Bin Packing via Discrepancy of Permutations

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Cargèse Workshop 2010

DISCRETE OPTIMIZATION

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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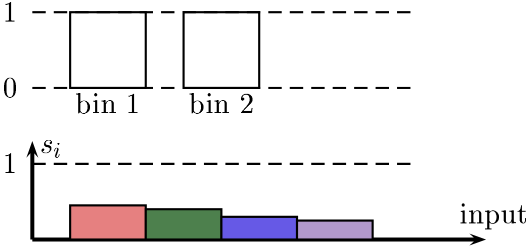
- Items with sizes $s_1, \ldots, s_n \in [0, 1]$

Goal: Pack items into minimum number of **bins** of size 1.



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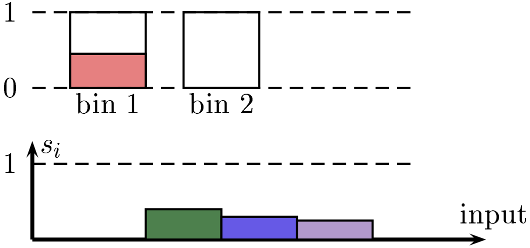

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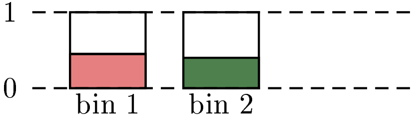

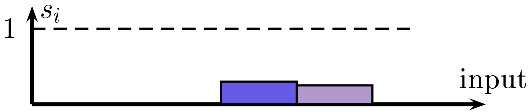

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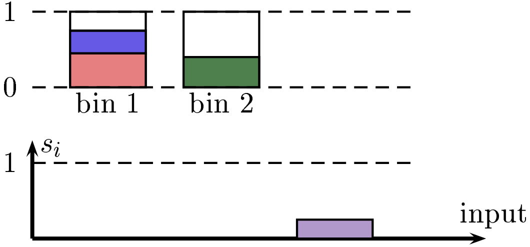

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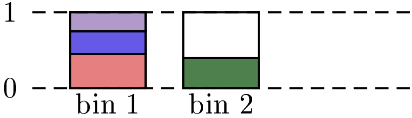

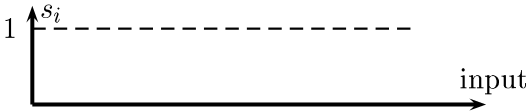

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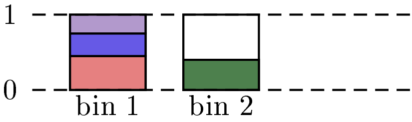

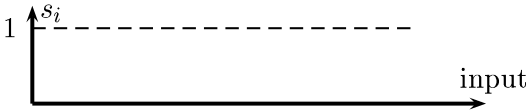

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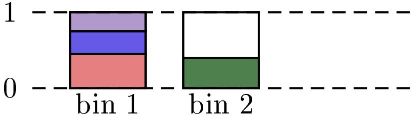

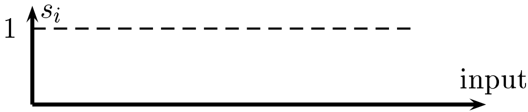

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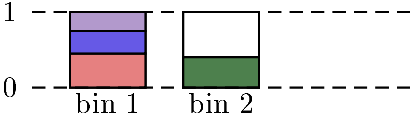

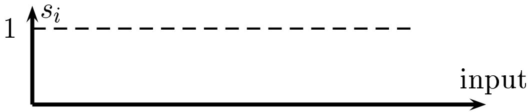

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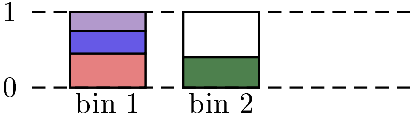

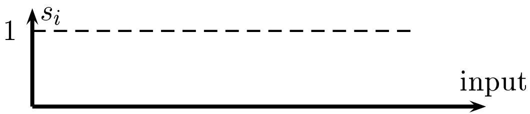

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- Strongly **NP**-hard even if $\frac{1}{4} < s_i < \frac{1}{2} \rightarrow$ **3-Partition**

