ADDITIONAL PROBLEMS FOR EC1 AND EC2

version of 30 July 2017

CHAPTER 1

1. (a) [2] Let $0 \leq k \leq 2$. Show that for $n \geq 3$, the number of permutations $w \in \mathfrak{S}_n$ whose number of inversions is congruent to k modulo 3 is independent of k. For instance, when $n = 3$ there are two permutations with 0 or 3 inversions, two with one inversion, and two with two inversions.

   (b) [2+] Let M be the multiset $\{1^{a_1}, \ldots, k^{a_k}\}$, where each a_i is a positive integer. Find a simple characterization of those sequences $(a_1, \ldots, a_k)$ for which the number of permutations of M with an even number of inversions is equal to the number with an odd number of inversions. Your condition should not involve any sums.

2. [3] Let $\sigma(n, k)$ be the number of surjections $[n] \to [k]$. Regarding n as fixed, let c_n be the value of k that maximizes $\sigma(n, k)$. Show that $c_n \sim n/(2 \log 2) \approx 0.7213475n$.

3. [2–] Let $f(n)$ be the number of partitions of n for which each part occurs at most twice. For instance, $f(5) = 5$, the partitions being 5, 41, 32, 311, 221. Let $g(n)$ be the number of partitions of n whose parts are not divisible by three. Show that $f(n) = g(n)$ for all $n \geq 0$.

4. [2+] Let $f(n)$ be the number of ways to choose a permutation $w \in \mathfrak{S}_n$ and then choose an element of each cycle of w. (Set $f(0) = 1$.) For instance, $f(1) = 1$ and $f(2) = 3$. Find a simple formula (no infinite sums, in particular) for $\sum_{n \geq 0} f(n) \frac{x^n}{n!}$. (You don't need to find a formula for $f(n)$.)

5. [2+] Find the number $f(n)$ of pairs (π, σ) of partitions of $[n]$ such that σ covers π in the lattice Π_n of partitions of $[n]$. For those not familiar with this poset terminology, this means that σ is obtained from π by merging two blocks of π into a single block. Express your answer in terms of Bell numbers. It should not involve any summation symbols or implied summations like $B(0) + B(1) + \cdots + B(n)$.

6. (a) [3–] For $0 \le k \le d$ define a polynomial $P_{d,k}(n)$ by

$$\sum_{n\ge 0} P_{d,k}(n)x^n = \frac{(1+x)^k}{(1-x)^{d+1}}.$$

Show that $P_{d,k}(n)$ has positive coefficients.

(b) [5–] Is there a nice combinatorial interpretation of the coefficients of $d!P_{d,k}(n)$? The case $k=d$ is especially interesting.

7. [3] For $S,T \subseteq [n-1]$, define $\beta_n(S,T)$ to be the number of permutations $w \in \mathfrak{S}_n$ satisfying $D(w)=S$ and $D(w^{-1})=T$. Let A_n be the $2^{n-1}\times 2^{n-1}$ matrix whose rows and columns are indexed by the subsets of $[n-1]$ (in some order), and whose (S,T)-entry is $\beta_n(S,T)$. Show that $\mathrm{rank}(A_n)=p(n)$, the number of partitions of n.

8. [2+] Evaluate the sum

$$F_n := \sum (-1)^{\lfloor \frac{n-1}{2}\rfloor - k},$$

where the sum is over all chains $\emptyset \subset S_1 \subset \cdots \subset S_k \subset [n]$ of subsets of $[n]$ such that $\#S_i$ is even for $1 \le i \le k$. NOTE. The notation $S \subset T$ means that S is a subset of T and $S \ne T$. The chain $\emptyset \subset [n]$ (the case $k=0$) contributes $(-1)^{\lfloor \frac{n-1}{2}\rfloor}$ to the sum.

