Directed Steiner Tree and the Lasserre Hierarchy

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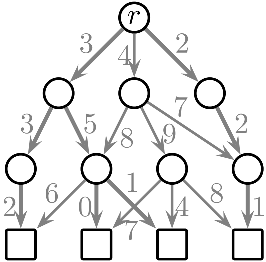
- directed weighted graph $G = (V, E, c)$



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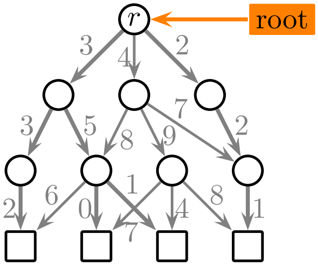

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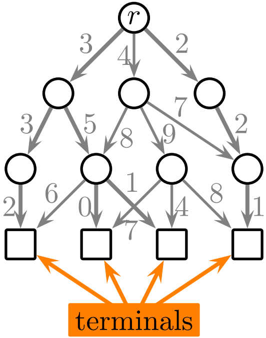

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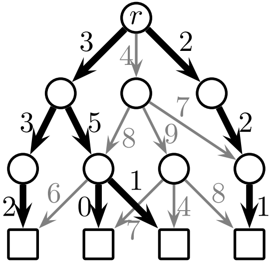

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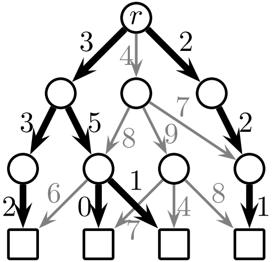

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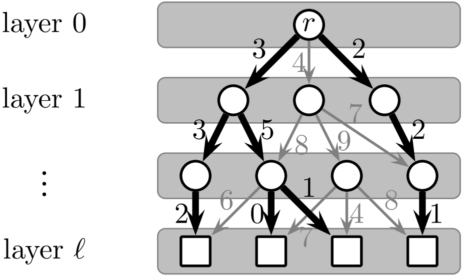

- W.l.o.g. G is **acyclic**
- Modulo $O(\log|X|)$ factor, may assume $\ell = \log|X|$ **levels** ($\exists \ell$-level tree of cost $\ell \cdot |X|^{1/\ell} \cdot OPT$ [Zelikovsky '97])

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Generalizes:

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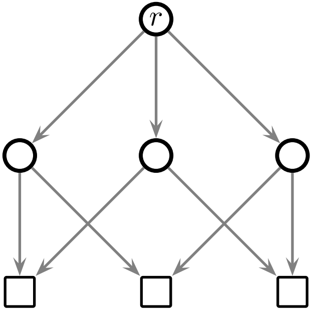
What about LPs?

A flow based LP



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Constraints:

$$\min \sum_{e \in E} c_e y_e$$

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\sum_{e \in \delta^+(v)} f_{s,e} - \sum_{e \in \delta^-(v)} f_{s,e} \quad &= \quad \begin{cases} 1 & v = r \\ -1 & v = s \\ 0 & \text{otherwise} \end{cases} && \forall s \in X \ \forall v \in V \\
f_{s,e} \quad &\leq \quad y_e && \forall s \in X \ \forall e \in E \\
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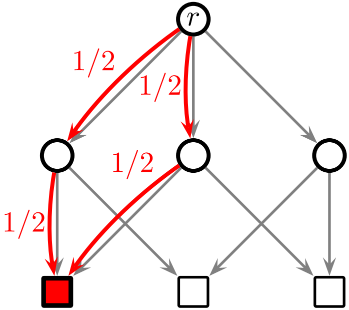

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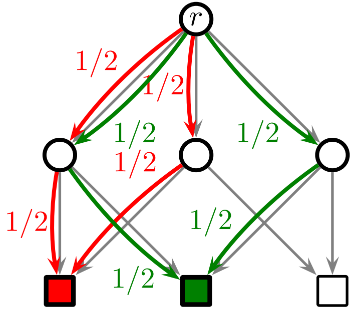

