# Approximating Bin Packing within $O(\log OPT \cdot \log\log OPT)$ bins

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TCS+ Online Seminar

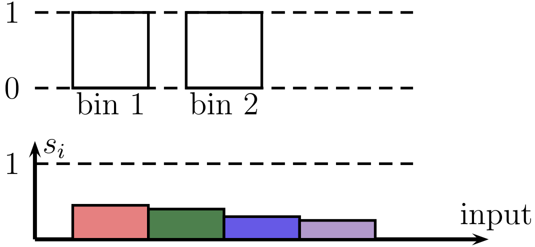
May 22, 2013

Massachusetts Institute of Technology

# Bin Packing

**Input:** Items with sizes $s_1, \ldots, s_n \in [0, 1]$

**Goal:** Pack items into minimum number of **bins** of size 1.



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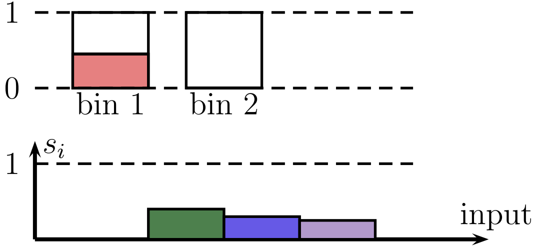

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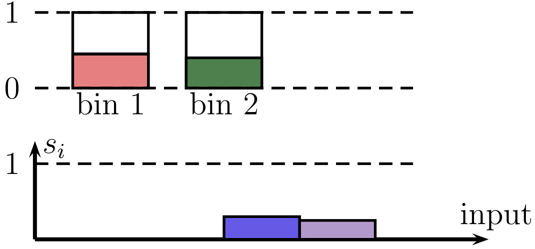

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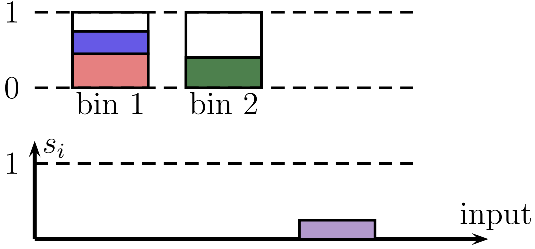

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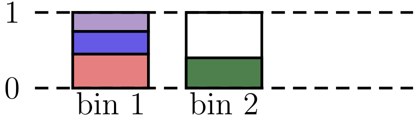


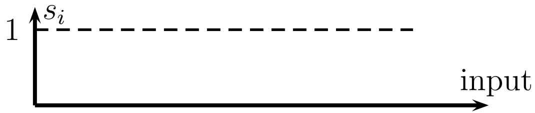

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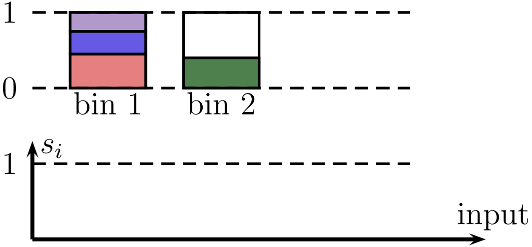


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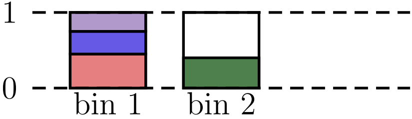


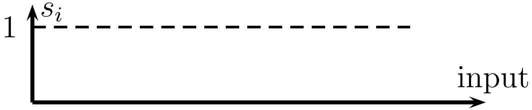


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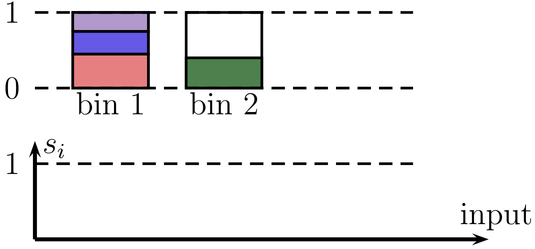


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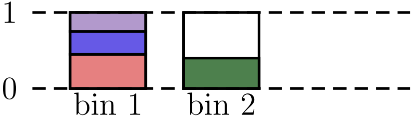


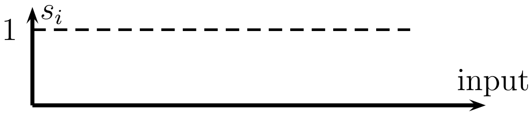


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- [Karmarkar & Karp '82]: $APX \leq OPT + O(\log^2 OPT)$ in poly-time

# The Gilmore Gomory LP relaxation

- $b_i = \#$items with size $s_i$
- **Feasible patterns**:

$$\mathcal{P} = \left\{ p \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^{n} s_i p_i \leq 1 \right\}$$

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$$\begin{aligned} \min \sum_{p \in \mathcal{P}} x_p & & & \\ \sum_{p \in \mathcal{P}} p_i \cdot x_p & \geq & b_i & \quad \forall i \in [n] \\ x_p & \geq & 0 & \quad \forall p \in \mathcal{P} \end{aligned}$$

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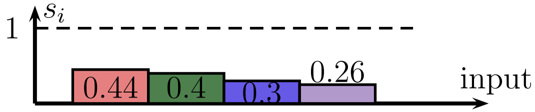
- Can find $x$ with $\mathbf{1}^T x \leq OPT_f + \delta$ in time poly$(\|b\|_1, \frac{1}{\delta})$

# The Gilmore Gomory LP - Example



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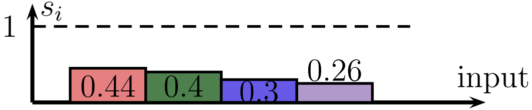


$$\min \mathbf{1}^T x$$

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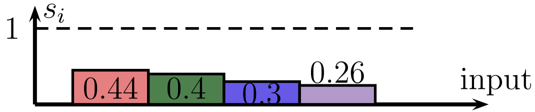


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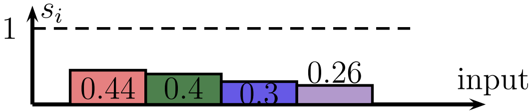


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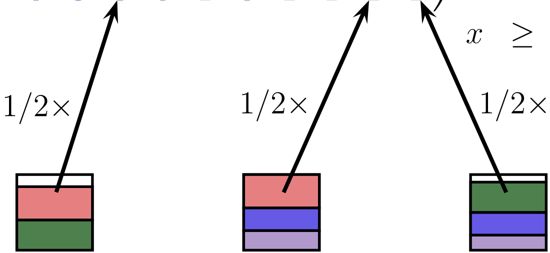


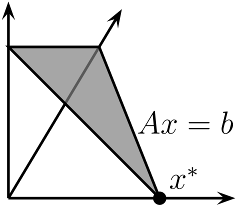
# Main result

- The $O(\log^2 OPT)$ bound of [Karmarkar & Karp '82] is based on:

**Fact:** Any feasible system ($b \in \mathbb{R}^n$)

$$Ax = b, \; x \geq \mathbf{0}$$

has a solution $x^*$ with $|\text{supp}(x^*)| \leq n$.