9. (a) [2+] For $\mathrm{id} \ne w \in \mathfrak{S}_n$, let $m_1(w)$ be the smallest element of the descent set $D(w)$. Set $m_1(\mathrm{id})=0$. Find the expected value $E_1(n)$ of $m_1(w)$ over all $w \in \mathfrak{S}_n$. Express your answer as a simple sum. Find $\delta_1 := \lim_{n\to\infty} E_1(n)$.

(b) [3] Let $m_k(w)$ denote the kth smallest element of the descent set $D(w)$. Set $m_k(w)=0$ if $\mathrm{des}(w)<k$. Let $\delta_k := \lim_{n\to\infty} E_k(n)$, where $E_k(n)$ is the expected value of $m_k(w)$ for $w\in\mathfrak{S}_n$. Find an explicit formula for $\sum_{k\ge1}\delta_k x^k$ and an asymptotic formula for δ_k as $k\to\infty$.

10. (a) [2+] Fix $n\ge 1$. Let $\mathrm{oa}(n)$ be the number of sets $S\subseteq[n-1]$ for which $\alpha_n(S)$ is odd. Find a simple formula for $\mathrm{oa}(n)$ involving the number $f(m)$ of ordered set partitions of an m-element set. (An exercise in Chapter 1 may prove useful.) NOTE. Though irrelevant here, we have by Example 3.18.10 that $\sum_{m\ge0} f(m)\frac{x^m}{m!} = 1/(2-e^x)$.

(b) [3–] Fix $n \geq 1$. Let $\mathrm{ob}(n)$ be the number of sets $S \subseteq [n-1]$ for which $\beta_n(S)$ is odd. Is $\mathrm{ob}(n)$ always a power of 2?

(c) [5–] What more can be said about the numbers $\mathrm{ob}(n)$?

NOTE. For some analogous problems, see Exercises 1.14(b), 1.15 and 7.15.

11. [3–] Let p be a prime. Find a simple description of all positive integers d with the following property:

$$A(d,k) \equiv (-1)^{k-1}\binom{d-1}{k-1} \pmod{p}, \text{ for all } 1 \leq k \leq d,$$

where $A(d,k)$ is an Eulerian number.

12. (a) [2+] Let $1 \leq a \leq b \leq c \leq d$ with $ad = bc$. Show that $\binom{a+d}{a} \leq \binom{b+c}{b}$.

(b) [5–] Show that the polynomial $\binom{\boldsymbol{b+c}}{\boldsymbol{b}} - \binom{\boldsymbol{a+d}}{\boldsymbol{a}}$ has nonnegative coefficients. Here $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ denotes a q-binomial coefficient.

(c) [5–] Show in fact that the coefficients of $\binom{\boldsymbol{b+c}}{\boldsymbol{b}} - \binom{\boldsymbol{a+d}}{\boldsymbol{a}}$ are unimodal.

13. Call two permutations $u, v \in \mathfrak{S}_n$ *equivalent* if v can be obtained from u by interchanging adjacent elements that differ by 1 (clearly an equivalence relation). For instance, the equivalence classes for $n = 3$ are $\{123, 213, 132\}$ and $\{231, 321, 312\}$.

(a) [3–] Let $f(n)$ be the number of equivalence classes in $\mathfrak{S}_n$, with $f(0) = 1$. Find a simple formula for $f(n)$ as a finite sum. Use this to express the generating function $F(x) = \sum_{n\geq 0} f(n)x^n$ in terms of the power series $G(x) = \sum_{n\geq 0} n!x^n$.

(b) [2+] Show that the size of every equivalence class is a product of Fibonacci numbers.

(c) [3–] Let $N(n)$ be the number of one-element equivalence classes in $\mathfrak{S}_n$. Express the generating function $\sum_{n\geq 0} N(n)x^n$ in terms of $G(x)$.

14. [2+] Show by simple combinatorial reasoning that the Bell number $B(n)$ is even if and only if $n \equiv 2 \pmod{3}$.