The Gilmore Gomory LP relaxation

- Feasible patterns:

$$\mathcal{P} = \{p \in \{0,1\}^n \mid s^T p \leq 1\}$$

- Gilmore Gomory LP relaxation:

$$\begin{aligned} \min \sum_{p \in \mathcal{P}} x_p & & & \\ \sum_{p \in \mathcal{P}} p \cdot x_p & \geq & \mathbf{1} & \\ x_p & \geq & 0 & \quad \forall p \in \mathcal{P} \end{aligned}$$

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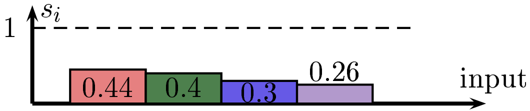

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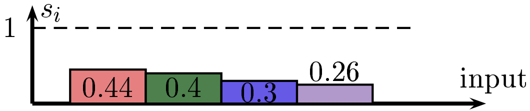

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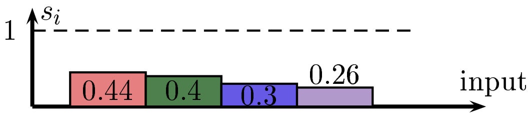

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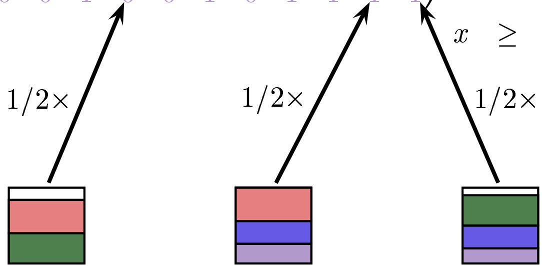

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Modified Integer Roundup Conjecture

$OPT \leq \lceil OPT_f \rceil + 1$

- True, if # of different item sizes ≤ 7 [Sebő, Shmonin '09]
- Best known general bound: $OPT \leq OPT_f + O(\log^2 n)$

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Question

Is additive gap **constant** (if $\frac{1}{4} < s_i < \frac{1}{2}$)?

Beck's Conjecture

3-Permutations Conjecture [Beck]

Given any **3 permutations** on n symbols, one can color the symbols with red and blue, such that in any interval of any of those permutations, the number of red and blue symbols differs by $O(1)$.

permutation 1: 4 6 1 5 7 2 8 3

permutation 2: 7 8 2 5 3 4 1 6

permutation 3: 2 1 6 4 8 5 3 7

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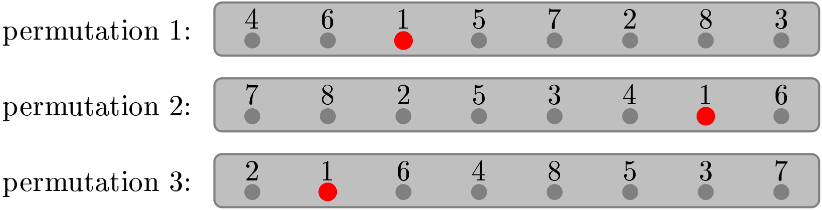

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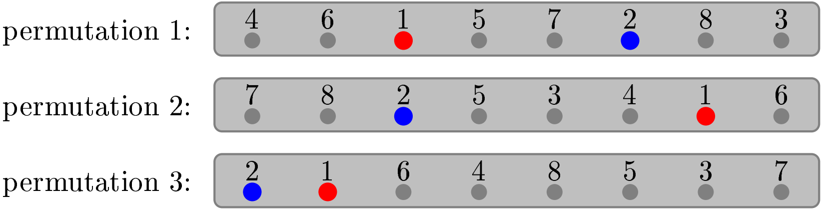

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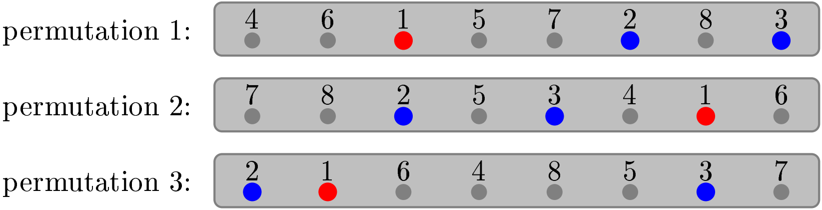

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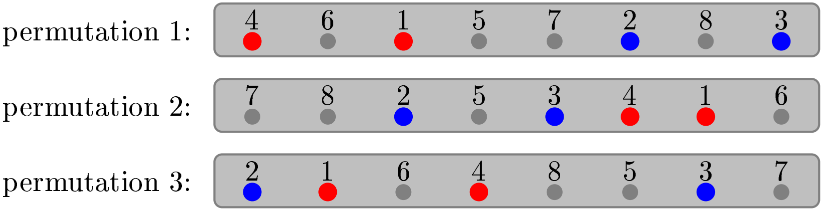

