}\,. \qquad (6)$$

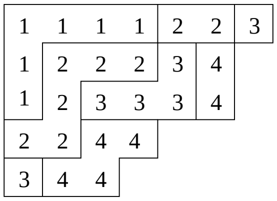
(c) [3] Prove the following "finite analogue" of (b). Let $g \in \Lambda(x_1, \dots, x_m) \otimes \Lambda(y_1, \dots, y_n)$. Then

$$g|_{x_1=t,\, y_1=-t} = g(x, y)|_{x_1=y_1=0}$$

if and only if g is a polynomial in the "variables" $p_i(x_1, \dots, x_m) + (-1)^{i-1} p_i(y_1, \dots, y_n)$, $i \geq 1$.

(d) [2–] Show that for any $f \in \Lambda$, $f(x/x)$ is a polynomial in the odd power sums $p_1, p_3, p_5, \dots$.

(e) [3–] Define a *supertableau* of shape λ to be an array T of positive integers of shape λ such that (i) the rows and columns are weakly increasing, and (ii) the diagonals from the upper left to lower right are strictly increasing (equivalently, there is no 2×2 square of equal entries). A maximal rookwise-connected subset of equal entries is called a *component* of T. Let $c(T)$ denote the number of components. For instance, if T is given by:



then T has one component of 1's, two components of 2's, three components of 3's, and two components of 4's, so $c(T) = 8$. Show that

$$s_\lambda(x/x) = \sum_T 2^{c(T)} x^T,$$

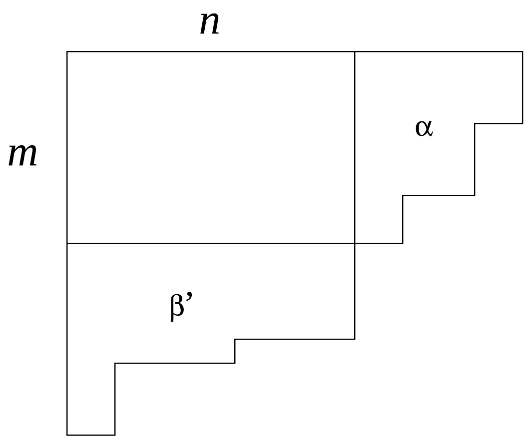
where T ranges over all supertableaux of shape λ and x^T has its usual meaning.

(f) [2] Let (n^m) denote an $m \times n$ rectangular shape. Show in two different ways that

$$s_{(n^m)}(x_1, \ldots, x_m/y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j).$$

The first proof (easy) should use Exercises 7.41 and 7.42. The second proof should use (b) above (in the easy "only if" direction) but no RSK, Cauchy identity, etc.

(g) [3–] More generally, let α, β be partitions with $\ell(\alpha) \leq m$ and $\ell(\beta) \leq n$. Let $[m, n, \alpha, \beta]$ denote the partition obtained by adjoining α to the right of (n^m) and β' below (n^m), as illustrated below.



Show that

$$s_{[m,n,\alpha,\beta]}(x_1,\ldots,x_m/y_1,\ldots,y_n)$$
$$= s_\alpha(x_1,\ldots,x_m)s_\beta(y_1,\ldots,y_n)\cdot\prod_{i=1}^m\prod_{j=1}^n(x_i+y_j).$$

60. (a) [3] Define a grading on the ring Λ of symmetric functions by setting $\deg(p_i)=1$ for all $i\geq 1$. Thus $\deg(p_\lambda)=\ell(\lambda)$. Let $\hat{s}_\lambda$ denote the terms of least degree appearing in the expansion of s_λ in terms of power sums. (It is an easy consequence of Exercise 7.52 and the Murnaghan-Nakayama rule that this least degree is equal to $\operatorname{rank}(\lambda)$.) For instance,

$$s_{221}=\frac{1}{24}p_1^5-\frac{1}{12}p_2p_1^3-\frac{1}{6}p_3p_1^2+\frac{1}{8}p_2^2p_1+\frac{1}{4}p_4p_1-\frac{1}{6}p_3p_2,$$

so

$$\hat{s}_{221}=\frac{1}{4}p_4p_1-\frac{1}{6}p_3p_2.$$

Let V_n denote the subspace of $\Lambda_{\mathbb{Q}}$ spanned by all $\hat{s}_\lambda$ such that $\lambda\vdash n$. Show that a basis for V_n is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda)=\ell(\lambda)\}$.

(b) [2] Deduce from (a) that $\dim V_n$ is the number of $\mu\vdash n$ whose parts differ by at least 2. (By the Rogers-Ramanujan identities, this is also the number of $\mu\vdash n$ whose parts are $\equiv \pm 1 \pmod 5$.)

(c) [3] Define the *augmented monomial symmetric function* $\tilde{m}_\lambda = r_1!r_2!\cdots m_\lambda$, where $\lambda=\langle 1^{r_1},2^{r_2},\ldots\rangle$. Let t_λ denote the result of substituting ip_i for p_i in the expansion of $\hat{s}_\lambda$ in terms of power sums. Suppose that $t_\lambda = \sum_\mu a_{\lambda\mu} p_\mu$. Show that

$$t_\lambda = \sum_\mu a_{\lambda\mu} \tilde{m}_\mu.$$

(d) [5–] Let W_n denote the space of all $f \in \Lambda^n_{\mathbb{Q}}$ such that if $f = \sum_\mu a_{\lambda\mu} p_\mu$, then $f = \sum_\mu a_{\lambda\mu} \tilde{m}_\mu$. Find $\dim W_n$. Does W_n have a nice basis?