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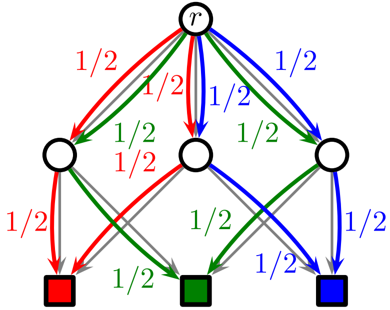

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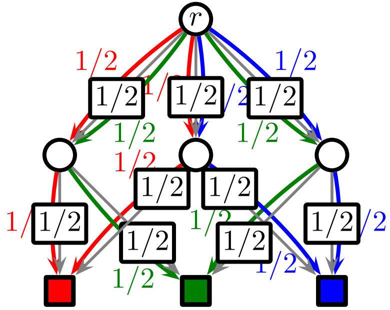

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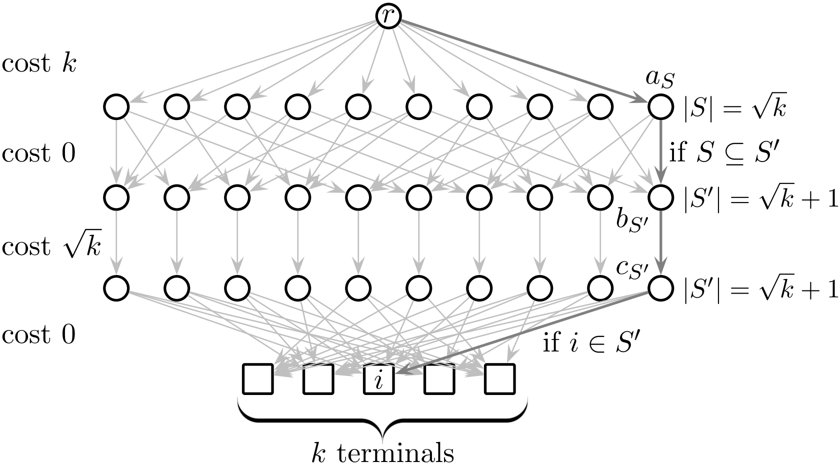
Integrality gap instance [Zosin - Khuller '02]



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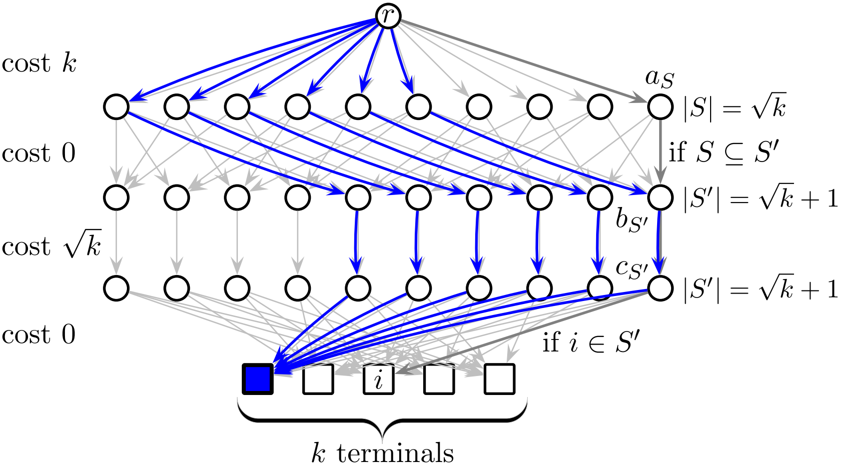

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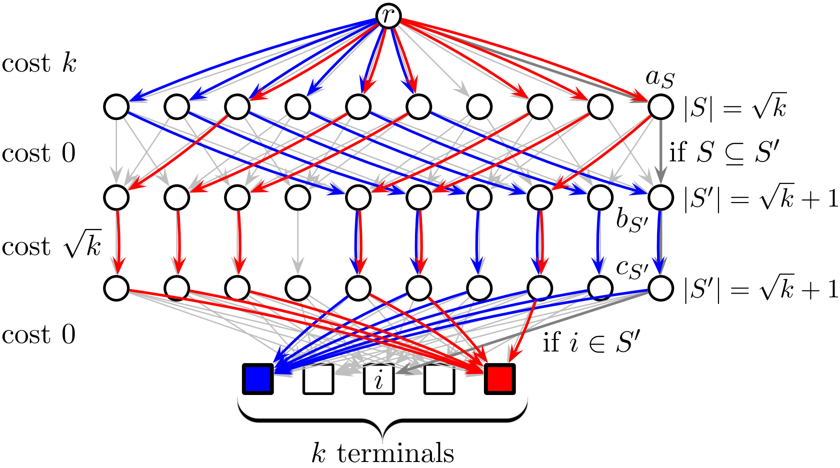