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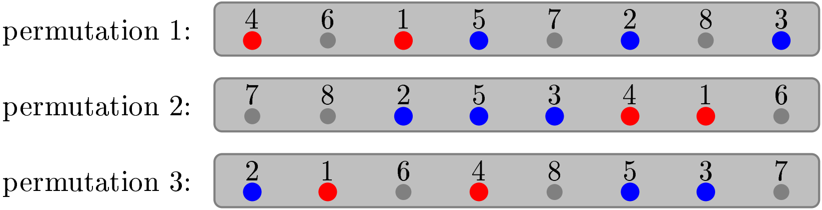

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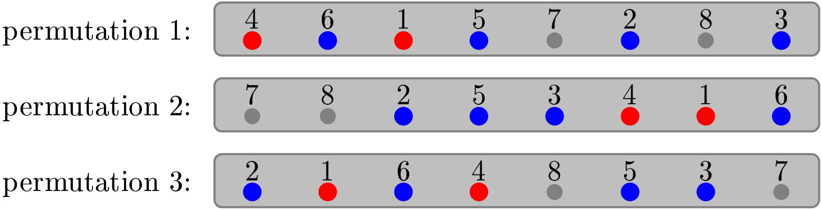

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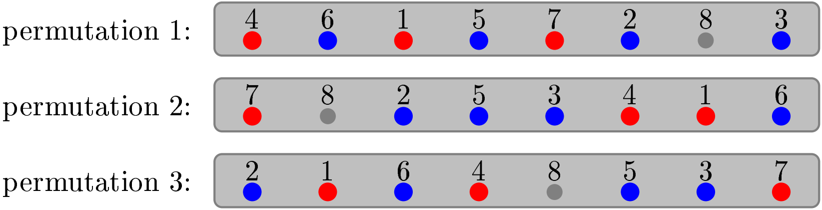

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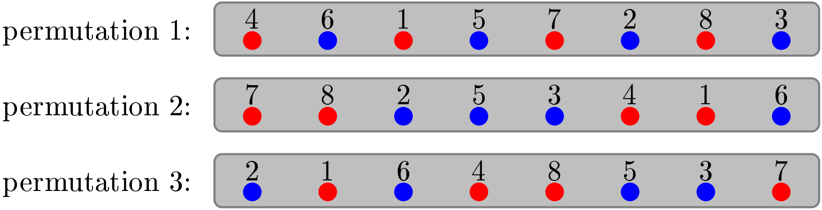

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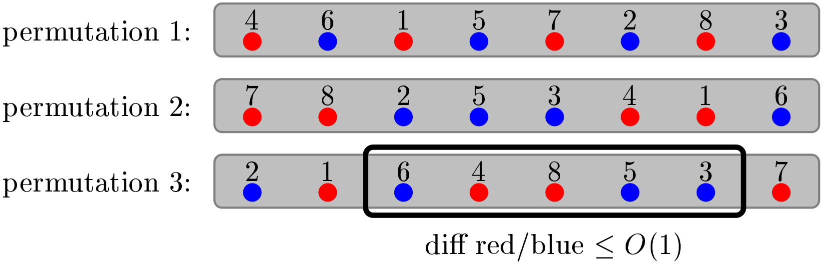

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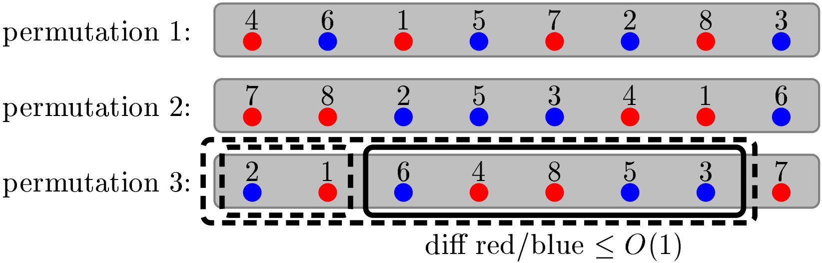