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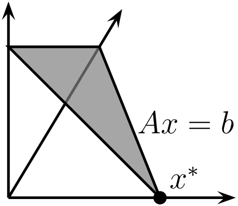


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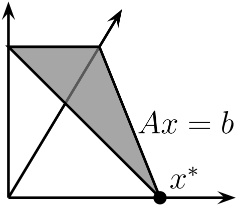


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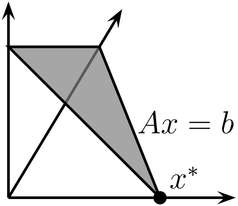


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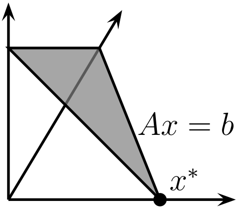


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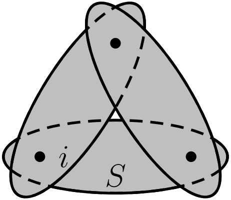
**$\Rightarrow$ Discrepancy theory**

# Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}, S_i \subseteq [n]$



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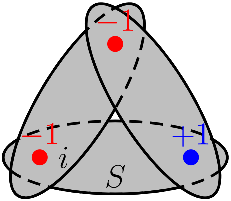

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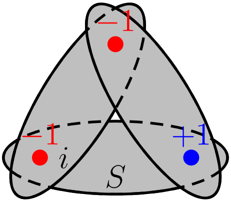

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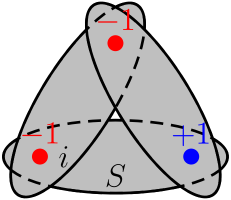


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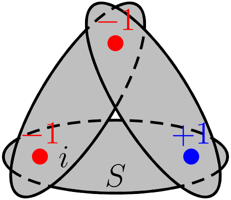


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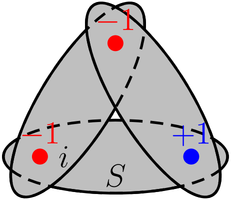


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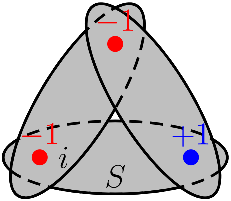


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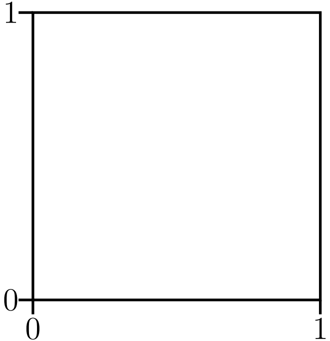
**$\Rightarrow$ Entropy method / Partial coloring method**

- Initially **non-constructive!**
  Recent algorithms by [Bansal '10, Lovett-Meka '12]

# Constructive Partial Coloring Lemma

**Lemma [Lovett-Meka '12]**



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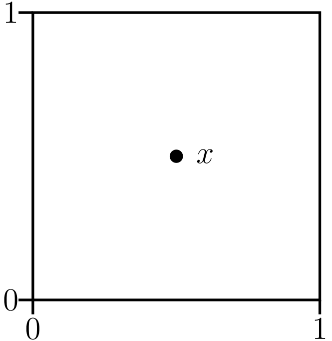
Given $x \in [0,1]^m$



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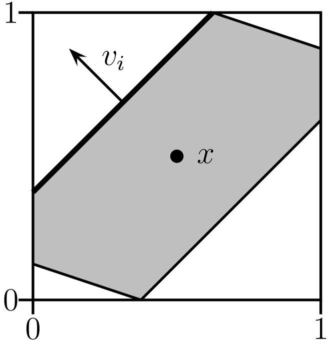

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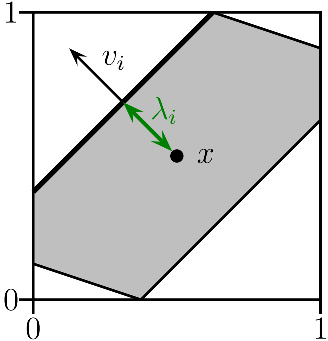
**Lemma [Lovett-Meka '12]**

Given $x \in [0,1]^m$, unit vectors $v_i$, parameters $\lambda_i \geq 0$

- $|\langle v_i, y - x \rangle| \leq \lambda_i \; \forall i$



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**Lemma [Lovett-Meka '12]**

Given $x \in [0,1]^m$, unit vectors $v_i$, parameters $\lambda_i \geq 0$

Then one can find $y \in [0,1]^m$ with

- $y_j \in \{0,1\}$ for at least half of the indices $j$
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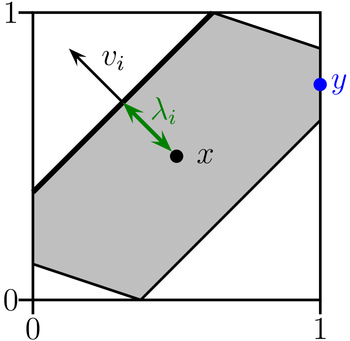

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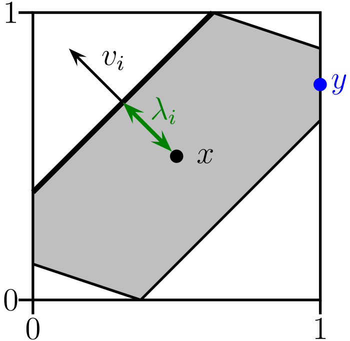

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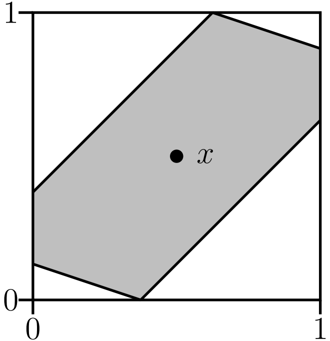
- **Algorithm:**
  (1) Perform Brownian motion with std. deviation 3 in each direction



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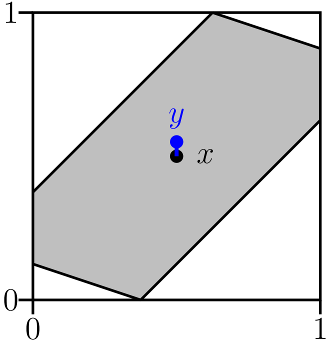

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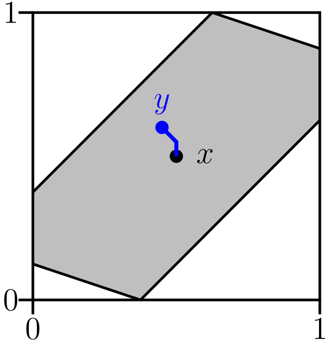

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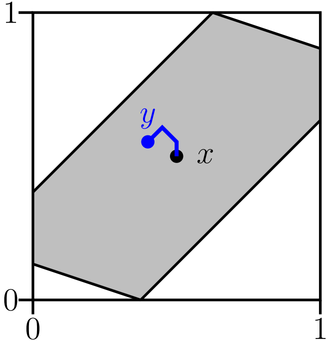

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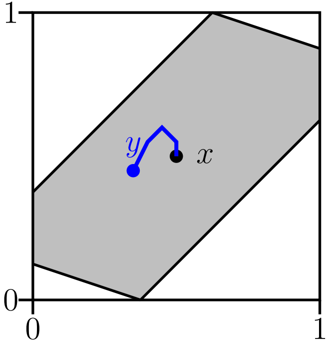

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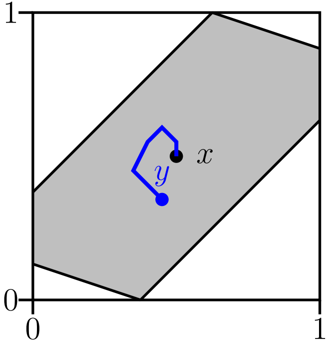

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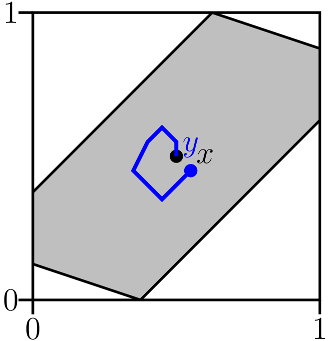