CHAPTER 2

15. [2+] Let $f(n)$ be the number of permutations $w \in \mathfrak{S}_{2n}$ such that we never have $w(i) = i$ for $1 \leq i \leq n$ (*not* $1 \leq i \leq 2n$). Find a formula for $f(n)$ involving a single summation symbol and find a simple expression (no summations) for $\lim_{n\to\infty} f(n)/(2n)!$.

CHAPTER 3

16. [2] Find a finite poset P with the following property, or show that no such P exists. The longest chain in P has m elements. P can be written as a union of two chains C_1 and C_2, but cannot be written in this way where $\#C_1 = m$.

17. (a) [2] How many nonisomorphic n-element posets contain an $(n-1)$-element antichain?

    (b) [2+] How many nonisomorphic n-element posets contain an $(n-1)$-element chain?

    (c) [2–] How many nonisomorphic n-element posets contain both an $(n-1)$-element antichain and an $(n-1)$-element chain?

18. (a) [3–] Find a finite poset P with the following property. The automorphism group $\mathrm{Aut}(P)$ of P acts transitively on the set M of minimal elements of P. Moreover, the restriction of $\mathrm{Aut}(P)$ to M does not contain a full cycle of the elements of M.

    (b) [5–] Does such a poset exist if all maximal chains have two elements?

19. [2+] Let $w = t_1, \dots, t_p$ be a permutation of the elements of a finite poset P. Call a permutation w' a *permissible swap* of w if it is obtained from w by interchanging some t_i and t_{i+1} where $t_i < t_{i+1}$. Clearly a sequence of permissible swaps must eventually terminate in a permutation v that has no permissible swaps. Show that v is independent of the sequence of permissible swaps.

20. [2+] For each permutation $w \in \mathfrak{S}_n$, let σ_w be the simplex in $\mathbb{R}^n$ defined by

$$\sigma_w = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, 0 \leq x_{w(1)} \leq x_{w(2)} \leq \cdots \leq x_{w(n)} \leq 1\}.$$

For any nonempty subset $S \subseteq \mathfrak{S}_n$, define

$$X_S = \bigcup_{w \in S} \sigma_w \subset \mathbb{R}^n.$$

Show that X_S is convex if and only if S is the set of linear extensions of some partial ordering of $[n]$.

21. [2+] Let $0 \leq p \leq 1$, and let P be a finite n-element poset with $\hat{0}$ and $\hat{1}$. Let $\sigma\colon P \to [n]$ be a linear extension of P. Define a random digraph D on the vertex set $[n]$ as follows. For each $s < t$ in P, choose the edge $s \to t$ of D with probability p.

    Now start at the vertex $\hat{0}$ of D. If there is an arrow from $\hat{0}$, then move to the vertex t for which $\hat{0} \to t$ is an edge of D and $\sigma(t)$ is as small as possible; otherwise stop. Continue this procedure (always moving from a vertex u to a vertex v for which $u \to v$ is an edge of D and $\sigma(v)$ is as small as possible) until unable to continue. What is the probability that we end at vertex $\hat{1}$? Try to give an elegant proof avoiding recurrence relations, linear algebra, etc.

22. (a) [2+] Let $f(n)$ be the average value of $\mu_P(\hat{0}, \hat{1})$, where P ranges over all (induced) subposets of the boolean algebra B_n containing $\hat{0}$ and $\hat{1}$. (The number of such P is 2^{2^n-2}.) Define the *Genocchi number* G_n by

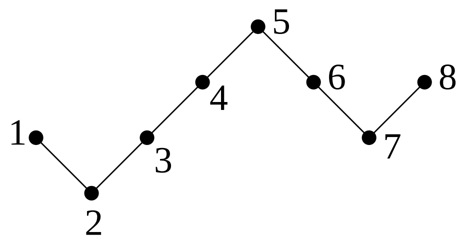
    $$\sum_{n \geq 0} G_n \frac{x^n}{n!} = \frac{2x}{1+e^x},$$

    as in Exercise 5.8(d). Show that $f(n) = 2G_{n+1}/(n+1)$.