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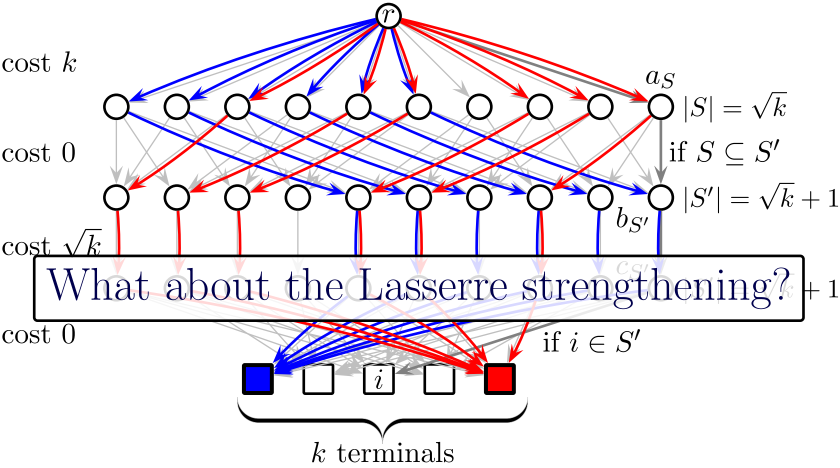

What about the Lasserre strengthening?

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Round-t Lasserre relaxation

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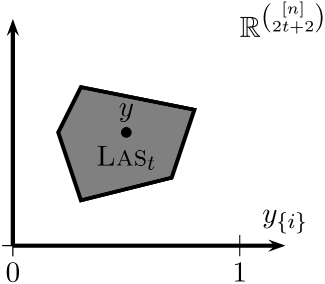
\begin{aligned}
(y_{I \cup J})_{|I|,|J| \leq t+1} &\succeq 0 \\
\Big(\sum_{i \in [n]} A_{\ell i} y_{I \cup J \cup \{i\}} - b_\ell y_{I \cup J} \Big)_{|I|,|J| \leq t} &\succeq 0 \quad \forall \ell \in [m] \\
y_\emptyset &= 1
\end{aligned}
$$

Properties of Lasserre hierarchy

Theorem

Let $K = \{x \in \mathbb{R}^n \mid Ax \geq b\}$; $\quad y \in \textsc{Las}_t(K)$; $\quad |I|, |J| \leq t$



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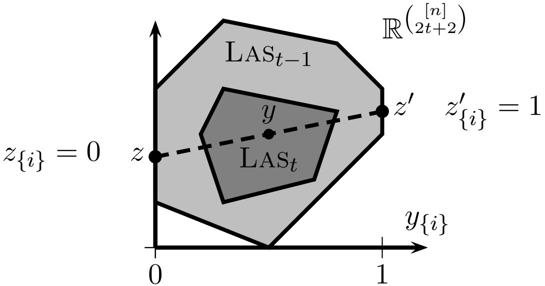

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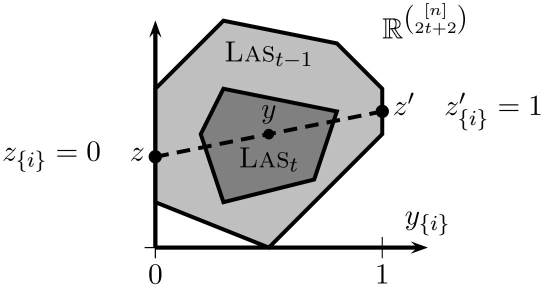