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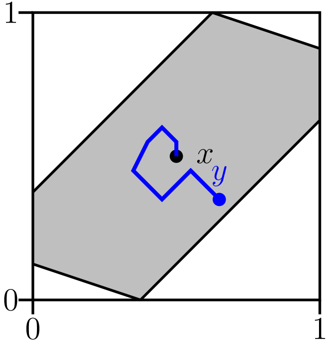

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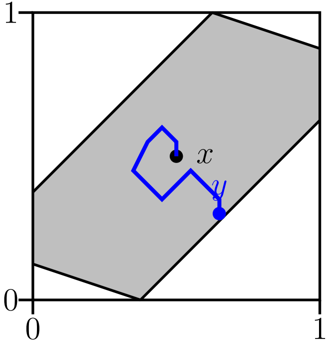

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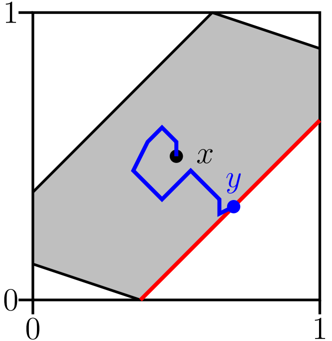

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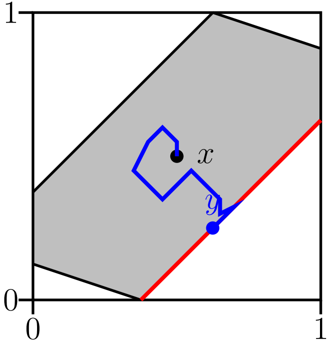

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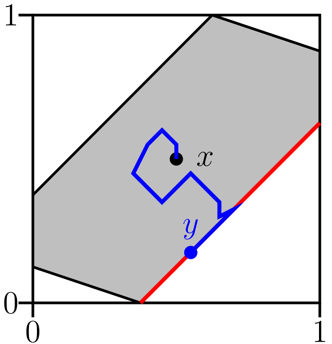

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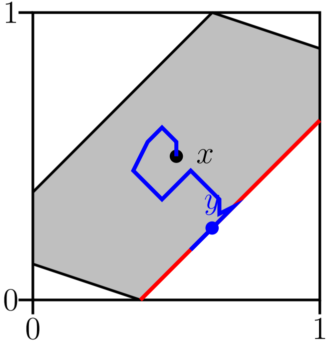

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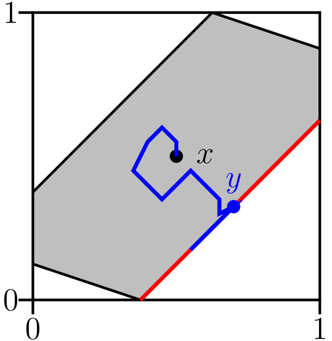

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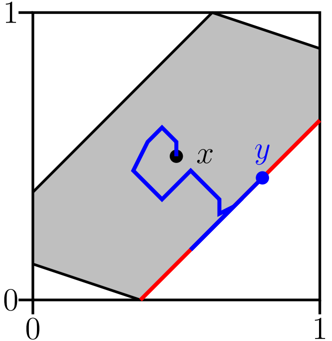

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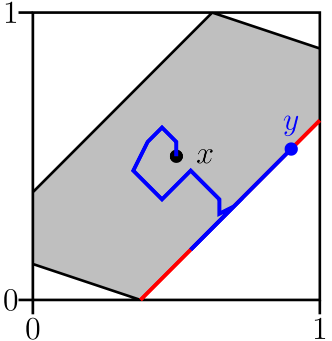

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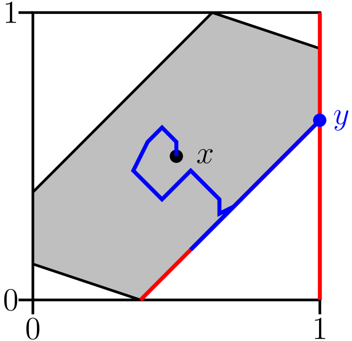

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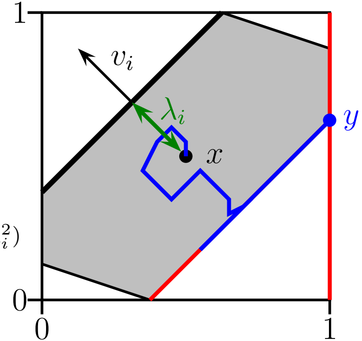

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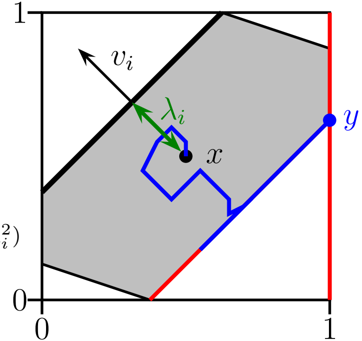

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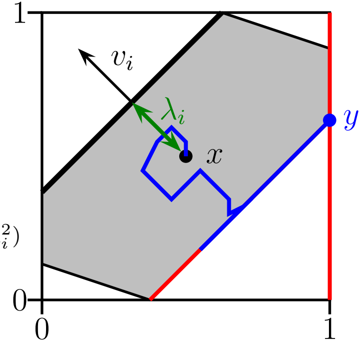


# The algorithm – a first attempt

(1) Compute a fractional LP solution $x$

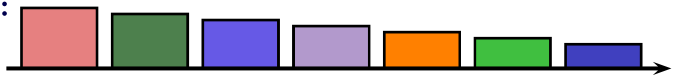
(2) FOR $\log n$ iterations DO

    (3) run the constructive partial coloring lemma to make half of the variables integral

# What property should $y$ satisfy?

**Input:**

$Ax$



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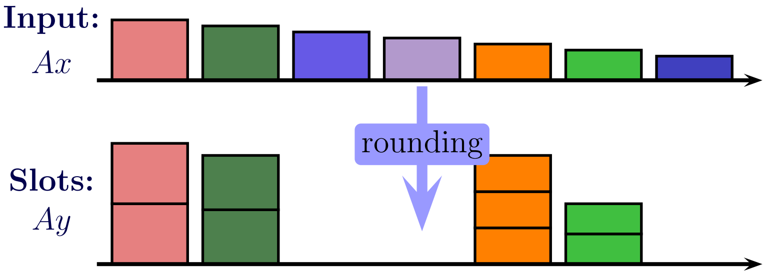

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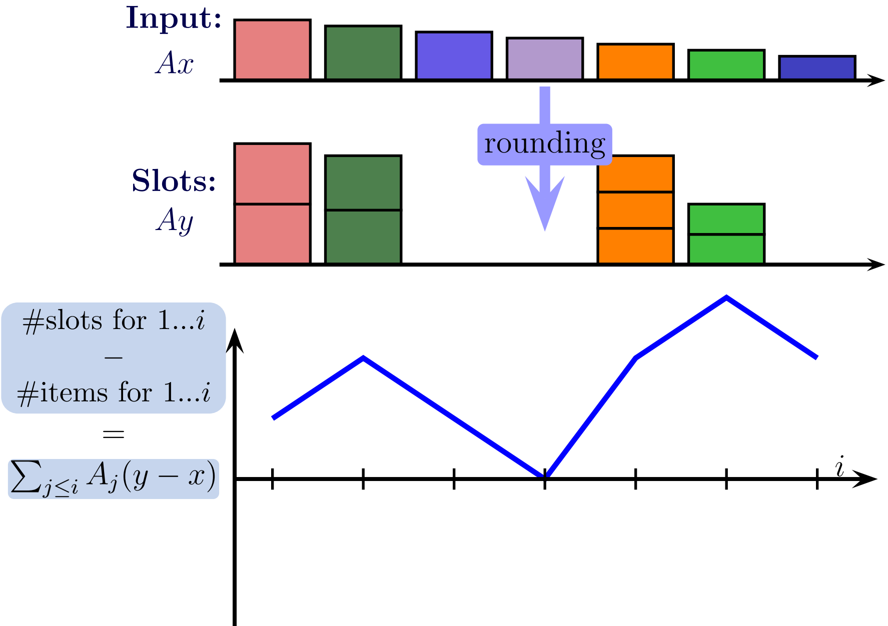