(e) [5–] Let $\varphi_k(s_\lambda)$ denote the terms of the least k degrees (that is, of degrees $\operatorname{rank}(\lambda)$, $\operatorname{rank}(\lambda)+1, \ldots,$ $\operatorname{rank}(\lambda)+k-1$) appearing in the expansion of s_λ in terms of power sums, so in particular $\hat{s}_\lambda = \varphi_1(s_\lambda)$. Let $V_n^{(k)}$ denote the subspace of $\Lambda^n_{\mathbb{Q}}$ spanned by all $\varphi_k(s_\lambda)$. Show that a basis for $V_n^{(2)}$ is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda) \geq \ell(\lambda) - 1\}$.

(f) [5–] Find a basis and/or the dimension of $V_n^{(k)}$ for $k \geq 3$. NOTE. It is *false* that a basis for $V_n^{(3)}$ is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda) \geq \ell(\lambda) - 2\}$.

61. (a) [3] Let t be an indeterminate. Let $\vartheta : \Lambda \to \Lambda[t]$ be the specialization (homomorphism) defined by

$$\vartheta(p_k) = t + \sum_{i=1}^{k} \binom{k}{i} p_i.$$

Show that

$$\vartheta(s_\lambda) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \left(\prod_{u \in \lambda/\mu} (t + c(u)) \right) s_\mu,$$

where $c(u)$ denotes the content of the square u.

62. [3–] Define a $\mathbb{Q}$-linear transformation $\varphi\colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[t]$ by

$$\varphi(s_\lambda) = \frac{\prod_{i=1}^{n} (t + \lambda_i + n - i)}{H_\lambda},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$ and H_λ denotes the product of the hook lengths of λ. Show that for any $\mu \vdash n$ with $\ell(\mu) = \ell$ and $m_1(\mu) = m$ (the number of parts of μ equal to 1), we have

$$\varphi(p_\mu) = (-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} t(t+1)\cdots(t+i-1).$$

63. [5] Let $\mathcal{I}$ be a collection of subintervals $\{i, i+1, \dots, i+j\}$ of $[n]$. (Without loss of generality we may assume that $\mathcal{I}$ is an *antichain*, i.e., if $I, J \in \mathcal{I}$ and $I \subseteq J$, then $I = J$.) Define

$$f_{\mathcal{I}}(x) = \sum_{i_1 i_2 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_1 i_2 \cdots i_n$ ranges over all n-tuples of positive integers such that if $j, k \in I \in \mathcal{I}$ and $j \neq k$, then $x_{i_j} \neq x_{i_k}$. Thus $f_{\mathcal{I}} \in \Lambda$. For instance, if $\mathcal{I} = \emptyset$ then $f_{\mathcal{I}} = e_1^n$, and if $\mathcal{I} = \{[n]\}$ then $f_{\mathcal{I}} = n!\, e_n$. Show that $f_{\mathcal{I}}$ is e-positive.

64. (a) [2+] Fix integers $1 \leq m \leq n$. Find simple formulas for the four sums

$$\begin{aligned} a(m,n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu f^\lambda c^\lambda_{\mu\nu} \\ b(m,n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu c^\lambda_{\mu\nu} \\ c(m,n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\nu f^\lambda c^\lambda_{\mu\nu} \\ d(m,n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\lambda c^\lambda_{\mu\nu}, \end{aligned}$$

where $c^\lambda_{\mu\nu}$ denotes a Littlewood-Richardson coefficient. Some of the formulas may involve the number $t(k)$ of involutions in $\mathfrak{S}_k$ for certain k.

(b) [2+] Let

$$e(m,n) = \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\nu c^\lambda_{\mu\nu}.$$

Show that

$$\sum_{m \geq 0} \sum_{k \geq 0} e(m, m+k) x^m \frac{y^k}{k!} = P(x) \exp\left(\frac{y}{1-x} + \frac{y^2}{2(1-x^2)} \right),$$

where $P(x) = \prod_{i \geq 1} (1 - x^i)^{-1}$.

(c) [5–] Do something similar for

$$f(m,n) = \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} c^\lambda_{\mu\nu}.$$

65. [3–] Show that

$$\sum_{\mu,\nu,\lambda} \left(c^{\lambda}_{\mu\nu}\right)^2 q^{|\lambda|} = \frac{1}{\prod_{i\geq 1}(1-2q^i)}. \tag{7}$$

66. [3–] Let $k \geq 1$ and

$$B_k(x) = \sum_{\ell(\lambda)\leq k} s_\lambda(x),$$

as in Exercise 7.16(a). Show that

$$B_k(x) = \frac{\sum_\mu (-1)^{c_\mu} s_\mu(x)}{\prod_i (1-x_i)\cdot \prod_{i<j}(1-x_ix_j)},$$

where μ ranges over all partitions whose Frobenius notation has the form

$$\mu = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+k & a_2+k & \cdots & a_r+k \end{pmatrix},$$

and where $c_\mu = (|\mu| - rk + r)/2$.