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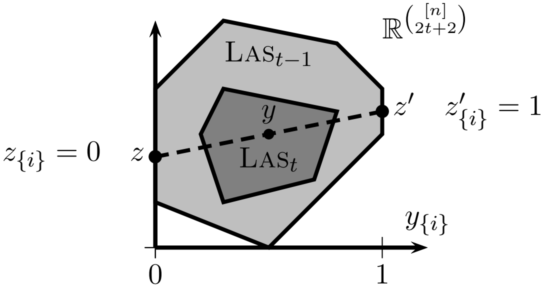

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- **Example:** For Knapsack take $S := \{\text{large items}\}$

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- Decomposition **not true** for *Sherali-Adams* or *Lovász-Schrijver* hierarchies

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(c) **Convergence:** $conv(K \cap \{0,1\}^n) = \textsc{Las}_n^{proj}(K)$

(d) **Monotonicity:** $I \supseteq J \implies 0 \leq y_I \leq y_J \leq 1$

(e) $y_I = 1 \iff \bigwedge_{i \in I}(y_{\{i\}} = 1)$.

(f) $(\forall i \in I : y_{\{i\}} \in \{0,1\}) \implies y_I = \prod_{i \in I} y_{\{i\}}$.

(g) $y_I = 1 \implies y_{I \cup J} = y_J$.

Our contribution

Theorem

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- Recall: gap is $\Omega(\sqrt{|X|})$ (for $\ell = 4$) without strengthening.
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Input: Group Steiner Tree instance

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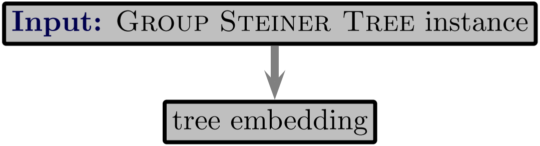

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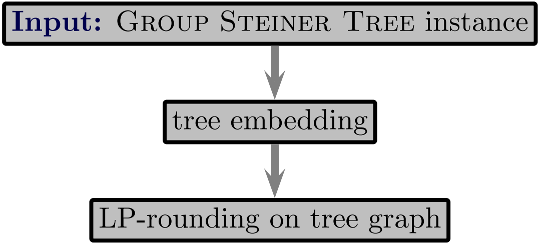

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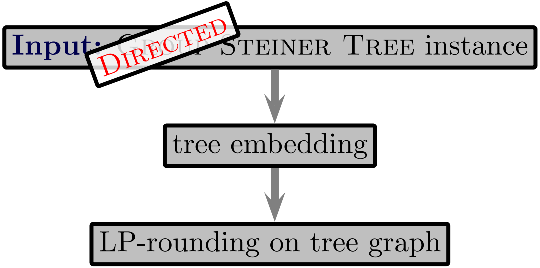

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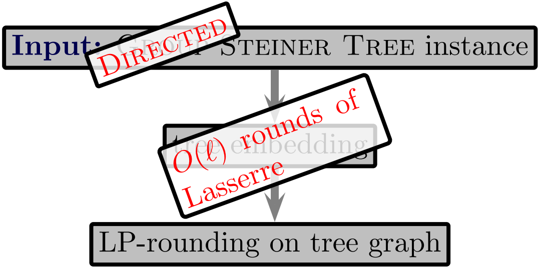

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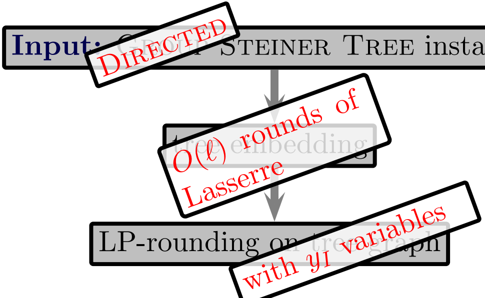


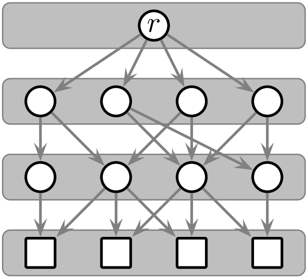
The rounding algorithm

- Let $Y \in \textsc{Las}_{O(\ell)}(\text{LP})$ (y_P value for $\{y_e \mid e \in P\}$-variables)