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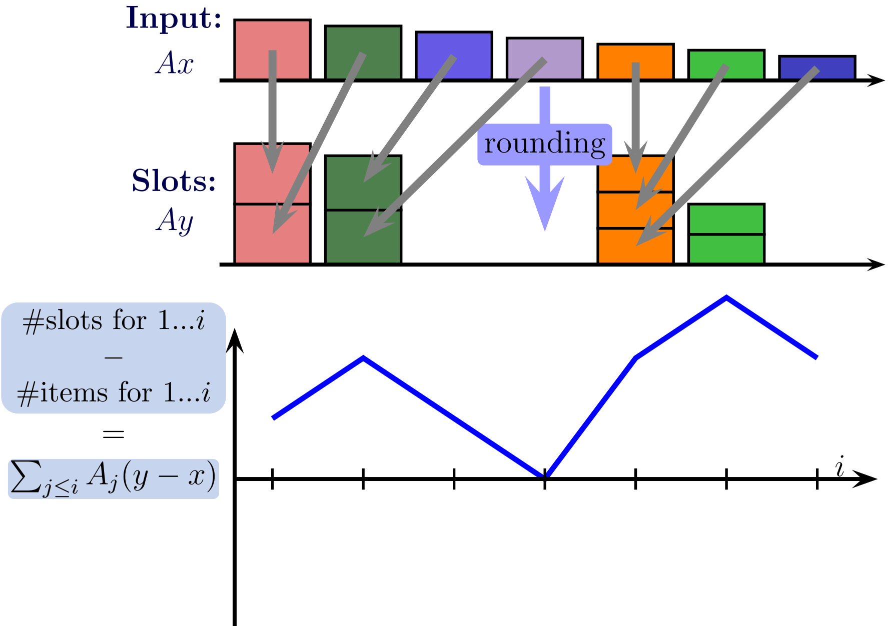

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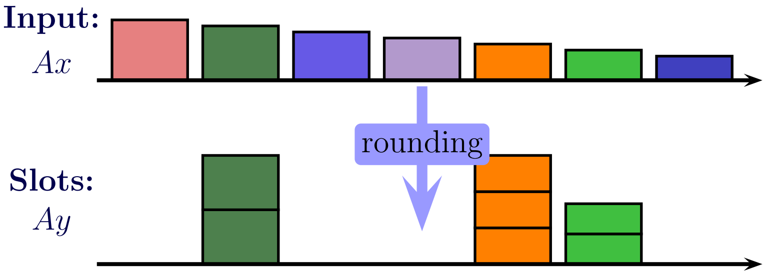
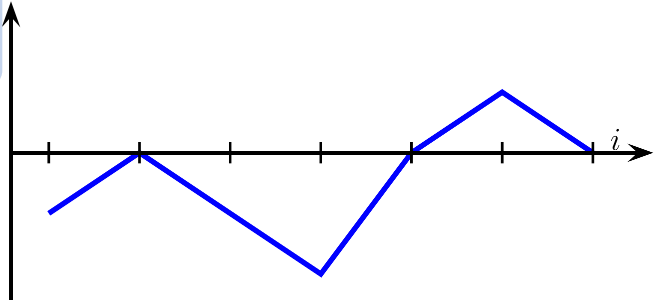



#slots for 1...$i$
−
#items for 1...$i$
=
$\sum_{j\leq i} A_j(y-x)$



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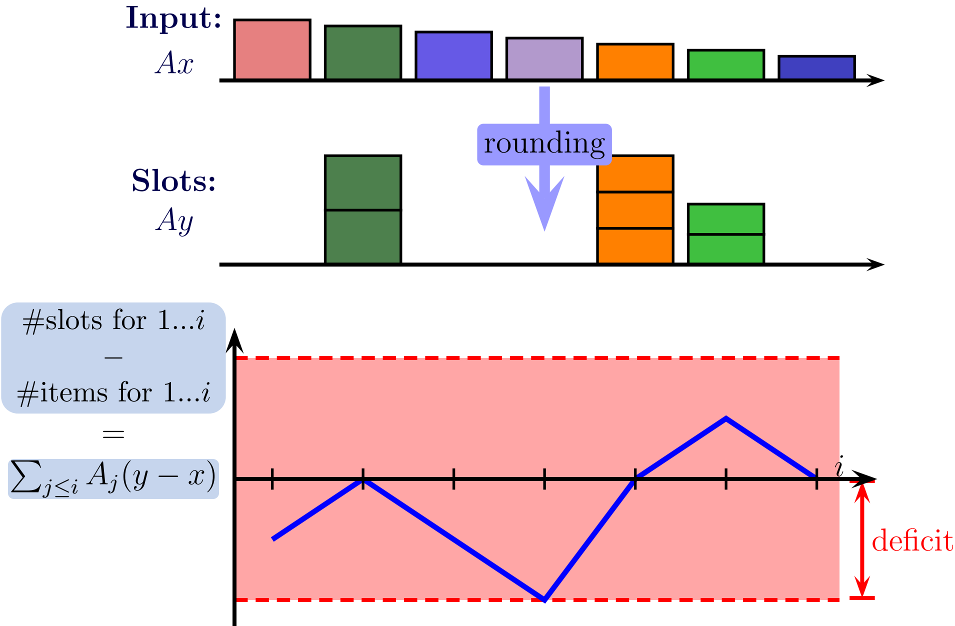
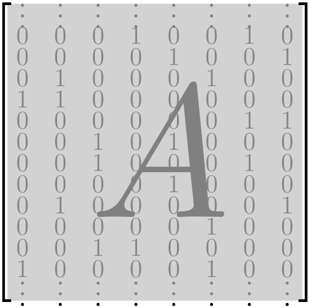


# Applying the Partial Coloring Lemma



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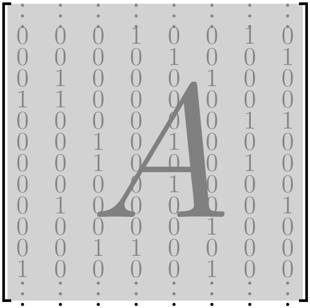


- **Given:** $x$. **Find:** $y$ with $|(\sum_{j\leq i} A_j)(x-y)|$ small $\forall i$
- Suppose $\frac{1}{k} \leq s_i \leq \frac{2}{k}$ for some $k \in \mathbb{N}$

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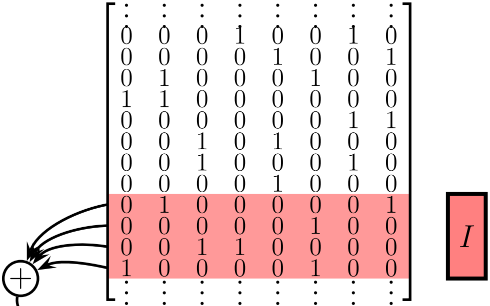


$$v_I = (1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 2 \;\; 0 \;\; 1) = \sum_{i \in I} A_i$$

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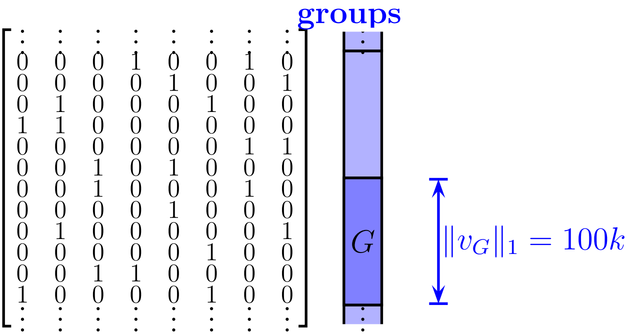