- W.l.o.g. consider intervals that start at beginning

Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}, S_i \subseteq [n]$

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- Every element in $\leq t$ sets: $\mathrm{disc}(\mathcal{S}) < 2t$ [Beck & Fiala '81]
  Conjecture: $\mathrm{disc}(\mathcal{S}) \leq O(\sqrt{t})$

Matrix discrepancy

- Matrix A

$$\mathrm{disc}(A) := \min_{x \in \{0,1\}^n} \|Ax - A \cdot (\frac{1}{2}, \ldots, \frac{1}{2})\|_\infty$$

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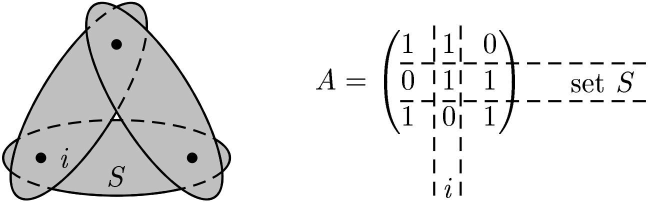

$$\mathrm{disc}(\mathcal{S}) \quad = \quad 2\cdot\mathrm{disc}(A)$$

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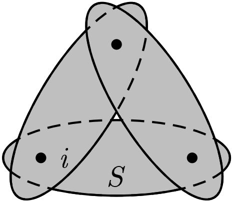


$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

set S

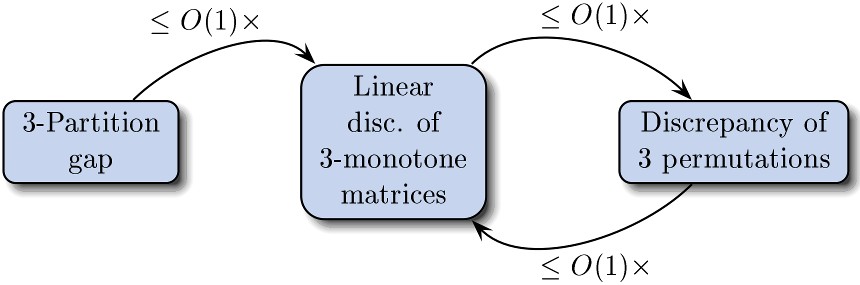
i

$$\text{disc}(\mathcal{S}) = 2 \cdot \text{disc}(A)$$

- **Linear discrepancy:**

$$\text{lindisc}(A) := \max_{y \in [0,1]^n} \min_{x \in \{0,1\}^n} \|Ax - Ay\|_\infty$$

Overview



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3-Partition gap
Linear disc. of 3-monotone matrices
Discrepancy of 3 permutations
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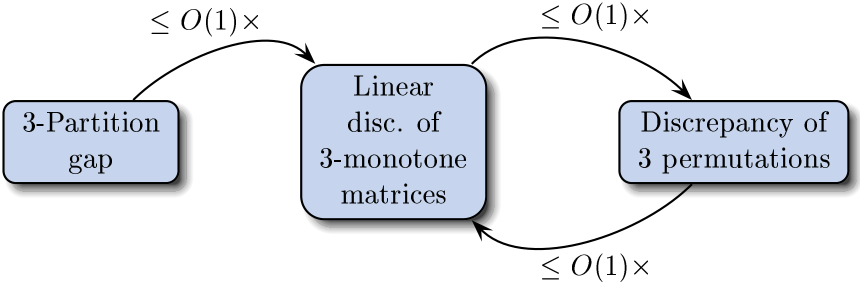

- A is **3-monotone**, if
  - columns are monotone increasing
  - entries are $\in \{0, \ldots, 3\}$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

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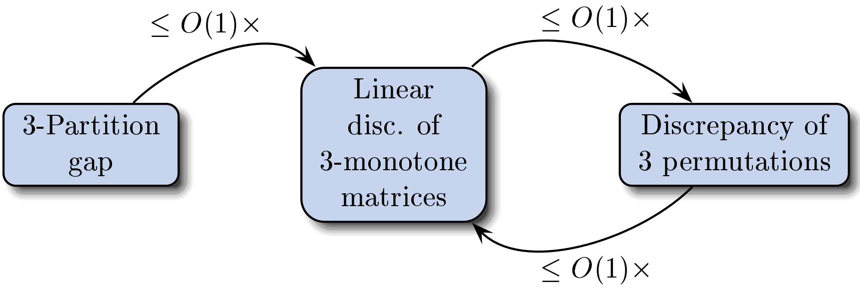

