18.S34 (FALL, 2010)

LINEAR ALGEBRA PROBLEMS

Useful ideas for evaluating determinants

1. Row reduction, expanding by minors, or combinations thereof; sometimes these are useful in combination with an induction argument.
2. Listing enough eigenvectors to account for all of the eigenvalues.
3. If M is a square matrix of rank 1, then $\det(I + M) = 1 + \operatorname{tr}(M)$. Generalization to higher rank is left as an exercise.
4. If you see two identical rows or columns, the determinant is zero. If everything in the matrix is a polynomial in some indeterminates, you may be able to use this to find some factors of the determinant.
5. If you can get the matrix into some sort of block form, that might help.

Fact: the Perron-Frobenius theorem A square matrix with positive entries has a unique eigenvector with positive entries. Moreover, the eigenvalue for that eigenvector has strictly bigger absolute value than any other eigenvalue. (If your matrix has some zero entries, you may be able to apply the theorem to a power of the matrix.)

Example 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix}.$$

Example 2 Compute the determinant of

$$A = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{bmatrix}.$$

1. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

   Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

2. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{bmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{bmatrix}.$$

   is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers.

3. Let M_n be the $(2n+1) \times (2n+1)$ for which

$$(M_n)_{ij} = \begin{cases} 0, & i = j \\ 1, & i - j \equiv 1, \ldots, n \pmod{2n+1} \\ -1, & i - j \equiv n+1, \ldots, 2n \pmod{2n+1}. \end{cases}$$

   Find the rank of M_n.

4. Let A be the $n \times n$ matrix with $A_{jk} = \cos(2\pi(j+k)/n)$. Find the determinant of $I + A$.

5. Let a_{ij} $(i, j = 1, 2, 3)$ be real numbers such that $a_{ij} > 0$ for $i = j$, and $a_{ij} < 0$ for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the quantities $a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3$ for $i = 1, 2, 3$ are either all positive, all negative, or all zero.

6. Let x, y, z be positive real numbers, not all equal, and define

$$a = x^2 - yz, \qquad b = y^2 - zx, \qquad c = z^2 - xy.$$

   Express x, y, z in terms of a, b, c. (Hint: can you find a linear algebra interpretation of a, b, c, by making a certain matrix involving x, y, z?)

7. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

8. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

9. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

10. A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some points the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on. (Hint: interpret as a linear algebra problem over the field of two elements.)

11. Suppose that A, B, C, D are $n \times n$ matrices (with entries in some field), such that AB^T and CD^T are symmetric, and $AD^T - BC^T = I$. Prove that $A^T D - C^T B = I$. (Hint: find a more "matricial" interpretation of the condition $AD^T - BC^T = I$.)

12. Let A be a $2n \times 2n$ skew-symmetric matrix (i.e., a matrix in which $A_{ij} = -A_{ji}$) with integer entries. Prove that the determinant of A is a perfect square. (Hint: prove a polynomial identity.)

13. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \ldots, n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

    HINT. Consider the $n \times n$ matrix $M_{n,k}$ defined by

    $$(M_{n,k})_{ij} = \begin{cases} 1, & |i - j| \leq k \\ 0, & \text{otherwise.} \end{cases}$$

14. Let A be an $n \times n$ real symmetric matrix and B an $n \times n$ positive definite matrix. (A square matrix over $\mathbb{R}$ is *positive definite* if it is symmetric and all its eigenvalues are positive.) Show that all eigenvalues of AB are real. HINT. Use the following two facts from linear algebra: (a) all eigenvalues of a real symmetric matrix are real, and (b) a positive definite matrix has a positive definite square root.

15. Let p be a prime, and let $A = (a_{ij})_{i,j=0}^{p-1}$ be the $p \times p$ matrix defined by

    $$a_{ij} = \binom{i+j}{i}, \quad 0 \leq i, j \leq p - 1.$$

    Show that $A^3 \equiv I \pmod{p}$, where I denotes the identity matrix. In other words, every entry of $A^3 - I$, evaluated over $\mathbb{Z}$, is divisible by p.