(1) $T := \{\emptyset\}$

(2) FOR all $P \in T$ and incident $e \in E$ DO

  (3) $\Pr[\text{add } P \cup \{e\} \text{ to } T] = \frac{y_{P \cup \{e\}}}{y_P}$



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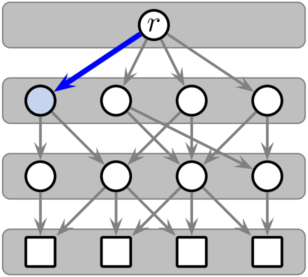

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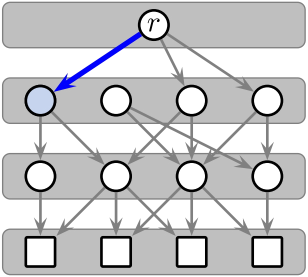

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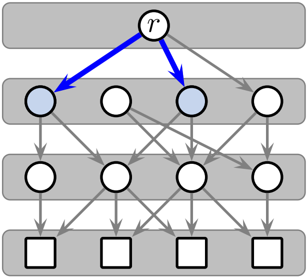

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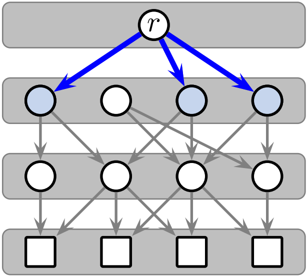

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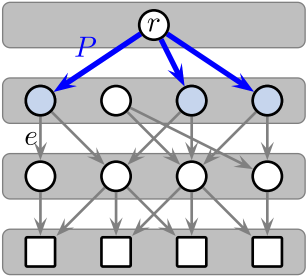

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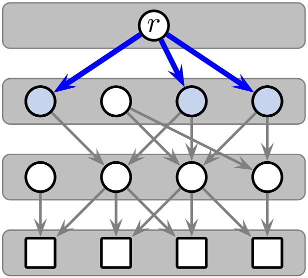

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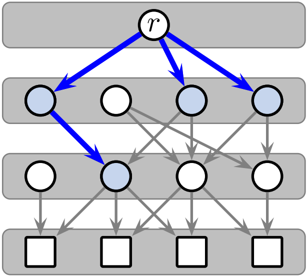

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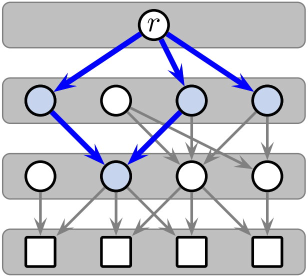

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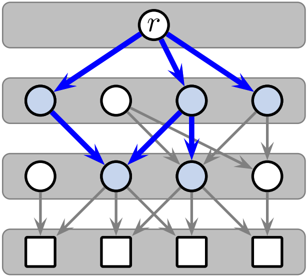

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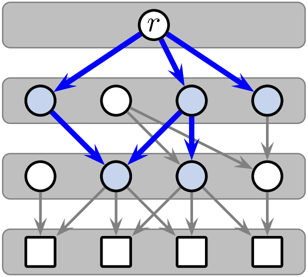

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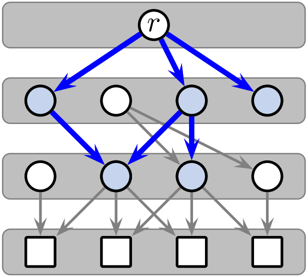

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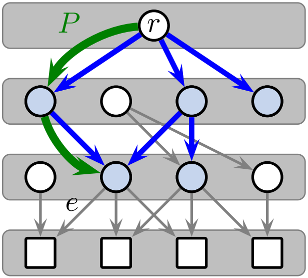

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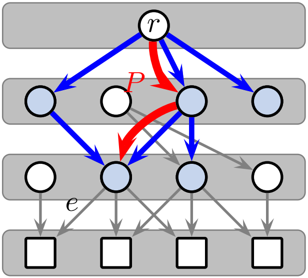