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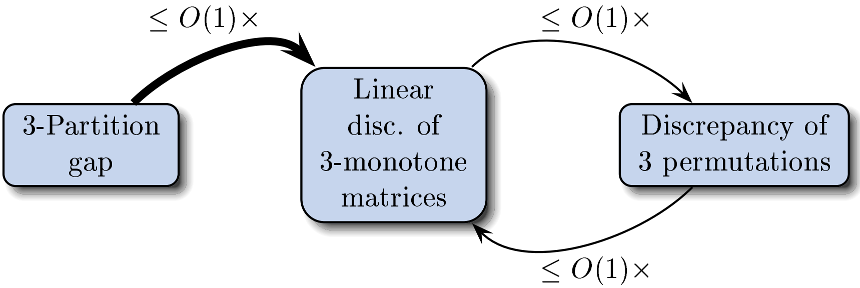

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Reduction: Gap $\rightarrow$ LinDisc

Lemma

Suppose $lindisc(A) \leq O(1)$ *for any 3-monotone matrix* A*. Then the* 3*-Partition gap is* $O(1)$*.*

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- y be opt. fractional 3-Partition sol., B pattern matrix

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

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- Add up rows $1, \ldots, i$ to obtain row i for new matrix A. Append row $(3, \ldots, 3)$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 3 & 3 \\ \hline 3 & 3 & 3 \end{pmatrix}$$

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- Add up rows $1, \ldots, i$ to obtain row i for new matrix A. Append row $(3, \ldots, 3)$
- A is 3-monotone $\Rightarrow x \in \{0,1\}^m : \|Ax - Ay\|_\infty = O(1)$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 3 & 3 \\ \hline 3 & 3 & 3 \end{pmatrix}$$

Reduction: Gap $\to$ LinDisc

Lemma

Suppose lindisc$(A) \leq O(1)$ for any 3-monotone matrix A. Then the 3-Partition gap is $O(1)$.

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- Due to last row: $\mathbf{1}^T x = \mathbf{1}^T y \pm O(1)$
- $B_i y = 1 \;\Rightarrow\; A_i y = i \;\Rightarrow\; A_i x = i \pm O(1)$
- Due to ith row: x reserves $i \pm O(1)$ slots for items $1, \ldots, i$

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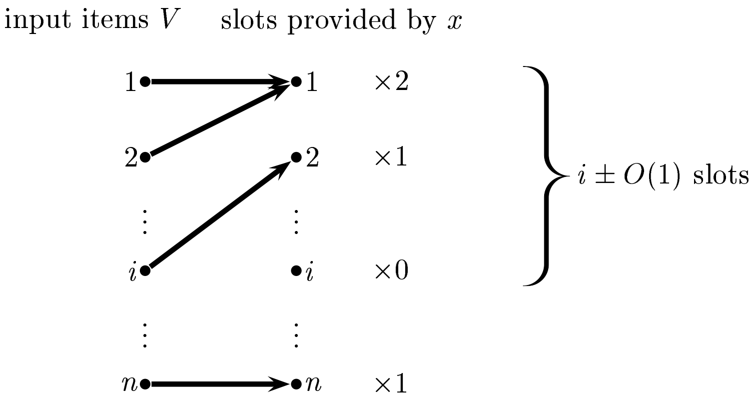
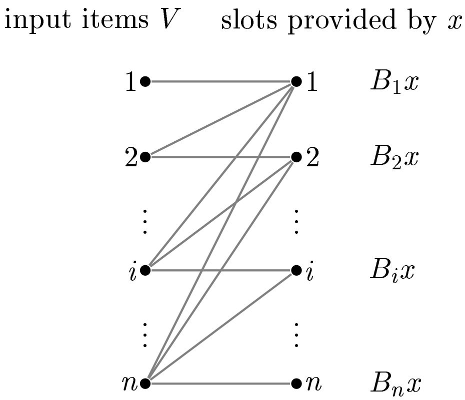
Reduction: Gap → LinDisc (2)

input items V	slots provided by x		
$1\bullet$	$\bullet 1$	$\times 2$	$i \pm O(1)$ slots
$2\bullet$	$\bullet 2$	$\times 1$	
$\vdots$	$\vdots$		
$i\bullet$	$\bullet i$	$\times 0$	
$\vdots$	$\vdots$		
$n\bullet$	$\bullet n$	$\times 1$	

Reduction: Gap → LinDisc (2)