67. [3+] Let λ, μ, ν be partitions and $n \in \mathbb{P}$. Show that $c^{\lambda}_{\mu\nu} \neq 0$ if and only if $c^{n\lambda}_{n\mu,n\nu} \neq 0$.

68. [4–] Let λ, μ, ν be partitions of length at most n. If A is an $n \times n$ hermitian matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, then write $\operatorname{spec}(A) = (\alpha_1, \ldots, \alpha_n)$. Show that the following two conditions are equivalent:

    - There exist $n\times n$ hermitian matrices A, B, C such that $A = B+C$, $\operatorname{spec}(A) = \lambda$, $\operatorname{spec}(B) = \mu$, and $\operatorname{spec}(C) = \nu$.
    - $c^{\lambda}_{\mu\nu} \neq 0$, where $c^{\lambda}_{\mu\nu}$ denotes a Littlewood-Richardson coefficient.

69. (a) [2–] Find all partitions $\lambda \vdash n$ such that $\chi^\lambda(\mu) \neq 0$ for all $\mu \vdash n$.

    (b) [5–] Find all partitions $\mu \vdash n$ such that $\chi^\lambda(\mu) \neq 0$ for all $\lambda \vdash n$.

70. [2+] Given $\lambda \vdash n$, let H_λ denote the product of the hook lengths of λ, so $H_\lambda = n!/f^\lambda$. Show that for $k \in \mathbb{N}$,

$$\sum_{\lambda\vdash n} H_\lambda^{k-2} = \frac{1}{n!}\#\{(w_1, w_2, \ldots, w_k) \in \mathfrak{S}_n^k \,:\, w_1^2 w_2^2 \cdots w_k^2 = 1\}.$$

    Hint. Use Exercises 7.69(b) (or more precisely, its solution) and 7.70.

71. (a) [2+] Show that

$$\sum_{n\geq 0}\sum_{\lambda\vdash n}(f^\lambda)^2\prod_{u\in\lambda}(t+c_u^2)\cdot\frac{x^n}{n!^2}=(1-x)^{-t},$$

where c_u denotes the content of the square u in the diagram of λ.

(b) [3+] Show that

$$\sum_{n\geq 0}\sum_{\lambda\vdash n}(f^\lambda)^2\prod_{u\in\lambda}(t+h_u^2)\cdot\frac{x^n}{n!^2}=\prod_{i\geq 1}(1-x^i)^{-1-t},$$

where h_u denotes the hook length of the square u in the diagram of λ.

(c) [3] Show that for any $r\geq 0$ we have

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}\prod_{i=0}^{r-1}(c_u^2-i^2)=\frac{(2r)!}{(r+1)!^2}(n)_{r+1}. \tag{8}$$

(d) [2] Deduce from equation (8) that

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}c_u^{2k}=\sum_{j=1}^{k}T(k,j)\frac{(2j)!}{(j+1)!^2}(n)_{j+1},$$

where $T(k,j)$ is a *central factorial number* (EC2, Exercise 5.8).

(e) [3] Show that for any $r\geq 0$ we have

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}\prod_{i=1}^{r}(h_u^2-i^2)=\frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}(n)_{r+1}. \tag{9}$$

(f) [2] Deduce from (e) that

$$\frac{1}{n!}\sum_{\lambda\vdash n}f_\lambda^2\sum_{u\in\lambda}h_u^{2k}=\sum_{j=1}^{k+1}T(k+1,j)\frac{1}{2j^2}\binom{2(j-1)}{j-1}\binom{2j}{j}(n)_j,$$

where $T(k+1,j)$ is as in (d).

(g) [3] Let $F = F(x) \in \Lambda_{\mathbb{Q}}$ be a symmetric function. Define

$$\Phi_n(F) = \frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2 F(h_u^2 \,:\, u \in \lambda).$$

Here $F(h_u^2 \,:\, u \in \lambda)$ denotes substituting the quantities h_u^2, where u is a square of the diagram of λ, for n of the variables of F, and setting all other variables equal to 0. Show that $\Phi_n(F)$ is a polynomial function of n.

(h) [3] Let $G(x;y)$ be a power series of bounded degree (say over $\mathbb{Q}$) that is symmetric separately in the x variables and y variables. Let

$$\Psi_n(G) = \frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2 G(c_u \,:\, u \in \lambda;\ \lambda_i - i \,:\, 1 \leq i \leq n).$$

Show that $\Psi_n(G)$ is a polynomial function of n.

72. Let $k \geq 1$ and $p \geq 2$. Show that the number of p-cores (as defined in Exercise 7.59(d)) with largest part k is $\binom{k+p-2}{p-2}$.

73. Fix a partition $\mu \vdash k$, and define $N(n;\mu) = \sum_{\lambda\vdash n} f^{\lambda/\mu}$. Let $t(j)$ denote the number of involutions in $\mathfrak{S}_j$.