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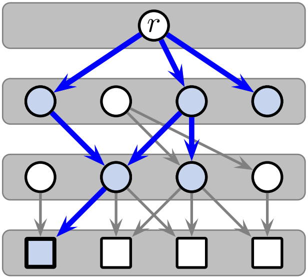

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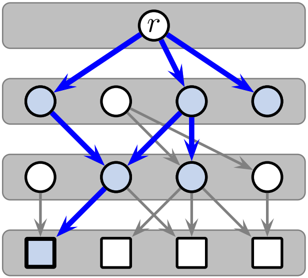

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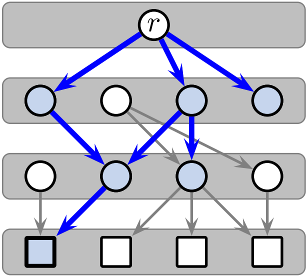

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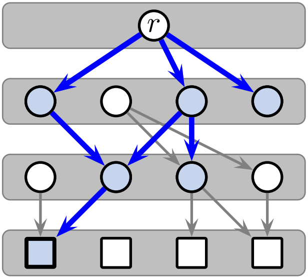

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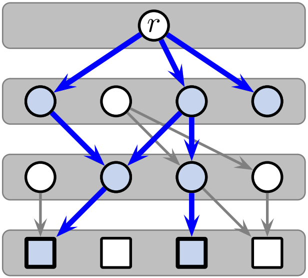

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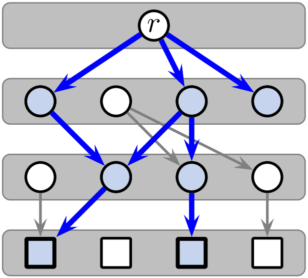

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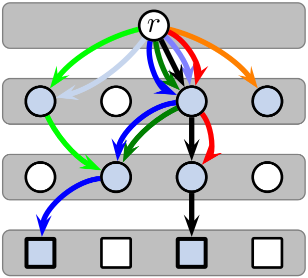

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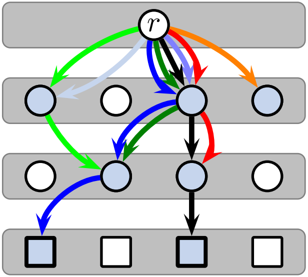

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Road map:

- Show $\Pr[e \in T] = y_{\{e\}}$

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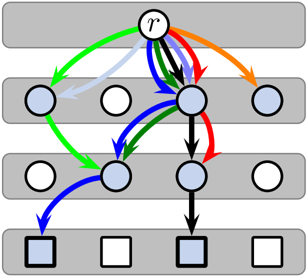

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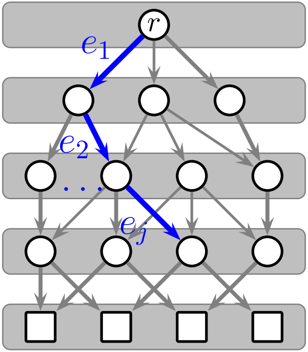
- Show $\Pr[e \in T] = y_{\{e\}}$
- $\Pr[s \text{ connected}] \geq \Omega(\frac{1}{\#\text{levels}})$ for each terminal s

Probability to sample a particular path

Lemma

For any root-path P: $\Pr[P \in T] = y_P$.

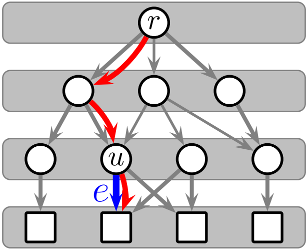
$$\Pr[P \in T] = y_{\{e_1\}} \cdot \frac{y_{\{e_1,e_2\}}}{y_{\{e_1\}}} \cdot \frac{y_{\{e_1,e_2,e_3\}}}{y_{\{e_1,e_2\}}} \cdot \ldots \cdot \frac{y_P}{y_{P \setminus \{e_j\}}} = y_P.$$