Reduction: Gap → LinDisc (2)



- Bipartite graph $G = (V \dot{\cup} U, E)$ with $(i,j) :\Leftrightarrow s_i \leq s_j$

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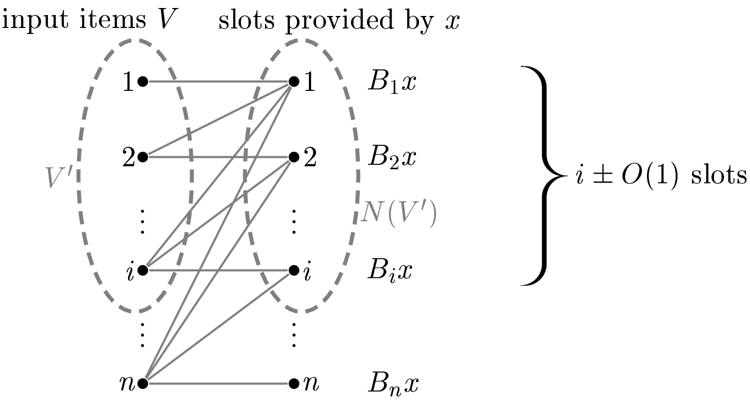

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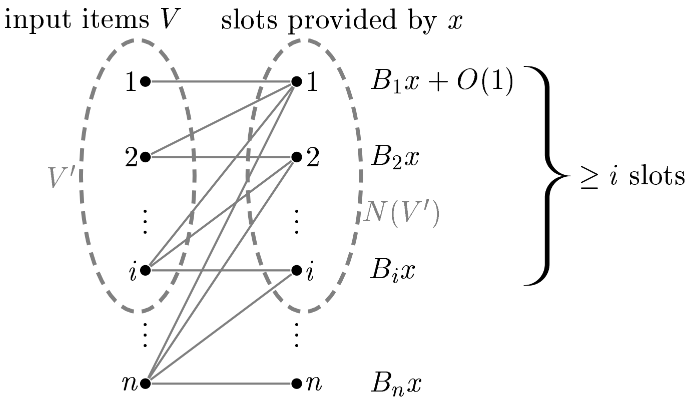

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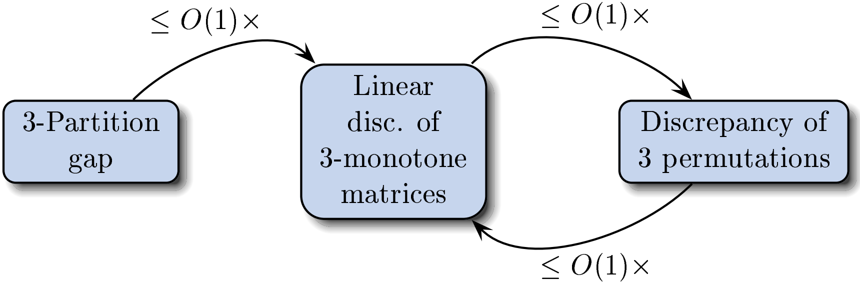
- $x + O(1)$ extra bins is feasible (costs $\leq OPT_f + O(1)$) □

Overview



$\leq O(1)\times$
3-Partition gap
Linear disc. of 3-monotone matrices
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Discrepancy of 3 permutations
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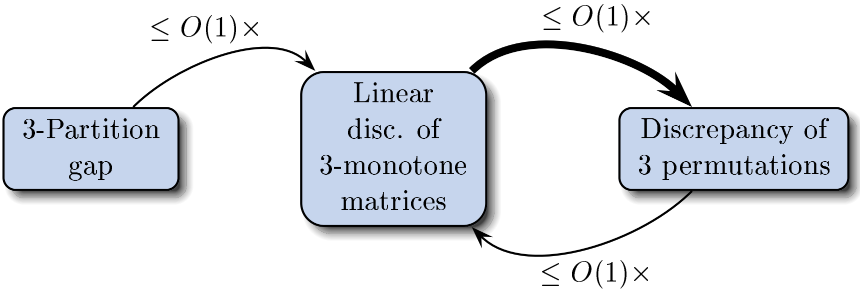
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Reduction: LinDisc $\to$ Perm.Disc.

Lemma

Let A be 3-monotone. Beck's Conjecture $\Rightarrow$ $lindisc(A) = O(1)$.