(a) [2+] Show that for all $n, k \geq 0$ we have

$$N(n+k;\mu) = \sum_{j=0}^{k}\binom{n}{j}\left(\sum_{\nu\vdash k-j} f^{\mu/\nu}\right)t(n-j).$$

(b) [3–] Let $\tilde{\nu}$ be the partition obtained from ν by replacing each even part $2i$ with i, i. Equivalently, if w is a permutation of cycle type ν, then w^2 has cycle type $\tilde{\nu}$. Show that for $n \geq k$,

$$N(n;\mu) = \sum_{j=0}^{k}\frac{t(n-j)}{(k-j)!}\sum_{\substack{\nu\vdash j\\ m_1(\nu)=m_2(\nu)=0}} z_\nu^{-1}\chi^\mu(\tilde{\nu}, 1^{k-j}).$$

For instance,

$$N(n;32) = N(n;221) = \frac{1}{24}(t(n) - 4t(n-3) + 6t(n-4)).$$

74. (a) [3–] Fix a partition $\mu \vdash k$. Given $\lambda \vdash n \geq k$, define

$$\widehat{\chi}^{\lambda}(\mu, 1^{n-k}) = \frac{(n)_k \chi^{\lambda}(\mu, 1^{n-k})}{\chi^{\lambda}(1^n)}.$$

Let $p \times q$ denote the partition with p parts equal to q. Fix a partition $w_\mu \in \mathfrak{S}_k$ of cycle type μ, and let $\kappa(w)$ denote the number of cycles of the permutation $w \in \mathfrak{S}_k$. Show that

$$\widehat{\chi}^{p \times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv=w_\mu} p^{\kappa(u)}(-q)^{\kappa(v)},$$

where the sum ranges over all $k!$ pairs $(u, v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ satisfying $uv = w_\mu$. Hint. Use the Murnaghan-Nakayama rule and Exercise 7.70.

(b) [3+] [to be inserted]

75. (a) [2+] Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Show that

$$P_n(q) := \sum_{w} q^{\kappa(w(1,2,\ldots,n))} = \frac{1}{n(n+1)}\left((q+n)_{n+1} - (q)_{n+1}\right).$$

where w ranges over all $(n-1)!$ n-cycles in $\mathfrak{S}_n$ and $w(1, 2, \ldots, n)$ denotes the product of w with the n-cycle $(1, 2, \ldots, n)$. For instance,

$$\begin{aligned} \sum_{\rho(w)=(3)} q^{\kappa(w(1,2,3))} &= \frac{1}{12}((q+3)_4 - (q)_4) \\ &= q^3 + q. \end{aligned}$$

Hint. Use Exercise 7.70.

(b) [2+] Show that all the zeros of $P_n(q)$ have real part 0.

(c) [5–] It follows from (a) that

$$P_n(q) = \frac{1}{\binom{n+1}{2}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} c(n+1, n-2i) q^{n-2i},$$

where $c(n+1, n-2i)$ denotes the number of permutations $w \in \mathfrak{S}_{n+1}$ with $n-2i$ cycles. Is there a bijective proof? (In fact, it isn't so obvious that $c(n+1, n-2i)$ is divisible by $\binom{n+1}{2}$. J. Burns has proved the stronger result that if $\lambda \vdash n+1$ and $\varepsilon_\lambda = -1$, then $(n+1)!/z_\lambda$ is divisible by $\binom{n+1}{2}$.)

(d) [3] Generalize (b) as follows. Fix $\lambda \vdash n$. Define

$$P_\lambda(q) = \sum_{\rho(w)=\lambda} q^{\kappa(w(1,2,\dots,n))},$$

where w ranges over all permutations in $\mathfrak{S}_n$ of cycle type λ. Show that all the zeros of $P_\lambda(q)$ have real part 0.

76. [3] Define two compositions α and β of n to be *equivalent* if $s_{B_\alpha} = s_{B_\beta}$ (as defined in §7.23). Describe the equivalence classes of this equivalence relation, showing in particular that the cardinality of each equivalence class is a power of two.

    NOTE. A "trivial" equivalence is given by

    $$(\alpha_1, \alpha_2, \dots, \alpha_k) \sim (\alpha_k, \dots, \alpha_2, \alpha_1).$$

    It is surprising that an equivalence class can have more than two elements, e.g., $\{(1,2,1,3,2), (2,3,1,2,1), (2,1,2,3,1), (1,3,2,1,2)\}$.

77. [3] Define the *rank* of a skew shape λ/μ to be the minimal number of border strips in a border strip tableau of shape λ/μ. It it easy to see that when $\mu = \emptyset$ this definition agrees with that on page 289. Let $|\lambda/\mu| = n$, and let ν be a partition of n satisfying $\ell(\nu) = \mathrm{rank}(\lambda/\mu)$. Show that $\chi^{\lambda/\mu}(\nu)$ is divisible by $m_1(\nu)!\, m_2(\nu)! \cdots$. (Incidentally, note that by the definition (7.75) of $\chi^{\lambda/\mu}(\nu)$ we have $\chi^{\lambda/\mu}(\nu) = 0$ if $\ell(\nu) < \mathrm{rank}(\lambda/\mu)$.)