Upper bounding the expected cost

Lemma

$$\sum_{P \text{ ending in } e} y_P \leq y_{\{e\}}$$



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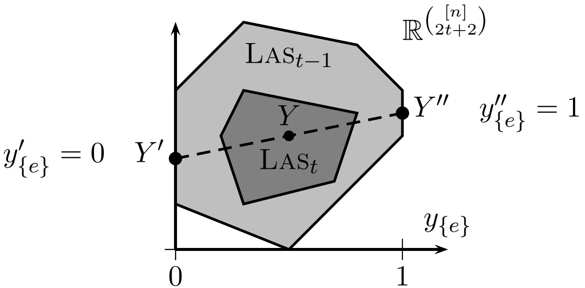

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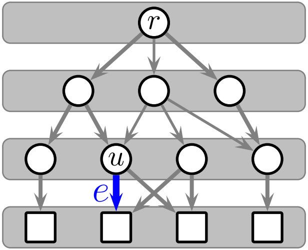

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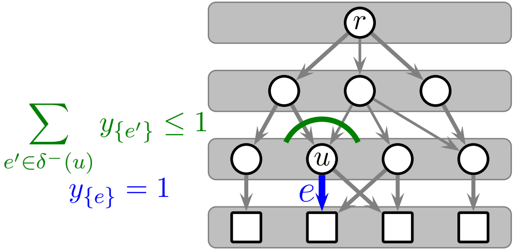

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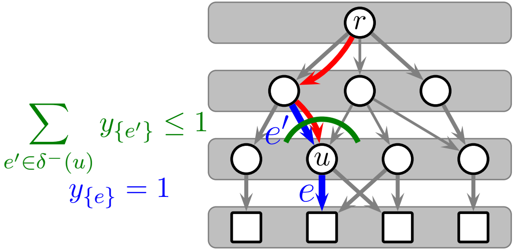

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$$\sum_{P \text{ ending in } e} y_P = \sum_{e' \in \delta^-(v)} \sum_{P \text{ ending in } e'} y_P \leq 1 \quad \square$$

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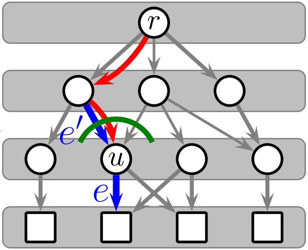

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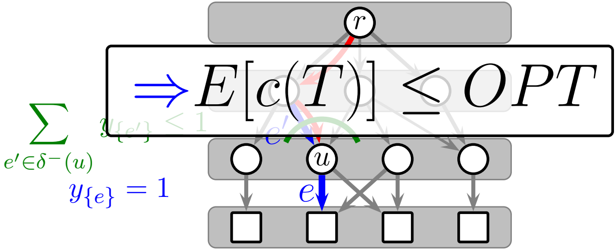
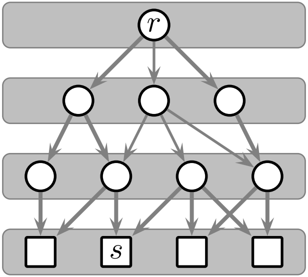


$$\Rightarrow E[c(T)] \leq OPT$$

Each terminal connected once in expectation

Lemma

For terminal s: $$\sum_{P \text{ ending in } s} y_P = 1.$$



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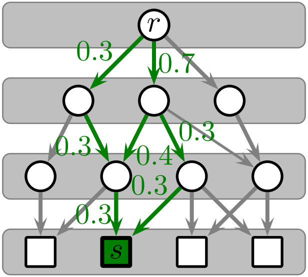

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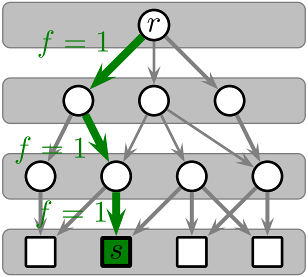

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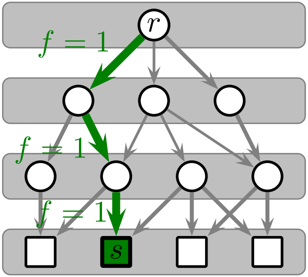