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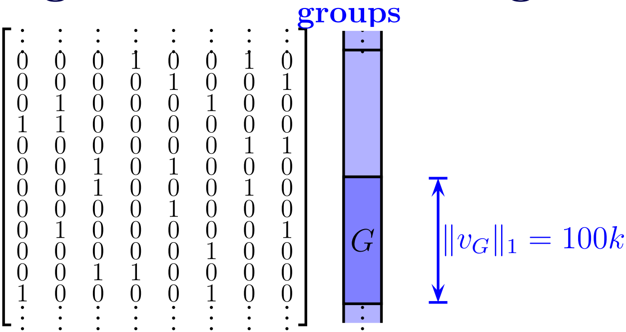


**Constructive Partial Coloring Lemma:**

- **Input:** $x$; $(v_G, \lambda_G = 0)\ \forall G$

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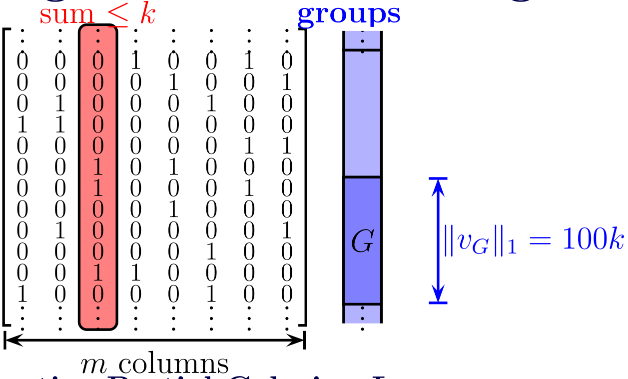


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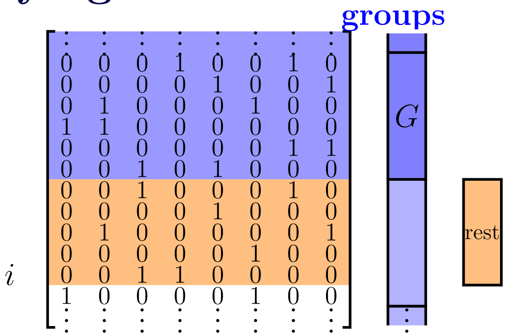


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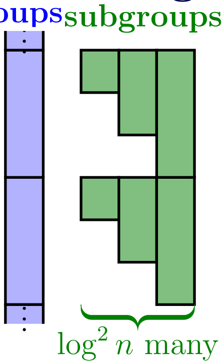
- **Condition:** $\sum_G e^{-\lambda_G^2/16} = \#\text{groups} \le \frac{m}{100}$
- **Output** $y$**:** $v_G(x-y) = 0\ \forall G$

$$\Big|\sum_{j\le i} A_j(x-y)\Big| = \Big|\sum_{G\subseteq\{1,\dots,i\}} \underbrace{v_G(x-y)}_{=0} + \underbrace{v_{\text{rest}}(x-y)}_{\le 100k}\Big| \le 100k$$

# Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$




**Constructive Partial Coloring Lemma:**

- **Input:** $x$; $(v_G, \lambda_G = 0)\ \forall G$; $(v_S, \lambda_S = \log\log n)\ \forall S$
- **Condition:** $\sum_G e^{-\lambda_G^2/16} = \#\text{groups} \leq \frac{m}{100}$
- **Output** $y$**:** $v_G(x-y) = 0\ \forall G$; $|v_S(x-y)| \leq \lambda_S \|v_S\|_2\ \forall S$

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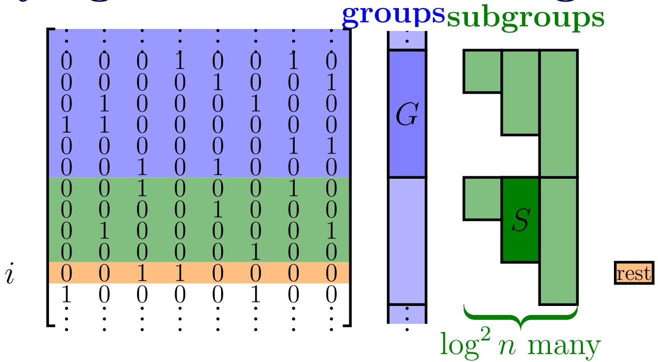


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$$\Big|\sum_{j\leq i} A_j(x-y)\Big| = \Big|\sum_{G\subseteq\{1,\dots,i\}} \underbrace{v_G(x-y)}_{=0} + \underbrace{v_S(x-y)}_{\lesssim\|v_S\|_2} + \underbrace{v_{\text{rest}}(x-y)}_{\lesssim\frac{1}{\log^2 n}k}\Big|$$

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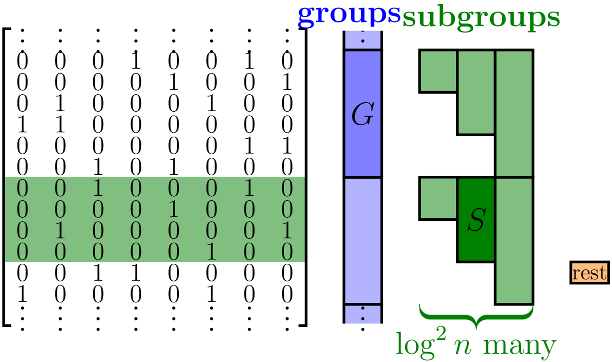


- Can get

$$\|v_S\|_2 \overset{\text{Hölder}}{\leq} \sqrt{\frac{\|v_S\|_\infty}{\|v_S\|_1}} \cdot \|v_S\|_1$$

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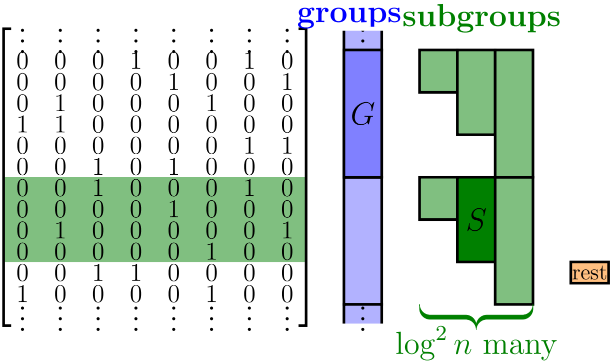


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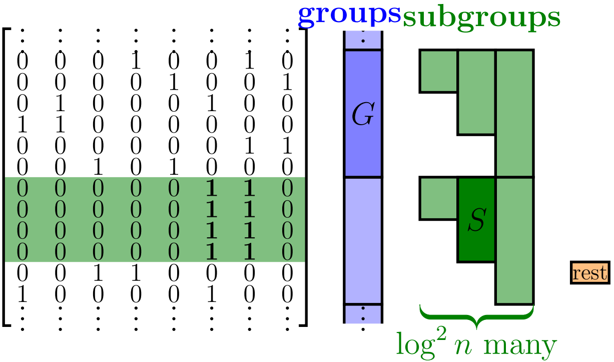