78. Let λ/μ be a skew shape, identified with its Young diagram $\{(i,j) : \mu_i < j \leq \lambda_i\}$. We regard the points (i,j) of the Young diagram as squares. An *outside top corner* of λ/μ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \notin \lambda/\mu$. An *outside diagonal* of λ/μ consists of all squares $(i+p, j+p) \in \lambda/\mu$ for which (i,j) is a fixed outside top corner. Similarly an *inside top corner* of λ/μ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \in \lambda/\mu$ but $(i-1,j-1) \notin \lambda/\mu$. An *inside diagonal* of λ/μ consists of all squares $(i+p, j+p) \in \lambda/\mu$ for which (i,j) is a fixed inside top corner. If $\mu = \emptyset$, then λ/μ has one outside diagonal (the main diagonal) and no inside diagonals. Figure 1 shows the skew shape 8874/411, with outside diagonal squares marked by + and inside diagonal squares by −. Let $d^+(\lambda/\mu)$ (respectively, $d^-(\lambda/\mu)$) denote the

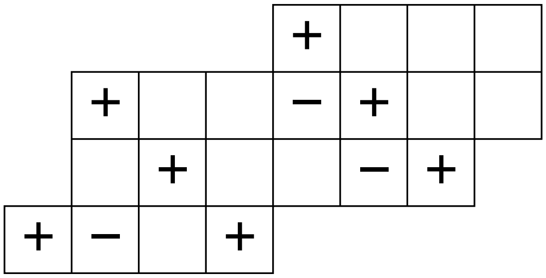


Figure 1: Outside and inside diagonals of the skew shape 8874/411

total number of outside diagonal squares (respectively, inside diagonal squares) of λ/μ.

Generalizing the code C_λ of Exercise 7.59, define the *code* $C_{\lambda/\mu}$ of λ/μ to be the two-line array whose top line is C_λ and whose bottom line is C_μ, where the indexing is "in phase." For instance,

$$C_{8874/411} = \begin{array}{cccccccccccccccc} \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \end{array}.$$

[3–, for the first four] Show that the following numbers are equal:

- The rank of λ/μ, as defined in Exercise 77 above.
- $d^+(\lambda/\mu) - d^-(\lambda/\mu)$
- The number of rows in the Jacobi-Trudi matrix for λ/μ (i.e., the matrix of Theorem 7.16.1) which don't contain a 1.
- The number of columns of $C_{\lambda/\mu}$ equal to $\substack{1\\0}$ (or to $\substack{0\\1}$).
- [3+] The largest power of t dividing the polynomial $s_{\lambda/\mu}(1^t)$.

79. [2+] Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Regard κ as a class function on $\mathfrak{S}_n$. Let $\lambda \vdash n$. Show that

$$\langle \kappa, \chi^\lambda \rangle = \begin{cases} \sum_{i=1}^n \frac{1}{i}, & \text{if } \lambda = (n) \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)}, & \text{if } \lambda = (p, q, 1^{n-p-q}),\ q > 0 \\ 0, & \text{otherwise.} \end{cases}$$

80. (a) [3] Define a class function $f_n : \mathfrak{S}_n \to \mathbb{Z}$ by

$$f_n(w) = n!(\kappa(w)+1)^{\kappa(w)-1},$$

where $\kappa(w)$ denotes the number of cycles of w. Show that f_n is a character of $\mathfrak{S}_n$.

(b) [3–] Let $F(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, so $F(x)^{\langle -1\rangle} = x^{1/x}$. Let the Taylor series expansion of $F(x)$ about $x = 1$ be given by

$$\begin{aligned} F(x) &= \sum_{n\geq 0} a_n \frac{(x-1)^n}{n!} \\ &= 1 + u + 2\frac{u^2}{2!} + 9\frac{u^3}{3!} + 56\frac{u^4}{4!} + 480\frac{u^5}{5!} + 5094\frac{u^6}{6!} + \cdots, \end{aligned}$$

where $u = x - 1$. Show that $\langle f_n, \mathrm{sgn}\rangle = a_n$, where sgn denotes the sign character of $\mathfrak{S}_n$. In particular, by (a) it follows that $a_n \geq 0$.

81. Let $E(\lambda)$ (respectively, $O(\lambda)$) be the number of SYT of shape λ whose major index is even (respectively, odd).

(a) [2+] Express the symmetric function

$$R_n = \sum_{\lambda\vdash n} (E(\lambda) - O(\lambda)) s_\lambda$$

in terms of the power sum symmetric functions.

(b) [2+] Deduce from (a) that if $\lambda \vdash n$, then $E(\lambda) = O(\lambda)$ if and only if one cannot place $\lfloor n/2 \rfloor$ disjoint dominos (i.e., two squares with an edge in common) on the diagram of λ.

(c) [2+] Show that (b) continues to hold for skew shapes λ/μ when $|\lambda/\mu|$ is even, but that the "only if" part can fail when $|\lambda/\mu|$ is odd.

(d) [2+] Let p be prime. Generalize (a)–(c) to the case $A_0(\lambda) = A_1(\lambda) = \cdots = A_{p-1}(\lambda)$, where $A_i(\lambda)$ denotes the number of SYT T of shape λ satisfying $\mathrm{maj}(T) \equiv i \pmod{p}$.