    (b) [2] It follows from (a) that $f(n) = 0$ when n is even. Give a noncomputational proof.

23. (a) [2] Let U_n be the set of all lattice paths λ of length $n-1$ (i.e., with $n-1$ steps), starting at $(0,0)$, with steps $(1,1)$ and $(1,-1)$. Thus $\#U_n = 2^{n-1}$. Regard the n integer points on the path λ as the elements of a poset P_λ, such that λ is the Hasse diagram of P_λ. Find $\sum_{\lambda \in U_n} e(P_\lambda)$.

    (b) [2+] Give P_λ the labeling ω_λ by writing the numbers $1, 2, \ldots, n$ along the path. For example, when $n = 8$ one possible pair $(P_\lambda, \omega_\lambda)$ is given by



Find $\sum_{\lambda\in U_n}\Omega_{P_\lambda,\omega_\lambda}(m)$ and $\sum_{\lambda\in U_n} W_{P_\lambda,\omega_\lambda}(q)$.

(c) [3–] Let V_n consist of those $\lambda\in U_n$ which never fall below the x-axis. It is well-known that $V_n=\binom{n-1}{\lfloor (n-1)/2\rfloor}$. Show that $\sum_{\lambda\in V_n} e(P_\lambda)$ is equal to the number of permutations $w\in\mathfrak{S}_n$ of odd order. A formula for this number is given in EC2, Exercise 5.10(c) (the case $k=2$).

(d) [5–] Is there a nice bijective proof or "conceptual proof" of (c)?

(e) [5–] Are there nice expressions for $\sum_{\lambda\in V_n}\Omega_{P_\lambda,\omega_\lambda}(m)$ and/or $\sum_{\lambda\in V_n} W_{P_\lambda,\omega_\lambda}(q)$?

(f) [3–] Now let W_n consist of all $\lambda\in V_{2n+1}$ that end at the x-axis. It is well-known that $\#W_n$ is the Catalan number $C_{n-1}=\frac{1}{n}\binom{2(n-1)}{n-1}$. Show that $\sum_{\lambda\in W_n} e(P_\lambda)$ is equal to the Eulerian-Catalan number $\mathrm{EC}_n = A(2n+1,n+1)/(n+1)$ of EC1, Exercise 1.53.

24. [2+] Let P be a finite poset with $\hat{0}$ and $\hat{1}$. For each $t\in P$ define a polynomial $f_t(x)$ with coefficients in $\mathbb{Z}[y]$ as follows:

$$\begin{aligned} f_{\hat{0}}(x) &= y\\ f_t(x+y) &= \sum_{s\le t} f_s(x). \end{aligned}$$

Express $f_{\hat{1}}(x)$ in terms of the zeta polynomial $Z_P(n)$.

CHAPTER 4

25. (a) [2+] Let $f_k(n)$ be the middle coefficient (i.e., the coefficient of $q^{\lfloor kn/2\rfloor}$) of the q-binomial coefficient $\binom{\boldsymbol{n+k}}{\boldsymbol{k}}$. Find a simple formula for the generating function $\sum_{n\ge 0} f_3(n)x^n$.

(b) [3–] Show that for any $k\in\mathbb{P}$, $f_k(n)$ is a quasipolyomial.

26. Let $f_k(n)$ denote the number of odd coefficients in the q-binomial coefficient $\binom{\boldsymbol{n}}{\boldsymbol{k}}$.

(a) [2+] Show that

$$\sum_{n\geq 2} f_2(n)x^n = \frac{x^2(1+x)}{(1-x)^2(1+x^2)}.$$

(b) [5–] Show that

$$\sum_{n\geq 3} f_3(n)x^n = \frac{P_3(x)}{\phi_1^2\phi_2^2\phi_3\phi_4^2\phi_6\phi_{12}},$$

where $P_3(x)$ has coefficients (beginning with the coefficient of x^3)

$$1, 4, 4, 8, 6, 4, 8, 4, 6, 8, 4, 4, 1,$$

and where ϕ_k is the kth cyclotomic polynomial, normalized to have constant term 1.