- Let $x \in [0,1]^n$ be given.
- Goal: Find $y \in \{0,1\}^n$ with $Ax \approx Ay$

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Theorem (Lovász, Spencer & Vesztergombi '86)

There is always a submatrix B of A with $lindisc(A) \leq 2 \cdot disc(B)$.

- Intuitively: Worst case is $x \in \{0, \frac{1}{2}\}^n$
- It suffices to show: $\text{disc}(A) = O(1)$

Reduction: LinDisc $\rightarrow$ Perm.Disc. (2)

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 3 & 2 \\ 3 & 2 & 3 & 2 \end{pmatrix}$$

Reduction: LinDisc → Perm.Disc. (2)

- Write $A = B^1 + B^2 + B^3$ with B^i 1-monotone

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 3 & 2 \\ 3 & 2 & 3 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{=B^1} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{=B^2} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}}_{=B^3}$$

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$\pi_1 = (3, 1, 2, 4)$ $\qquad$ $\pi_2 = (\mathbf{1}, \mathbf{3}, \mathbf{4}, 2)$ $\qquad$ $\pi_3 = (1, 3, 2, 4)$

Reduction: LinDisc $\to$ Perm.Disc. (2)

- Write $A = B^1 + B^2 + B^3$ with B^i 1-monotone
- Column order of B^i induces permutation π_i
- Let $\chi : [n] \to \{\pm 1\}$ be coloring that's good for $\pi_1, \ldots, \pi_3$.

$$\text{disc}(A) \leq \|A\chi\|_\infty \overset{\text{triangle ineq}}{\leq} \sum_{i=1}^{3} \underbrace{\|B^i\chi\|_\infty}_{=O(1)} = O(1) \qquad \square$$

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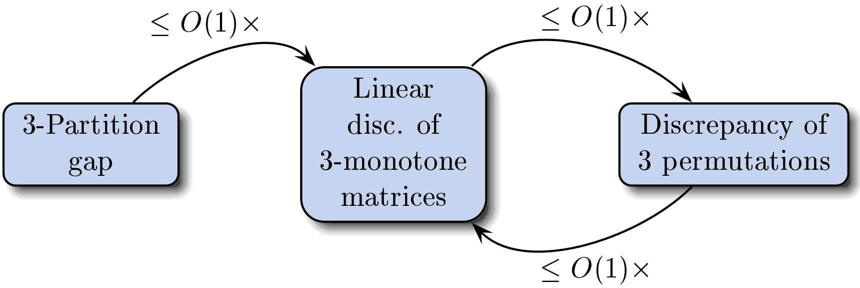
Open problem (1)



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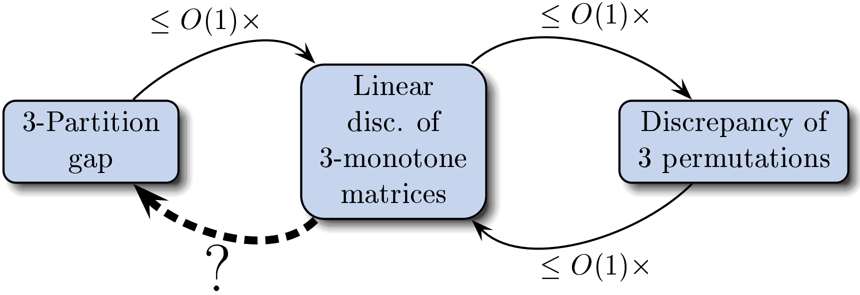

Open problem (2)

- Define

$$\Delta(n) := \max\left\{ \mathrm{disc}(A) \mid \begin{array}{c} A \in [0,1]^{n \times n}, \\ A \text{ has monotone columns} \end{array} \right\}$$

- **Example:**

$$A = \begin{pmatrix} 0.1 & 0.0 & 0.5 \\ 0.4 & 0.7 & 0.9 \\ 0.5 & 0.9 & 1.0 \end{pmatrix}$$

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Lemma

For **any** *Bin Packing instance*

$$OPT \leq OPT_f + O(\log n) \cdot \Delta(n).$$

- We can prove $\Delta(n) \leq O(\log n)$.

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Question

Is $\Delta(n) = O(1)$?

The end

Thanks for your attention

- *Bin Packing via Discrepancy of Permutations*
  (F. Eisenbrand, D. Pálvölgyi, T. Rothvoß - to appear in SODA'11; http://arxiv.org/abs/1007.2170)