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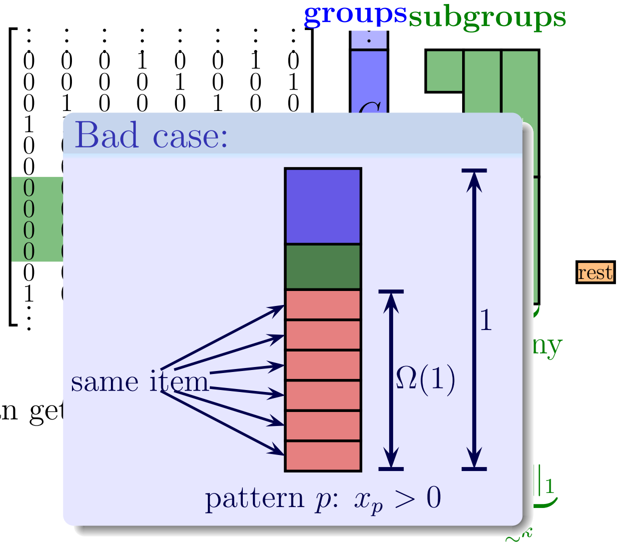
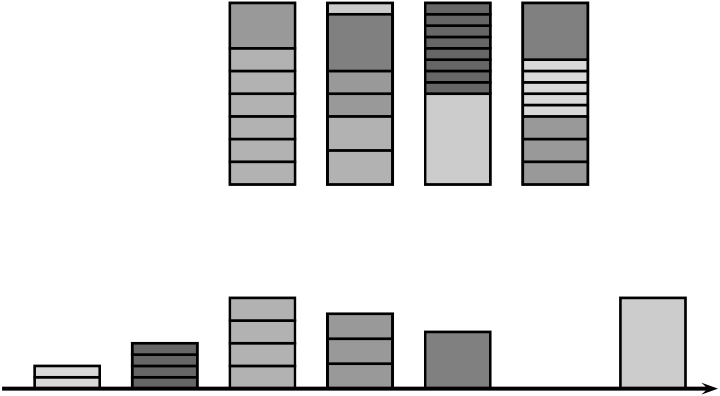



- Can get

# Gluing items

**fractional solution:**

**Input:**



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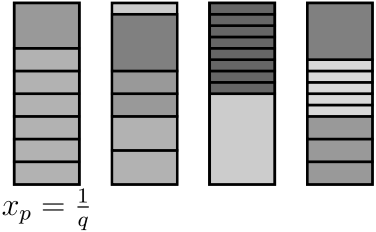


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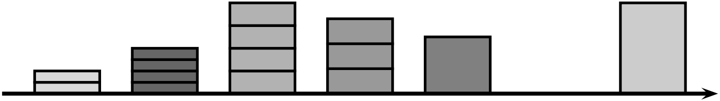

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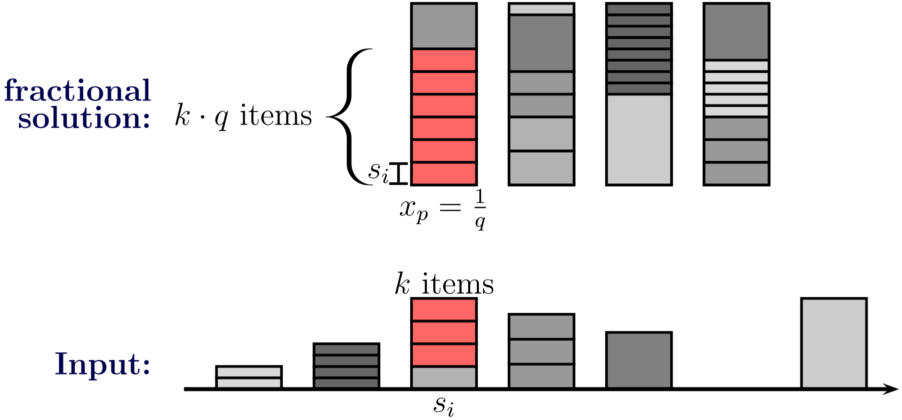

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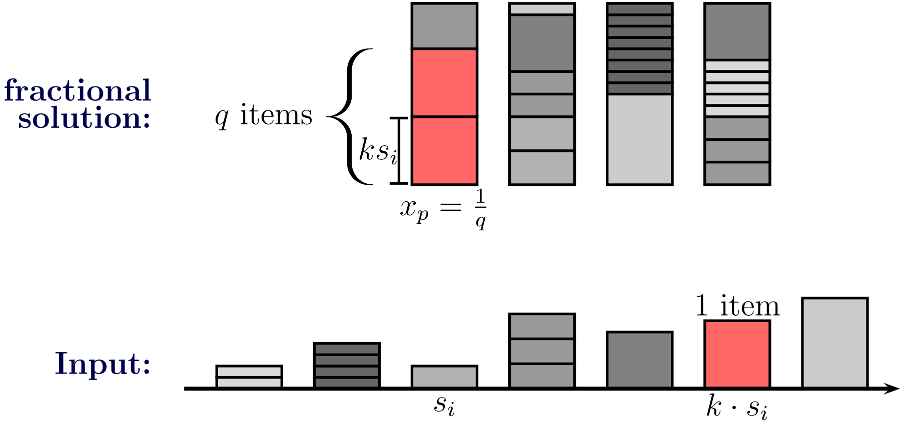

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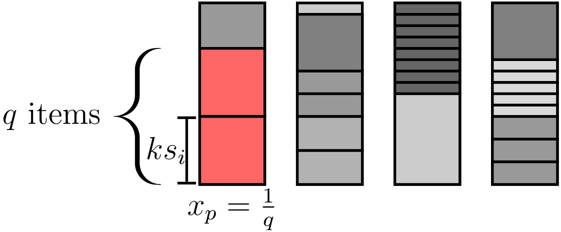


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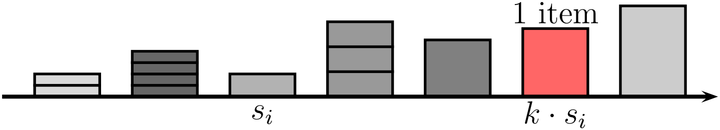


**Observations:**

- Doesn't change feasibility or objective function

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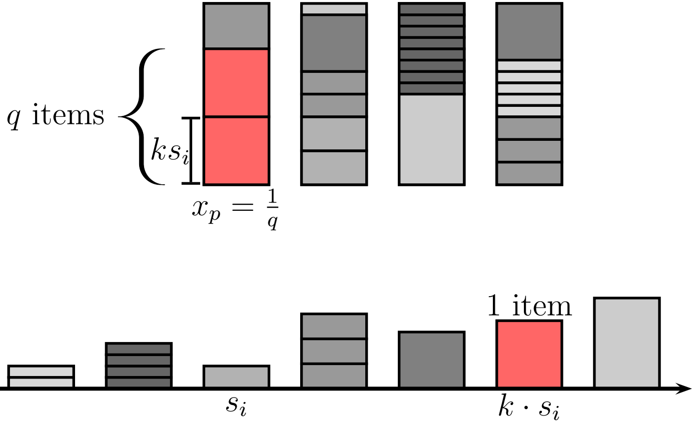
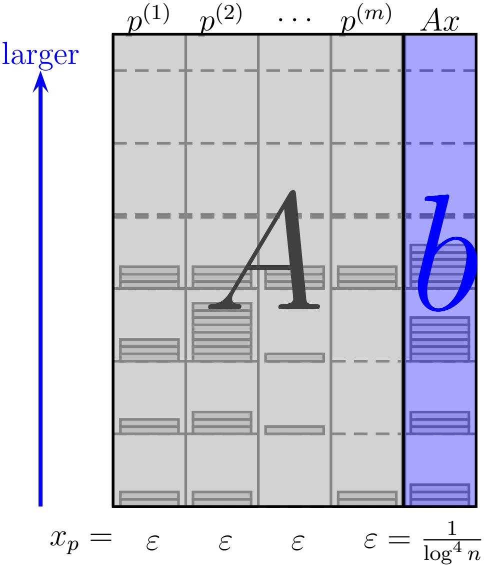