16. Let A be an $n \times n$ real matrix with every row and column sum equal to 0. Let $A[i, j]$ denote A with row i and column j removed. Show that $\det A[i, j]$ is independent of i and j. Can you express this determinant in terms of the eigenvalues of A?

17. Find the unique sequence $a_0, a_1, \ldots$ of real numbers such that for all $n \geq 0$ we have

    $$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

    (When $n = 0$ the second matrix is empty and by convention has determinant one.)

18. (a) Let $A = A(n)$ be the $n \times n$ real matrix given by

$$A_{ij} = \begin{cases} 1, & j = i+1 \ (1 \le i \le n-1) \\ 1, & j = i-1 \ (2 \le i \le n) \\ 0, & \text{otherwise.} \end{cases}$$

Let $V_n(x) = \det(xI - A)$, so $V_0(x) = 1$, $V_1(x) = x$, $V_2(x) = x^2 - 1$, $V_3(x) = x^3 - 2x$. Show that $V_{n+1}(x) = xV_n(x) - V_{n-1}(x)$, $n \ge 1$.

(b) Show that

$$V_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Deduce that the eigenvalues of $A(n)$ are $2\cos(j\pi/(n+1))$, $1 \le j \le n$.

19. Given $v = (v_1, \dots, v_n)$ where each $v_i = 0$ or 1, let $f(v)$ be the number of even numbers among the n numbers

$$v_1+v_2+v_3, v_2+v_3+v_4, \dots, v_{n-2}+v_{n-1}+v_n, v_{n-1}+v_n+v_1, v_n+v_1+v_2.$$

For which positive integers n is the following true: for all $0 \le k \le n$, exactly $\binom{n}{k}$ vectors of the 2^n vectors $v \in \{0,1\}^n$ satisfy $f(v) = k$?

20. Let $M(n)$ denote the space of all real $n \times n$ matrices. Thus $M(n)$ is a real vector space of dimension n^2. Let $f(n)$ denote the maximum dimension of a subspace V of $M(n)$ such that every nonzero element of V is invertible.

(a) (easy) Show that $f(n) \le n$.

(b) (fairly easy) Show that if n is odd, then $f(n) = 1$.

(c) (extremely difficult) For what n does $f(n) = n$?

(d) (even more difficult) Find a formula for $f(n)$ for all n.

Solution to Example 1 Of course you can do this by hand, but here is a more instructive method that generalizes easily. As a function of the x_i, $\det(A)$ is a polynomial of degree 3. Moreover, it vanishes whenever $x_i = x_j$ because then the matrix A has two equal columns. So as a polynomial $\det(A)$ is divisible by $(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$, which also has degree 3, so the two must agree up to a scalar factor. Moreover, they both have the same coefficient of $x_1 x_2^2$ since you get that from the main diagonal in A and from all the first terms when expanding the product. Hence

$$\det(A) = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1).$$

Solution to Example 2 Put $\zeta = \exp(2\pi i/3)$. Now note that the column vectors

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \zeta & \zeta^2 \end{bmatrix}, \begin{bmatrix} 1 & \zeta^2 & \zeta \end{bmatrix}$$

are eigenvectors with respective eigenvalues

$$a_0 + a_1 + a_2, a_0 + \zeta a_1 + \zeta^2 a_2, a_0 + \zeta a_2 + \zeta^2 a_1.$$

We claim that the determinant is the product of these three eigenvalues: this is certainly true if the eigenvalues are distinct. But that is true if we take a_0, a_1, a_2 to be elements of the field of rational functions in those three indeterminates, so we get the equality

$$A = (a_0 + a_1 + a_2)(a_0 + \zeta a_1 + \zeta^2 a_2)(a_0 + \zeta a_2 + \zeta^2 a_1)$$

as a polynomial identity, and so it holds whatever the a_i are. Alternatively, you can use Example 1 to show that the three eigenvectors we wrote down are linearly independent, so the determinant is the product of their eigenvalues whether or not those eigenvalues are distinct.