82. (a) [5] A problem superficially similar to 81(b) is the following. We can regard an SYT of shape λ (or more generally, a linear extension of a finite poset P) as a permutation of the squares of λ (or the elements of P), where we fix some particular SYT T to correspond to the identity permutation. Define an *even* SYT to be one which, regarded as a permutation, is an even permutation, and similarly *odd* SYT. For which λ is the number of even SYT the same as the number of odd SYT? (It's easy to see that the answer does not depend on the choice of the "identity SYT" T.) This problem has been solved for rectangular shapes by a difficult argument (rating [3] or even [3+]).

(b) [3] Given an SYT T with n squares, let w_T be the permutation of $[n]$ obtained by reading the elements of T in the usual reading order (left-to-right, top-to-bottom). Write $\mathrm{sgn}(T) = \mathrm{sgn}(w_T)$, i.e., $\mathrm{sgn}(T) = 1$ if w_T is an even permutation, and $\mathrm{sgn}(T) = -1$ if w_T is an odd permutation. Show that

$$\sum_T \mathrm{sgn}(T) = 2^{\lfloor n/2 \rfloor},$$

where T ranges over all SYT with n squares.

83. (a) [3–] Let $g_\lambda = \sum_\pi x_1^{c_1(\pi)} x_2^{c_2(\pi)} \cdots$, where π ranges over all reverse plane partitions of shape λ, and $c_i(\pi)$ is the number of columns of π that contain the part i. Show that g_λ is an (inhomogeneous) symmetric function whose highest degree part is s_λ.

(b) [3] Define an *elegant* SSYT of skew shape λ/μ to be an SSYT of shape λ/μ for which the numbers in row i lie in the interval $[1, i-1]$. In particular, there are no elegant SSYT of shape λ/μ if the first row of λ/μ is nonempty. Let f_λ^μ be the number of elegant SSYT of shape λ/μ. Show that

$$g_\lambda = \sum_{\mu \subseteq \lambda} f_\lambda^\mu s_\mu.$$

In particular, g_λ is Schur positive.

Example. Let $\lambda = (2,1)$. Then there is one elegant SSYT of the empty shape $(2,1)/(2,1)$ and one elegant SSYT of shape $(2,1)/(2)$. Hence $g_{2,1} = s_{2,1} + s_2$.

(c) [3] For $k \geq 0$ and $n \geq 1$, let $g_n^{(k)} = \sum_{j=0}^{n} \binom{k-1+j}{j} h_{n-j}$. For instance, $g_n^{(0)} = h_n$ and $g_n^{(1)} = h_n + g_{n-1} + \cdots + h_1 + 1$. Set $g_0^{(k)} = 1$ and $g_{-n}^{(k)} = 0$ for $n > 0$. Show that if $\lambda = (\lambda_1, \ldots, \lambda_m)$, then

$$g_\lambda = \det \left(g_{\lambda_i - i + j}^{(i-1)} \right)_{i,j=1}^{m}.$$

84. (a) [2] A *set-valued tableau* of shape λ/μ is a filling of the diagram of λ/μ with nonempty finite subsets of $\mathbb{P}$ such that if each subset is replaced by one of its elements, then an SSYT always results. If T is a set-valued tableau, then let $x^T = x_1^{c_1(T)} x_2^{c_2(T)} \cdots$, where $c_i(T)$ is the number of boxes of T containing i. Set $|T| = \sum c_i(T)$, the total number of elements appearing in all the boxes. Define

$$G_{\lambda/\mu}(x) = \sum_T (-1)^{|T| - |\lambda/\mu|} x^T,$$

where T ranges over all set-valued tableaux of shape λ/μ. For instance,

$$G_{1^n} = e_n - n e_{n+1} + \binom{n+1}{2} e_{n+2} - \binom{n+2}{3} e_{n+3} + \cdots.$$

Show that $G_{\lambda/\mu}$ is a symmetric formal power series (i.e., an element of the completion $\hat{\Lambda}$ of the ring Λ of symmetric functions) whose least degree part is $s_{\lambda/\mu}$.

(b) [3] Let f_λ^μ have the meaning of the previous problem. Show that

$$s_\mu = \sum_{\lambda \supseteq \mu} f_\lambda^\mu G_\lambda.$$

For instance,

$$s_{1^n} = e_n = G_n + n G_{n+1} + \binom{n+1}{2} G_{n+2} + \binom{n+2}{3} G_{n+3} + \cdots.$$

(c) [2] Deduce from (a) and (b) that $\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu}$, where g_λ has the meaning of the previous problem.