(c) [3] Show that $f_k(n)$ a quasipolynomial for fixed k. More generally, if $f_{k,p,j}(n)$ is the number of coefficients of $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ congruent to j modulo the prime p, then for fixed k, p, j the function $f_{k,p,j}(n)$ is a quasipolyomial in n.

27. [2+] Let $f(n)$ denote the number of sequences $a_1a_2\cdots a_n$ with terms $1, 2, 3$ such that no two "cyclically consecutive" elements are equal, i.e., we cannot have $a_i = a_{i+1}$ (subscripts taken modulo n), and such that we cannot have 3 cyclically followed by 1. Give a simple formula for $f(n)$ in terms of the Lucas numbers L_n, defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$. Use the transfer-matrix method.

28. [3–] Fix integers $d \geq 0$, $N \geq 1$. Let $f(n)$ be an *integer-valued* quasipolynomial of degree d and quasiperiod N. Suppose that $f(n) = cn^d + O(n^{d-1})$ for some constant $c > 0$. Write

$$\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$

where P and Q are relatively prime polynomials. What is the smallest possible value of c? What is the least possible degree of $Q(x)$ for which this value of c is achieved?

29. [2+] Let $g(n)$ be the number of ways to tile a $2 \times n$ rectangle with $a \times b$ rectangles for any integers $a, b \geq 1$. (Set $g(0) = 1$.) Show that

$$\sum_{n \geq 0} g(n)x^n = \frac{(1-x)(1-3x)}{1-6x+7x^2}.$$

NOTE. The next six problems (ending in Problem 35) involve walks in *undirected* graphs G. A walk in an undirected graph G can be converted into a walk in a digraph D_G by replacing each nonloop edge of G adjacent to vertices u and v by the two directed edges $u \to v$ and $v \to u$. The adjacency matrix of G is denoted $\boldsymbol{A}(G)$.

30. (a) [2+] Let H_n be the complete bipartite graph K_{nn} with n vertex-disjoint edges removed. Thus H_n has $2n$ vertices and $n(n-1)$ edges, each of *degree* (number of incident edges) $n-1$. Show by linear algebra that the eigenvalues of G are ± 1 ($n-1$ times each) and $\pm(n-1)$ (once each).

    (b) [3–] Give a combinatorial proof. (The proof need not be a bijection, but it should use only combinatorial reasoning.)

31. [2+] Let $n \geq 1$. The *complete p-partite graph* $K(n,p)$ has vertex set $V = V_1 \cup \cdots \cup V_p$ (disjoint union), where each $\#V_i = n$, and an edge from every element of V_i to every element of V_j when $i \neq j$. (If $u, v \in V_i$ then there is no edge uv.) Thus $K(1,p)$ is the complete graph K_p, and $K(n,2)$ is the complete bipartite graph K_{nn}.

    (a) Use equation (4.36) of EC1 to find the number of closed walks of length ℓ in $K(n,p)$.

    (b) Deduce from (a) the eigenvalues of $K(n,p)$.

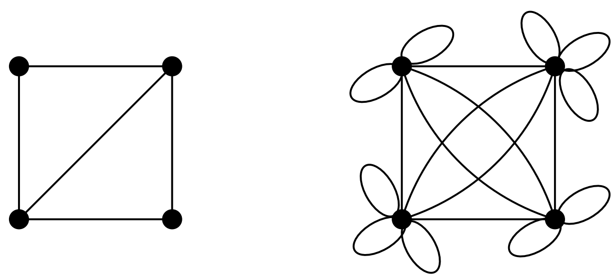
32. [2+] Let G be any finite simple graph, with eigenvalues $\lambda_1, \ldots, \lambda_p$. ("Simple" means no loops or multiple edges.) Let $G(n)$ be the graph obtained from G by replacing each vertex v of G with a set V_v of n vertices, such that if uv is an edge of G, then there is an edge from every vertex of V_u to every vertex of V_v (and no other edges). For instance, $K_p(n) = K(n,p)$. Find the eigenvalues of $G(n)$ in terms of $\lambda_1, \ldots, \lambda_p$.