**Observations:**

- Doesn't change feasibility or objective function
- Any solution to new instance induces solution to original one

# Making instance well spread



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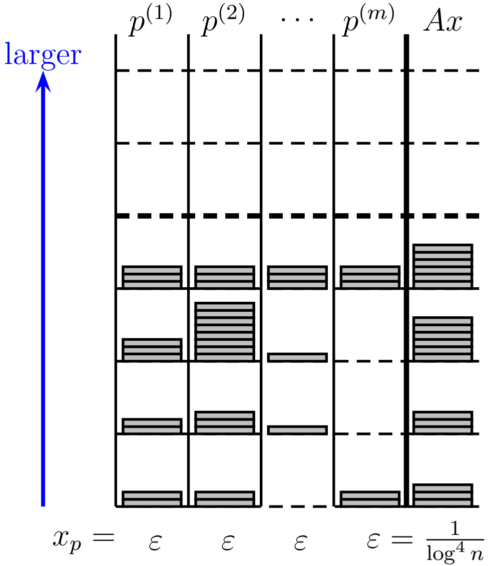

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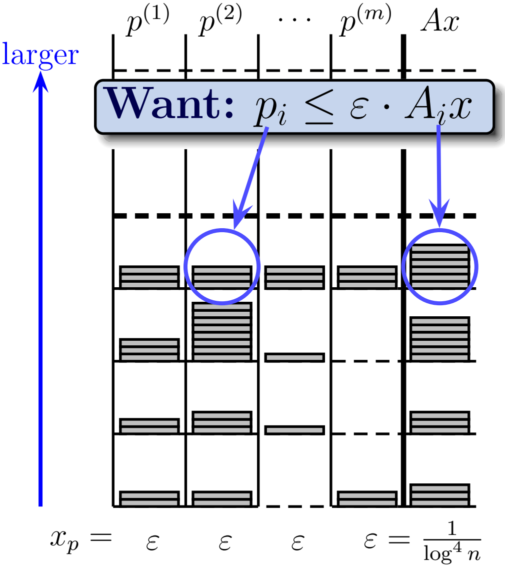

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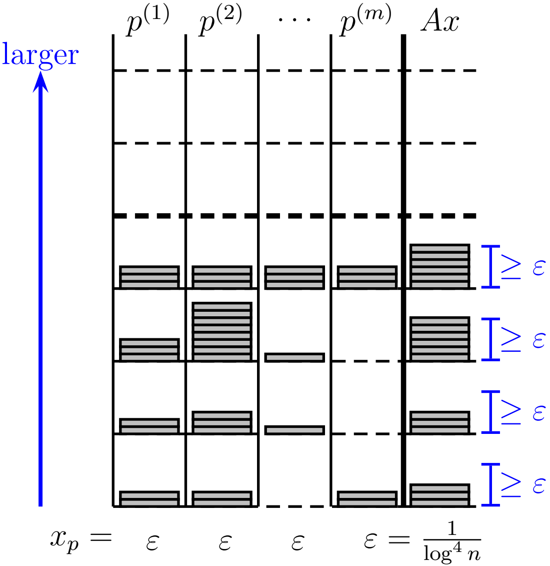

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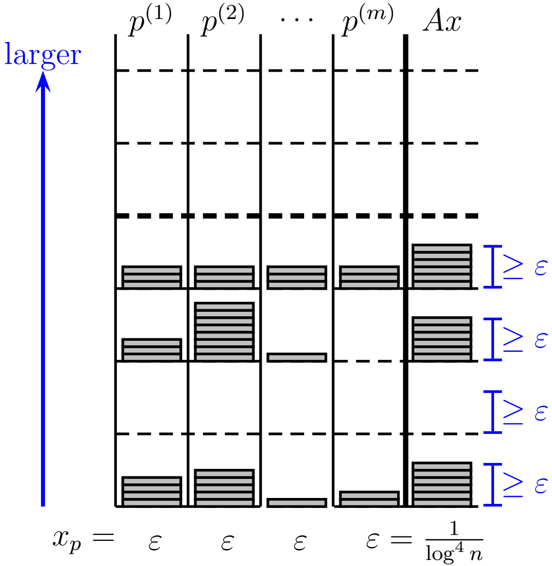


- **Cost:** $O(\varepsilon \cdot \log n)$

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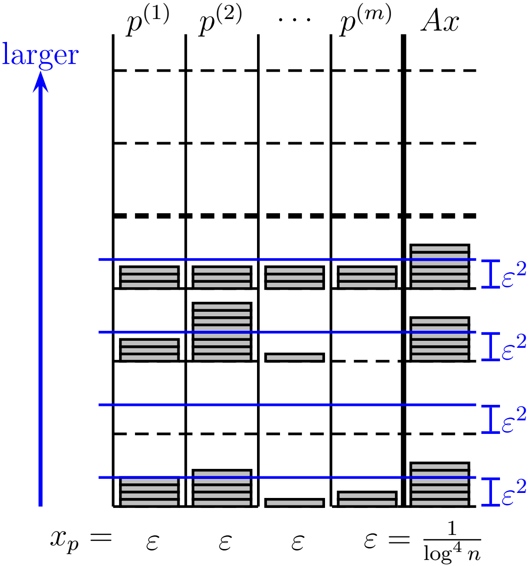


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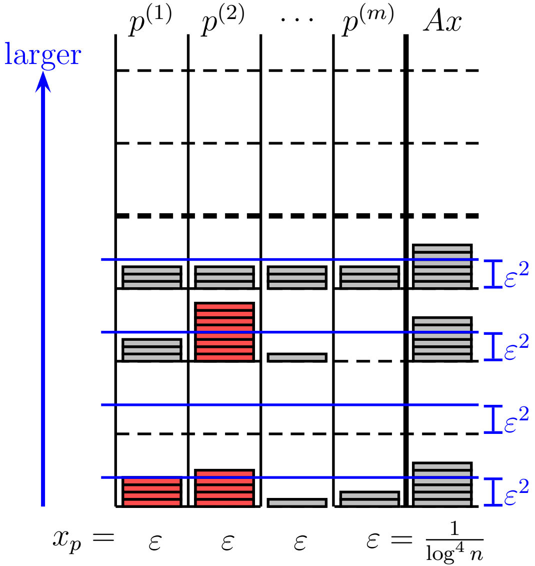


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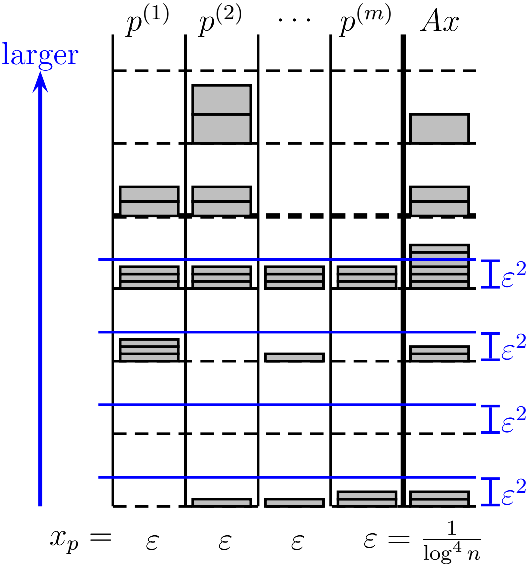