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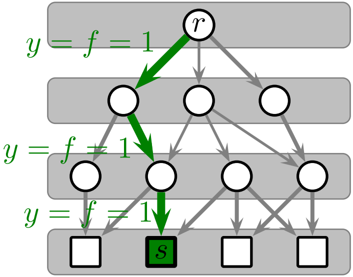

Each terminal connected once in expectation

Lemma

For terminal s: $\sum_{P \text{ ending in } s} y_P = 1.$

- No feasible frac. flow with $|\{e : f_{s,e} = 1\}| > \ell$
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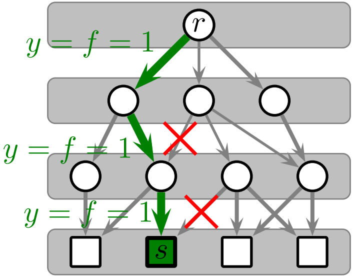

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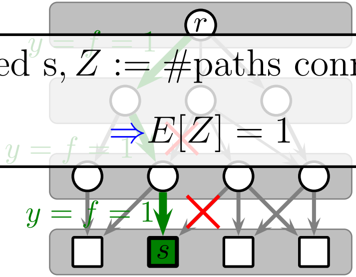


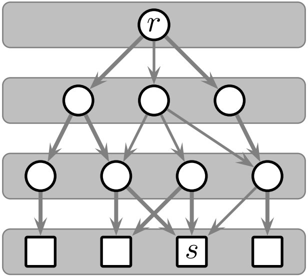
For fixed s, $Z :=$ #paths connecting s

$\Rightarrow E[Z] = 1$

Upper bounding the conditional expectation

Lemma

$E[Z \mid Z \geq 1] \leq \ell + 1.$



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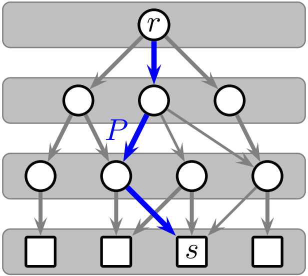

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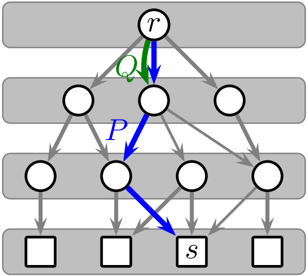

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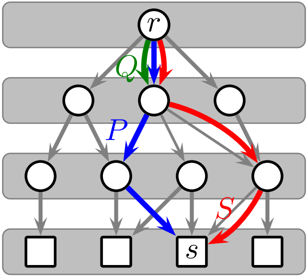

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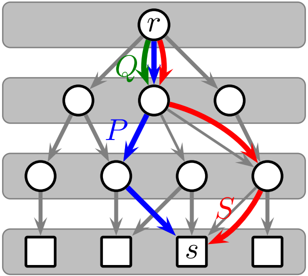
$$\sum_{S : S \supseteq Q, s \in S} \Pr[S \in T \mid Q \in T] \overset{\text{cond. prob.}}{\leq} \sum_{S : S \supseteq Q, s \in S} \frac{y_S}{y_Q} \overset{\substack{\text{as in previous} \\ \text{lemma}}}{\leq} 1 \quad \square$$



Done...

- Recall: $Z = \#$paths connecting a fixed terminal s

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$\Pr[Z \geq 1] \geq \frac{1}{\ell+1}$.

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$1 = E[Z]$

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- Recall: $Z = \#$paths connecting a fixed terminal s

Lemma

$\Pr[Z \geq 1] \geq \frac{1}{\ell+1}$.

$$1 = E[Z] = \Pr[Z = 0] \cdot \underbrace{E[Z \mid Z = 0]}_{=0} + \Pr[Z \geq 1] \cdot \underbrace{E[Z \mid Z \geq 1]}_{\leq \ell+1} \qquad \square$$

Open problems

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Is there a convex relaxation for Directed Steiner Tree that

- has $\mathrm{polylog}(|\mathbf{X}|)$ integrality gap
- can be solved in **polytime**?

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Thanks for your attention