33. [2] Let G be a (finite) graph with vertices $v_1, \dots, v_p$ and eigenvalues $\lambda_1, \dots, \lambda_p$. By Corollary 4.7.4 there are real numbers $c_1(i,j), \dots, c_p(i,j)$ such that for all $\ell \geq 1$,

$$\left(\boldsymbol{A}(G)^\ell\right)_{ij} = \sum_{k=1}^{p} c_k(i,j)\lambda_k^\ell.$$

Show that $c_k(i,i) \geq 0$. Show also that if $i \neq j$ then we can have $c_k(i,j) < 0$. (The simplest possible example will work.)

34. [2] Let G be a finite graph with eigenvalues $\lambda_1, \dots, \lambda_p$. Let $G^\star$ be the graph with the same vertex set as G, and with $\eta(u,v)$ edges between vertices u and v (including $u = v$), where $\eta(u,v)$ is the number of walks in G of length two from u to v. For example,



G $\quad\quad$ G^*

Find the eigenvalues of $G^\star$ in terms of those of G.

35. [2+] Let G be a finite simple (no loops or multiple edges) graph with at least two vertices. Suppose that for some $\ell \geq 1$, the number of walks of length ℓ between any two vertices u, v (including $u = v$) is odd. Show that there is a nonempty subset S of the vertices such that S has an even number of elements, and such that every vertex v of G is adjacent to an even number of vertices in S. (A vertex v is adjacent to itself if and only if there is a loop at v.)

CHAPTER 5

36. [3–] Let $T(x) = \sum_{n \geq 0} n^{n-1}\frac{x^n}{n!}$ and $U(x) = \sum_{n \geq 0} n^n \frac{x^n}{n!}$. Show that

$$U(x)^3 - U(x)^2 = \frac{T(x)}{(1-T(x))^3} = \sum_{n \geq 0} n^{n+1}\frac{x^n}{n!}.$$

37. [2] Let $h(n)$ be the number of ways n children can divide up into groups, where each group consists of a nonempty subset of children standing in a circle, with some children (at least one) inside the circle. This is just like Example 5.2.3, except that the circles can contain any positive number of children, not just one (perhaps not very physically realistic). As usual set $h(0) = 1$. For instance, $h(1) = 0$, $h(2) = 2$, $h(3) = 6$, $h(4) = 30$. Find $E_h(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!}$. Your answer should not involve logarithms.

38. (a) [2+] Let $h(n)$ be the number of ways n children can form a set of concentric circles by holding hands. For instance, with 12 children, we could have four of them forming a circle C_1 (in $3!\binom{12}{4}$ ways). Inside C_1 is a circle C_2 of just one child. Inside C_2 is a circle C_3 of two children. *Outside* C_1 is a circle C_4 of two children, and inside C_4 is a circle C_5 of the remaining three children. Show that

$$\begin{aligned} E_h(x) &= (1-x)^{-1/(1+\log(1-x))} \\ &= 1 + x + 4\frac{x^2}{2!} + 24\frac{x^3}{3!} + 190\frac{x^4}{4!} + 1860\frac{x^5}{5!} + \cdots . \end{aligned}$$

(b) [2+] Now the children can arrange themselves into circles in *any* way. That is, inside any circle C is a disjoint union (possibly empty) of circles with a similar structure inside each of them. Show that

$$\begin{aligned} E_h(x) &= 1 + \left(1 - (1+x)^{-1/(1+x)}\right)^{\langle -1\rangle} \\ &= 1 + x + 4\frac{x^2}{2!} + 27\frac{x^3}{3!} + 260\frac{x^4}{4!} + 3280\frac{x^5}{5!} + \cdots . \end{aligned}$$