(d) [3] For $k \geq 0$ and $n \geq 1$, let $G_n^{(k)} = \sum_{i,j \geq 0} (-1)^j \binom{k+i-2}{i} s_{(n+i,1^j)}$. For instance, $G_n^{(1)} = s_n - s_{(n,1)} + s_{(n,1,1)} - \cdots$. Set $G_0^{(k)} = 1$ and $G_{-n}^{(k)} = 0$ for $n > 0$. Show that if $\lambda = (\lambda_1, \ldots, \lambda_m)$, then

$$G_\lambda = \det \left(G_{\lambda_i - i + j}^{(m-i+1)} \right)_{i,j=1}^{m}.$$

85. [3] Let L_λ be the symmetric function of Exercise 7.89(f), where $\lambda \vdash n$. Let $\alpha \in \mathrm{Comp}(n)$, and let B_α be the corresponding border strip (as defined on page 383). Show that

$$\langle L_\lambda, s_{B_\alpha} \rangle = \#\{w \in \mathfrak{S}_n \,:\, \rho(w) = \lambda,\ D(w) = S_\alpha\},$$

where $D(w)$ denotes the descent set of w.

86. (a) [2+] Fix $n \geq 1$. Given $S, T \subseteq [n-1]$, let

$$\beta(S,T) = \#\{w \in \mathfrak{S}_n \,:\, D(w) = S,\ D(w^{-1}) = T\}.$$

Let $f(n) = \max_{S,T \subseteq [n-1]} \beta(S,T)$. Show that there is some $S \subseteq [n-1]$ for which $f(n) = \beta(S,S)$.

(b) [5–] Show that $f(n) = \beta(S,S)$, where $S = \{1,3,5,\ldots\} \cap [n-1]$.

87. [3–] Let $y := \sum_\lambda s_\lambda$. Show that

$$y * y = \exp\left(\sum_{n \geq 1} \frac{p_{2n-1}}{(2n-1)(1-p_{2n-1})} \right) \cdot \left(\prod_{n \geq 1} (1 - p_n^2) \right)^{-1/2},$$

where $*$ denotes internal product.

88. [5–] Let $|\lambda/\mu| = n$ and

$$\begin{aligned} f^{\lambda/\mu}(q) &= (1-q)(1-q^2)\cdots(1-q^n) s_{\lambda/\mu}(1,q,q^2,\ldots) \\ &= \sum_T q^{\mathrm{maj}(T)}, \end{aligned}$$

where T ranges over all skew SYT of shape λ/μ. (See Proposition 7.19.11.) We can regard $f^{\lambda/\mu}(q)$ as the "natural" q-analogue of $f^{\lambda/\mu}$. Investigate when $f^{\lambda/\mu}(q)$ has unimodal coefficients. This isn't always the case (e.g., $\lambda = (2,2)$, $\mu = \emptyset$) but it does seem to be unimodal in certain cases, such as when $\mu = \emptyset$ and λ is an arithmetic progression ending with 1.

89. [2–] We follow the notation of Sections 7.19 and 7.23. Let $\alpha \in \mathrm{Comp}(n)$ and $\lambda \vdash n$. Show that $\langle s_{B_\alpha}, s_\lambda \rangle$ is equal to the number of SYT of shape λ and descent set S_α.

90. (a) [2–] For a sequence $u = u_1 \cdots u_n$ of positive integers, define the *descent set* $D(u)$ in analogy to permutations, i.e.,
$$D(u) = \{i \,:\, u_i > u_{i+1}\} \subseteq [n-1].$$
Given $S \subseteq [n-1]$, define
$$f_S = \sum x_{u_1} \cdots x_{u_n},$$
where $u_1 \cdots u_n$ ranges over all sequences u of positive integers of length n satisfying $D(u) = S$. Show that $f_S = s_{B_{\mathrm{co}(S)}}$, using the notation of Sections 7.19 and 7.23.

    (b) [2+] Let $\mathcal{S}_k$ denote the set of all finite sequences $u_1 u_2 \cdots u_n$ of positive integers containing no strictly decreasing factor of length k, i.e., we never have $u_i > u_{i+1} > \cdots > u_{i+k-1}$. Show that
$$\sum_{u_1 \cdots u_n \in \mathcal{S}_k} x_{u_1} \cdots x_{u_n}$$
$$= \frac{1}{1 - e_1 + e_k - e_{k+1} + e_{2k} - e_{2k+1} + e_{3k} - e_{3k+1} + \cdots}.$$

91. (a) [3–] Let L_α be as in (7.89). Suppose that $s_\lambda = f + g$, where $f, g \in \Lambda$ and f, g are L-positive. Show that $f = 0$ or $g = 0$.

    (b) [2+] Give an example of an L-positive symmetric function that isn't s-positive.

92. [2–] Let $\mathfrak{A}_n$ denote the alternating group of degree n (regarded as a subgroup of $\mathfrak{S}_n$). Express the cycle index $Z_{\mathfrak{A}_n}$ as a linear combination of Schur functions.

93. [2+] Let χ be a character of $\mathfrak{S}_n$. Let $\mathrm{ch}(\chi) = \sum_{\mu \vdash n} c_\mu m_\mu$. Show that
$$c_\mu = \langle \chi|_\mu, 1_{\mathfrak{S}_\mu} \rangle,$$
the multiplicity of the trivial character $1_{\mathfrak{S}_\mu}$ of the Young subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$ in the restriction $\chi|_\mu$ of χ to $\mathfrak{S}_\mu$. In particular, if χ is a permutation representation then c_μ is the number of orbits of $\mathfrak{S}_\mu$.