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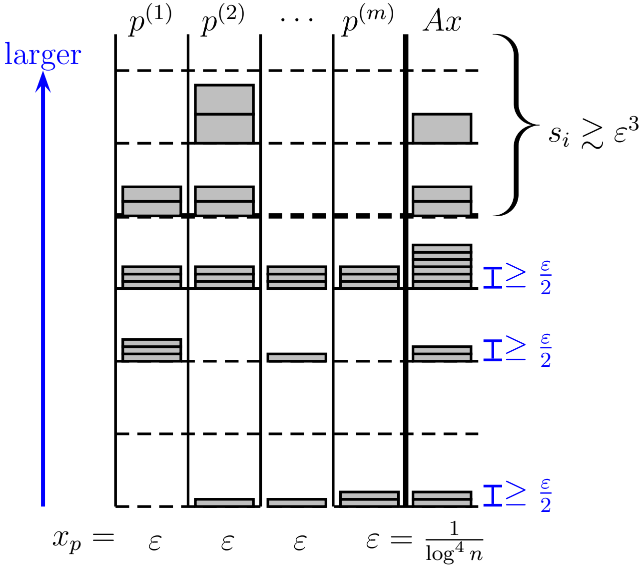


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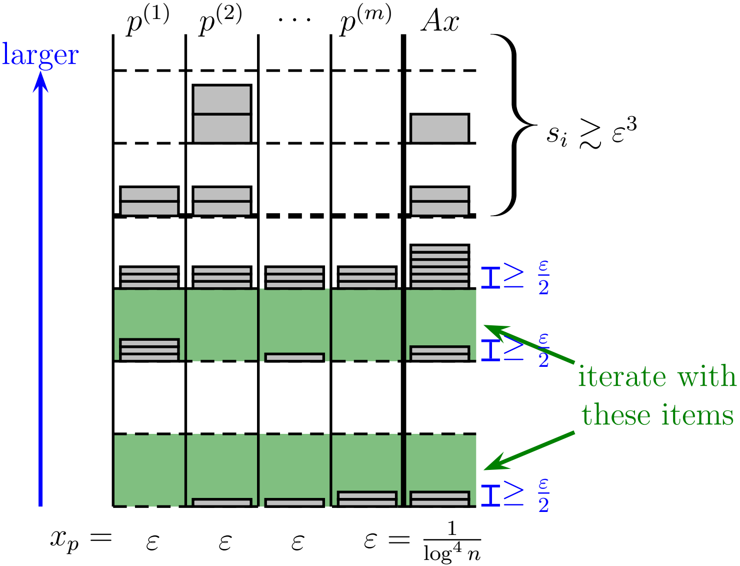


- **Cost:** $\log n \cdot O(\varepsilon \cdot \log n)$

# The complete algorithm

(1) Compute a fractional LP solution $x$

(2) FOR $\log n$ iterations DO

  (3) round $x$ s.t. $x_p \in \frac{\mathbb{Z}}{\log^4 n}$

  (4) apply gluing lemma

  (5) run the constructive partial coloring lemma to make half of the variables integral

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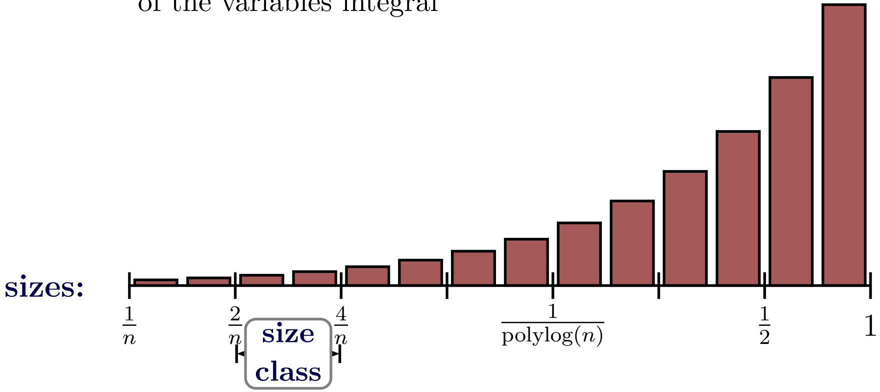

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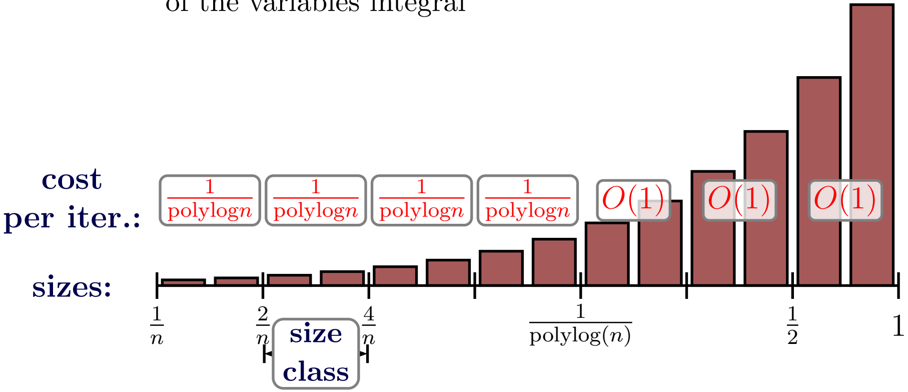

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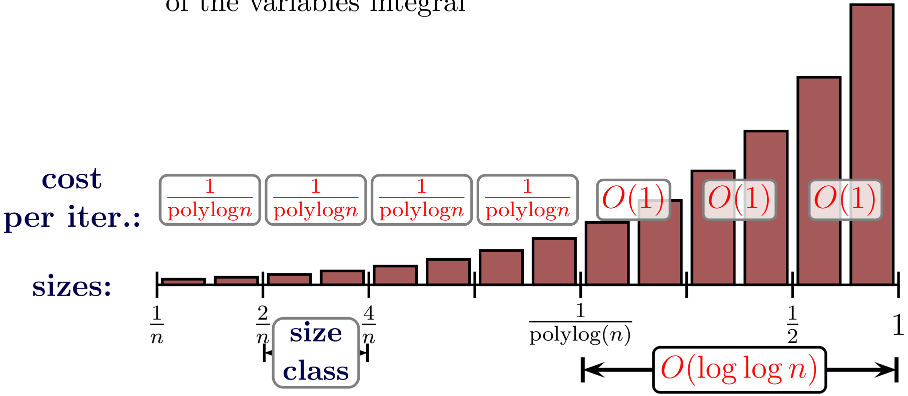

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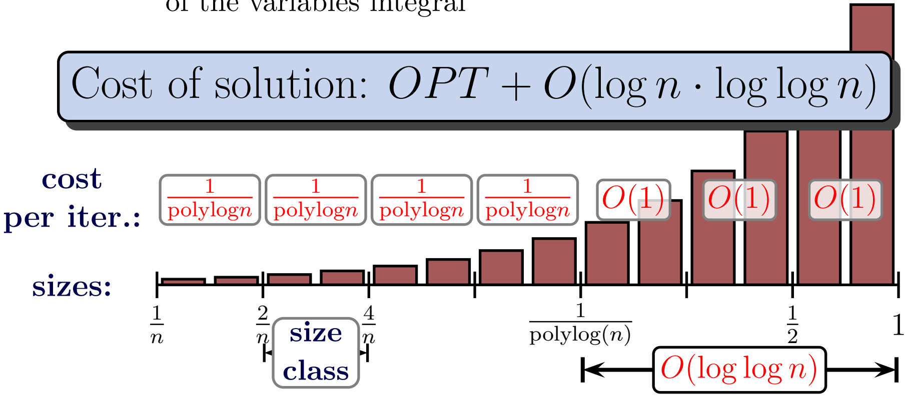
**Cost of solution: $OPT + O(\log n \cdot \log\log n)$**



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**Open question I**

Close the gap

$$1 \le \text{additive integrality gap} \le O(\log n \cdot \log\log n)$$

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## Open question I

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## Open question II

Are there other applications of techniques from **discrepancy theory** such as

- Partial coloring method
- Banaszczyk's vector balancing theorem

in **approximation algorithms**?

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Thanks for your attention