39. [2+] Let $H(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!}$, where $h(n)$ is the number of certain structures that can be put on an n-set as in Section 5.1. (Thus each structure is uniquely a disjoint union of connected structures.) Let $r(n)$ (respectively, $s(n)$) be the number of ways of putting a structure on an n-set and then putting the connected components into a cycle (respectively, linearly ordering them). Express $E_r(x)$ and $E_s(x)$ in terms of $H(x)$. Use this to express $E_s(x)$ in terms of $E_r(x)$. Then give a simple explanation of this last formula.

40. Let $f(n)$ be the number of distinct graphs G (allowing multiple) edges on the vertex set $[2n]$ such that the edges of G can be partitioned into

two complete matchings. Thus G has $2n$ edges. Find a simple formula for the generating function

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + 6\frac{x^2}{4!} + \cdots .$$

41. [2+] Fix $k \geq 1$. Choose an unrooted tree T on the vertex set $[n]$ uniformly at random. What is the probability $p_k(n)$ that vertex 1 has degree k (i.e., has exactly k neighbors)? Find $\lim_{n\to\infty} p_k(n)$.

42. (a) [2+] Let $f(n)$ be the number of ways to choose a rooted tree T on $[n]$ and then for each vertex v of T, either do nothing or choose a child of v. (Thus if v is an endpoint we have only one choice—do nothing.) For instance, $f(1) = 1$, $f(2) = 4$, $f(3) = 33$. Find a formula for $f(n)$ as a simple sum.

    (b) [3–] Give a simple combinatorial proof.

CHAPTER 6

43. (a) [3+] Show that the power series

$$F(x) = \sum_{n\geq 0} \frac{(10n)!\, n!}{(5n)!\,(4n)!\,(2n)!}x^n$$

is algebraic.

    (b) [3+] Do the same for

$$G(x) = \sum_{n\geq 0} \frac{(30n)!\, n!}{(15n)!\,(10n)!\,(6n)!}x^n.$$

Find the (minimal) degree of $G(x)$.

44. (a) [2+] What curious property does the following power series possess?

$$F(x) = x - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \frac{1}{8}x^5 + \frac{13}{64}x^7 - \frac{145}{256}x^9 + \frac{1305}{1024}x^{10}$$
$$-\frac{1587}{1024}x^{11} + \frac{8379}{8192}x^{12} - \frac{2009}{16384}x^{13} + \cdots .$$

(b) [5–] Is there a "reasonable" formula for the coefficients?

45. (a) [2+] Let $f(n)$ be the number of plane trees with n vertices such that if a vertex u has exactly one child v, then v is an endpoint. Let

$$F(x) = \sum_{n\geq 1} f(n)x^n = x + x^2 + x^3 + 3x^4 + \cdots .$$

Find an explicit formula for $F(x)$ and $F(x)^{\langle -1\rangle}$.

(b) [5–] Is there some simple combinatorial explanation for the relationship between these two generating functions? Can this phenomenon be generalized?

46. (a) [2+] Let t be an indeterminate. Find the coefficients of the generating function $F(x,y) = 1/(1-x-y)^t$.

(b) [3–] Find a simple formula (involving a single finite sum) for the diagonal $D(z) = \mathcal{D}F(x,y)$ when t is a positive integer.

47. [5–] Let $f(n)$ be an integer-valued unbounded P-recursive function. Show that $f(n)$ is composite for infinitely many positive integers n. (This surely must be true, since otherwise there is a simple recurrence for generating arbitrarily large primes. Perhaps the result is already known, but I have been unable to find it in the literature.)