94. (a) [1+] Let X be a nonempty subset of $\mathfrak{S}_n$. Suppose that the cycle indicator Z_X is s-positive. Show that X contains the identity element of $\mathfrak{S}_n$.

    (b) [5] What can be said about subsets X of $\mathfrak{S}_n$ for which Z_X is s-positive or h-positive? (See equation (7.120), Exercise 7.111(c,d), and Problem 95 below for some information.)

95. Let G be a subgroup of $\mathfrak{S}_n$ for which the cycle indicator Z_G is h-positive.

    (a) [2+] Show that $Z_G = h_\lambda$ for some $\lambda \vdash n$.

    (b) [3–] Show in fact that G is conjugate to the Young subgroup $\mathfrak{S}_\lambda$.

96. [2] Let $\mathfrak{I}_n$ denote the set of all indecomposable permutations in $\mathfrak{S}_n$, as defined in Exercise 1.32(a). Let $\tilde{Z}_{\mathfrak{I}_n}$ denote the augmented cycle indicator of $\mathfrak{I}_n$, as defined in Definition 7.24.1. Show that

$$\sum_{n\geq 1} \tilde{Z}_{\mathfrak{I}_n} x^n = 1 - \frac{1}{\sum_{n\geq 0} n! h_n x^n},$$

a direct generalization of Exercise 1.32(a).

97. [3–] Give a super-analogue of Theorem 7.24.4 (Pólya's theorem). More precisely, when $Z_G(x/y)$ is expanded as a linear combination of the $m_\lambda(x)m_\mu(y)$'s, give a combinatorial interpretation of the coefficients.

98. (a) [2+] Let T be an SYT of shape $\lambda \vdash n$. We can regard the tableau evac(T) (as defined in Appendix 1) as a permutation of the entries $1, 2, \ldots, n$ of T. Show that this permutation is even if and only if the integer $\binom{n}{2} + (\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$ is even, where $\mathcal{O}(\mu)$ denotes the number of odd parts of the partition μ. (Note that this condition depends only on the shape λ of T.)

    (b) [3] Let $e(n)$ denote the number of permutations $\lambda \vdash n$ for which evac(T) is an even permutation of T, for some (or every) SYT T of shape λ. Let $p(n)$ denote the total number of partitions of n. Show that $e(n) = (p(n) + (-1)^{\binom{n}{2}} f(n))/2$, where

$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1} \frac{1 + x^{2i-1}}{(1 - x^{4i})(1 + x^{4i-2})^2}.$$

99. [2] Express $\operatorname{ex} f[g]$ in terms of $\operatorname{ex} f$ and $\operatorname{ex} g$, where ex denotes the exponential specialization and $f[g]$ denotes plethysm.

100. [2+] Expand the plethysm $h_2[h_n]$ in terms of Schur functions.

101. The *plethystic inverse* of $f \in \Lambda$ is a symmetric function $g \in \Lambda$ satisfying $f[g] = g[f] = p_1$ (the identity element of the operation of plethysm). (See Exercise 7.88(d).) It is easy to see that if g exists, then it is unique. Moreover, g exists if and only if f has constant term 0 and $[p_1]f \neq 0$.

    (a) [2] Describe the plethystic inverse of $f = \sum_{n\geq 1} a_n p_1^n$, where $a_1 \neq 0$, in terms of "familiar" objects.

    (b) [2] Let $f = \sum_{n\geq 1} a_n p_n$, where $a_1 \neq 0$. Describe the plethystic inverse of f in terms of Dirichlet convolution. The *Dirichlet convolution* $f * g$ of two functions $a, b : \mathbb{P} \to \mathbb{C}$ is defined by

$$a * b(n) = \sum_{d|n} f(d)g(n/d).$$

102. [2–] The group $\mathrm{GL}(n, \mathbb{C})$ acts on the space $\mathrm{Mat}(n, \mathbb{C})$ of $n \times n$ complex matrices by left multiplication. Express the character of this action as a linear combination of irreducible characters.

CHRONOLOGY OF NEW PROBLEMS (beginning 4/13/02)

Some items prior to November 22, 2007, may be missing from this list.

67. April 13, 2002
68. April 13, 2002
98. May 5, 2002
65. June 8, 2003
10. June 10, 2003
79. October 6, 2003
80. October 6, 2003
58. October 10, 2003
60. October 10, 2003
87. October 10, 2003
76. October 13, 2003
4. July 3, 2004
78. August 17, 2004
64. (b) January 1, 2005
31. February 13, 2005
40. April 16, 2005
39. April 17, 2005
47. December 13, 2005
56. December 13, 2005
96. December 31, 2005

57. January 3, 2006
90. August 2, 2006
17. October 22, 2006
66. August 7, 2007
26. September 4, 2007
83. September 29, 2007
84. September 29, 2007
24. November 22, 2007
93. February 13, 2008
83(c). March 14, 2008
84(d). March 14, 2008
72. March 25, 2008
86. April 26, 2008
71. June 29, 2008
3. July 11, 2008
71. (expanded) July 15, 2008
62. July 15, 2008
36. October 9, 2008
37. February 20, 2009
34. February 20, 2009
28. March 22, 2009
29. March 22, 2009
52(b). April 4, 2009
14. November 17, 2009