48. (a) [2+] Let $f_d(n)$ be the number of walks in the first quadrant of $\mathbb{Z}^d$ (i.e., all coordinates nonnegative) starting at the origin and with steps $\pm e_i$, where e_i is the ith unit coordinate vector. Show that for fixed d, the function $f_d(n)$ is P-recursive.

(b) [3–] Find a simple formula for $f_2(n)$ and a three-term linear recurrence with polynomial coefficients satisfied by $f_2(n)$.

49. [2] Let $f\colon \mathbb{N} \to \mathbb{Q}$ be P-recursive, and let d be the least integer for which there is a recurrence

$$P_d(n)f(n+d) + P_{d-1}(n)f(n+d-1) + \cdots + P_0(n)f(n) = 0, \quad n \geq 0,$$

with $P_i(n) \in \mathbb{C}[n]$ and $P_d(n) \neq 0$. Show that there exists such a recurrence with $P_i(n) \in \mathbb{Z}[n]$.

50. (a) [3–] Show that $f(n) = n^n$ is not P-recursive.

(b) [2+] Does there exist a P-recursive function $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(n) \sim n^n$, i.e., $\lim_{n\to\infty} \frac{f(n)}{n^n} = 1$?

51. [2] Let $f\colon \mathbb{N} \to \mathbb{C}$ be P-recursive. Show that $\frac{\log f(n)}{n \log n}$ is bounded as $n \to \infty$. Thus for instance 2^{n^2} is not P-recursive.

52. (a) [3–] Show that for $\alpha \in \mathbb{R}$ there exists a P-recursive function $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(n) \sim n^\alpha$.

    (b) [2–] Let A be the set of all $\alpha \in \mathbb{R}$ for which there exists a P-recursive function $f\colon \mathbb{N} \to \mathbb{Z}$ satisfying $f(n) \sim n^\alpha$. Show that A is a submonoid of the additive reals.

    (c) [3] Show that $2^{-k} \in A$ for all $k \in \mathbb{N}$ and that $\sqrt{17} \in A$.

    (d) [5–] What more can be said about the monoid A?

53. [2+] Let $f(n) = n! + \frac{1}{n!}$. Using just hand computation, find a nontrivial linear recurrence with polynomial coefficients satisfied by $f(n)$.

54. [2] Let $u \in K[[x]]$ be D-finite, say

$$p_d(x)u^{(d)} + p_{d-1}(x)u^{(d-1)} + \cdots + p_1(x)u' + p_0(x)u = 0 \qquad (12)$$

as in EC2, equation (6.31). Find a nontrivial homogeneous linear differential equation with polynomial coefficients satisfied by $u+1$. (You can write the coefficients in any form that makes it clear that they are polynomials in x.)

55. (a) [2] Let $1 \le n_1 < n_2 < n_3 < \cdots$, and let

$$F(x) = \sum_{i\ge 0} a_i x^{n_i} \in K[[x]],$$

where $\mathrm{char}(K) = 0$ and $a_i \neq 0$ for all $i \ge 0$. Show that if

$$\limsup_{i\to\infty} (n_{i+1} - n_i) = \infty,$$

then $F(x)$ is not D-finite. For instance, $\sum_{n\ge 0} x^{\binom{n}{2}}$ is not D-finite.

    (b) [2] Show that (a) need not be true if $\mathrm{char}(K) > 0$. Show in fact that (a) can fail for algebraic $F(x)$.

(c) [5–] Suppose that $\limsup_{i\to\infty} n_i^{1/i} = \infty$. Is it true that then $F(x)$ is not D-finite for *any* field K? (Perhaps this is already known.)

56. [3+] Fix a subset $\mathcal{F}$ of $\mathfrak{S}_k$, and let $\mathrm{Av}_n(\mathcal{F})$ denote the number of permutations $w \in \mathfrak{S}_n$ that avoid the patterns in $\mathcal{F}$ (in the sense of `en.wikipedia.org/wiki/Permutation_pattern`). Show that $\mathrm{Av}_n(\mathcal{F})$ need not be D-finite.